Posets with each interval Homotopy Equivalent to a Ball or Sphere

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(joint work with Karola Mészáros)

Talk Plan: 1. SB-labelings: Definition & Consequences
2. Applications: weak order, finite distributive lattices, Tamari lattices, Pieri posets
3. Proof Ideas
4. Open Questions: Dominance Order, ...

Question (Björner & Greene): Why do so many posets P have Möbius function satisfying $\mu_P(u,v) = 0, \pm 1$ for all $u \leq v$ in P? Is there a unifying explanation?

Some Well-Known Examples:

weak Bruhat order, Tamari lattice, dominance order, finite distributive lattices, Cambrian lattices, ...

Doorway to Topology (Philip Hall, Rota):

$$\mu_P(u,v) = \tilde{\chi}\,(\underbrace{\Delta(u,v)}_{})$$

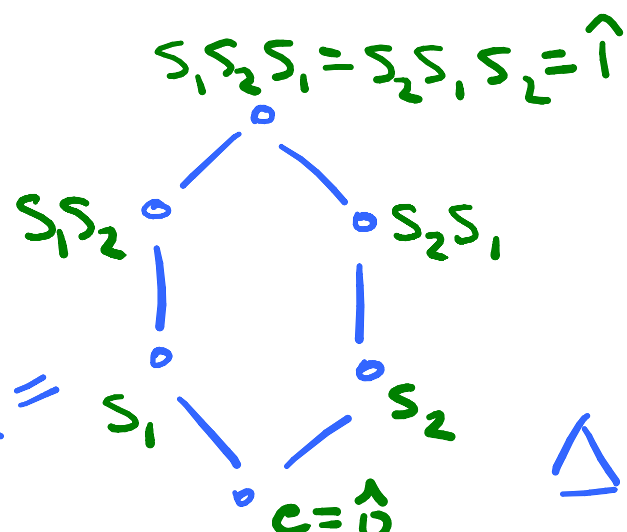
reduced Euler characteristic
=
$-1 + \#\text{vertices} - \#\text{edges} + \ldots$

"order complex" of $\{z \in P \mid u < z < v\}$, i.e. simplicial complex whose faces are poset chains

Recall: The order complex, denoted $\Delta(L)$, for a poset L is the simplicial complex whose i-dimensional faces are the chains $u_0 < u_1 < \dots < u_i$ of comparable elements in L.

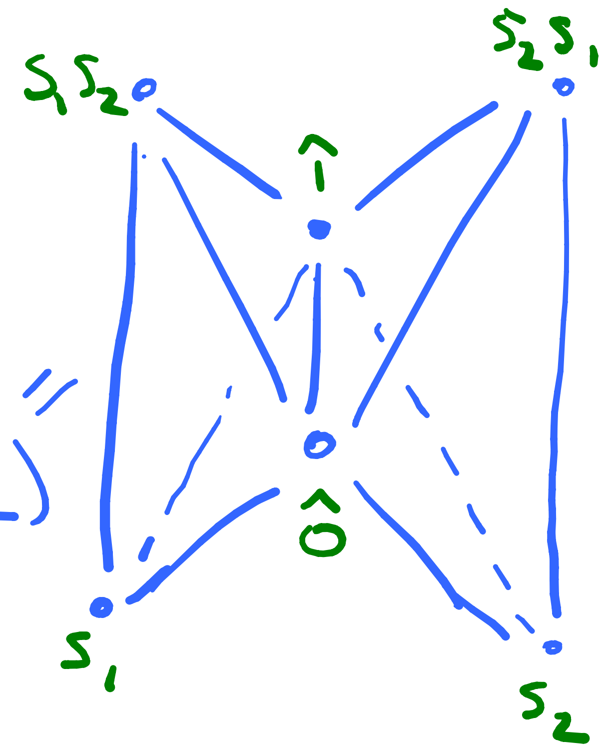
e.g.

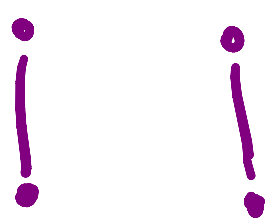
$L =$



$=$ weak Bruhat order

$\Delta(L) =$



$\Delta(\underbrace{\hat{0},\hat{1}}_{\text{subposet strictly between } \hat{0} \text{ and } \hat{1}}) =$  $\simeq S^0$

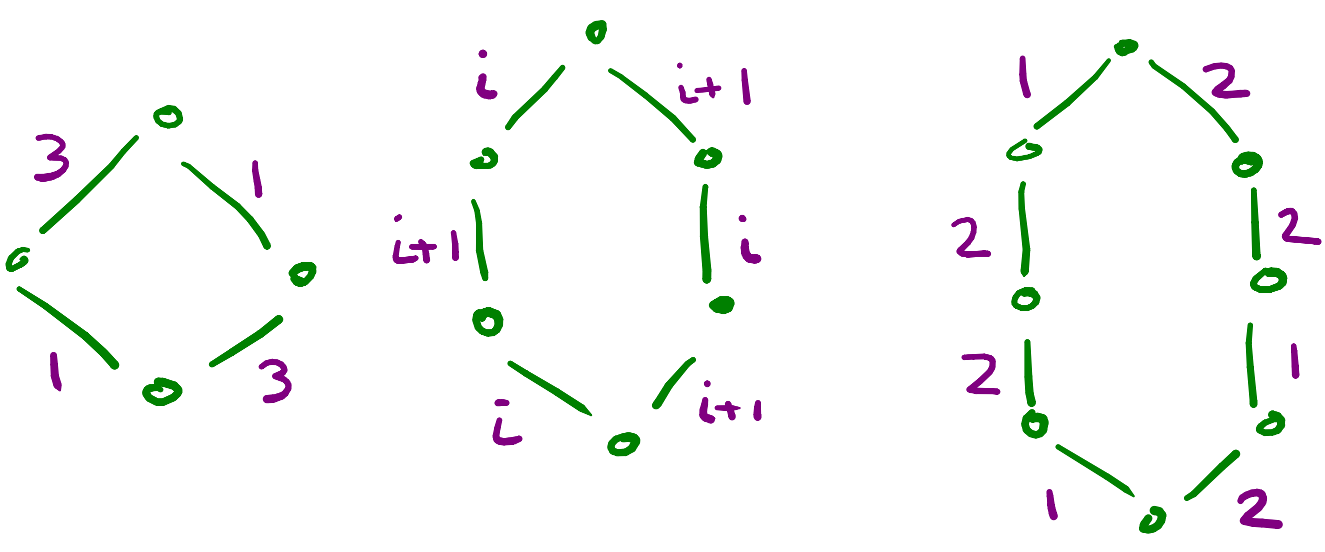
SB-labeling (Index 2 formulation):

Let λ be an edge labeling of a finite lattice L s.t. for all $u,v,w \in L$ such that $u \lessdot v$ & $u \lessdot w$

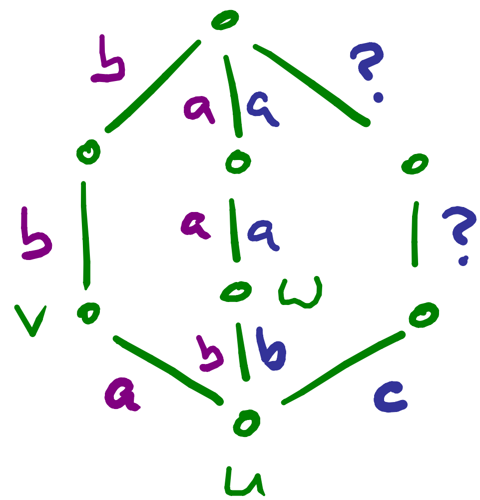
(1) $\lambda(u,v) \neq \lambda(u,w)$ for $v \neq w$

(2) Every saturated chain M on $[u, v \vee w]$ uses exactly the label set $\{\lambda(u,v), \lambda(u,w)\}$, using each label with a positive multiplicity.

e.g.



Non-Example:



Main Results:

Thm 1.: An edge labeling λ on finite lattice L is SB-labeling (index 2 formulation) $\iff$ λ is SB-labeling (general index formulation).

(therefore call either one "SB-labeling")

Thm 2: If finite lattice L has edge labeling λ which is SB-labeling, then $\Delta_L(u,v)$ is homotopy equivalent to ball or sphere for each $u < v$.

SB-Labeling (General Index Formulation)

- Given a finite lattice L with atoms $A(L)$, an edge-labeling with label set S is an lower SB-labeling if:
  - (1) $A(L) \subseteq S$ and $\lambda(\hat{0}, a) = a$ for each $a \in A(L)$
  - (2) If $x \in L$ satisfies $x = \underbrace{a_{i_1} \vee \dots \vee a_{i_r}}_{\text{join of atoms}}$ then all saturated chains M on $[\hat{0}, x]$ use exactly the labels $\{a_{i_1}, \dots, a_{i_r}\}$ each with positive multiplicity.
- If these conditions are met for every interval $[u,v]$ then λ is an SB-labeling.

"Sphere" or "Ball"

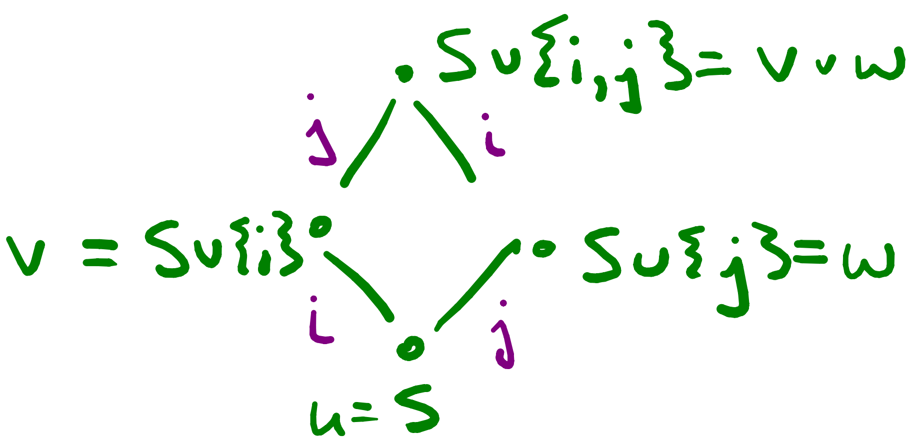
1st Example: Finite Distributive Lattices

- Use that $L = J(P)$ = poset of order ideals of P ordered by set containment

- Let $\lambda(S \lessdot S \cup \{i\}) = i$

  order ideals

- Labeling is also an EL-labeling with



2nd Example: Weak Bruhat Order

Idea: Use $\lambda(u \lessdot s_i u) = s_i$ justified by following results (see e.g. Björner-Brenti):

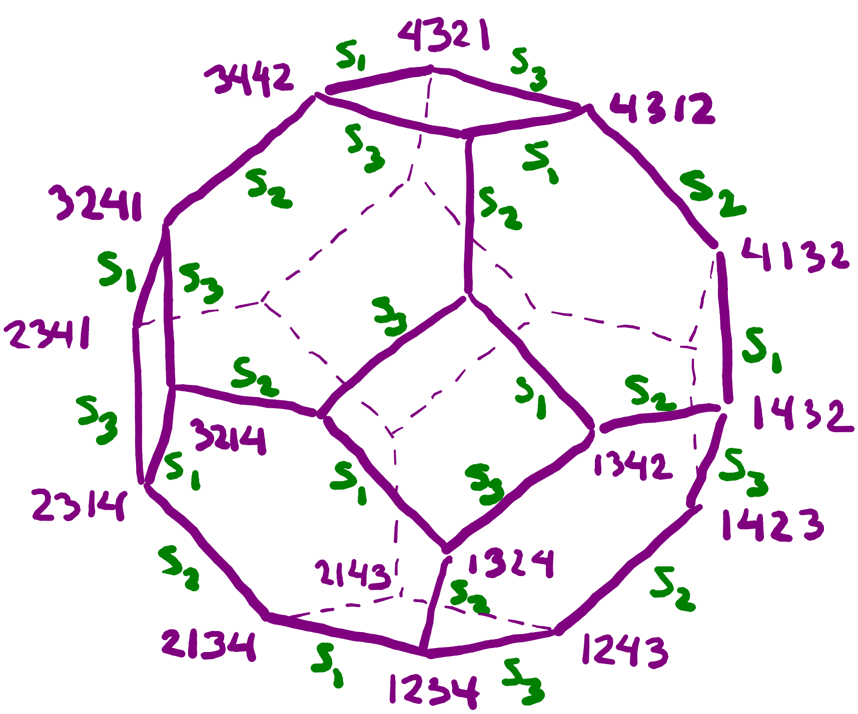
- x is a join of atoms $\Longleftrightarrow$ $x = w_0(W_J)$ for a parabolic subgroup W_J
  $\Longleftrightarrow$ all simple reflections in a reduced expression for x may appear rightmost

- intervals $[u, v \vee w]$ for $u \lessdot v$ and $u \lessdot w$ have two saturated chains given by braid relation

  $$s_i s_j \ldots = s_j s_i \ldots$$

  since

  $$[u, x] \cong [e, u^{-1}x]$$



Homotopy type 1st due to Björner & Edelman.

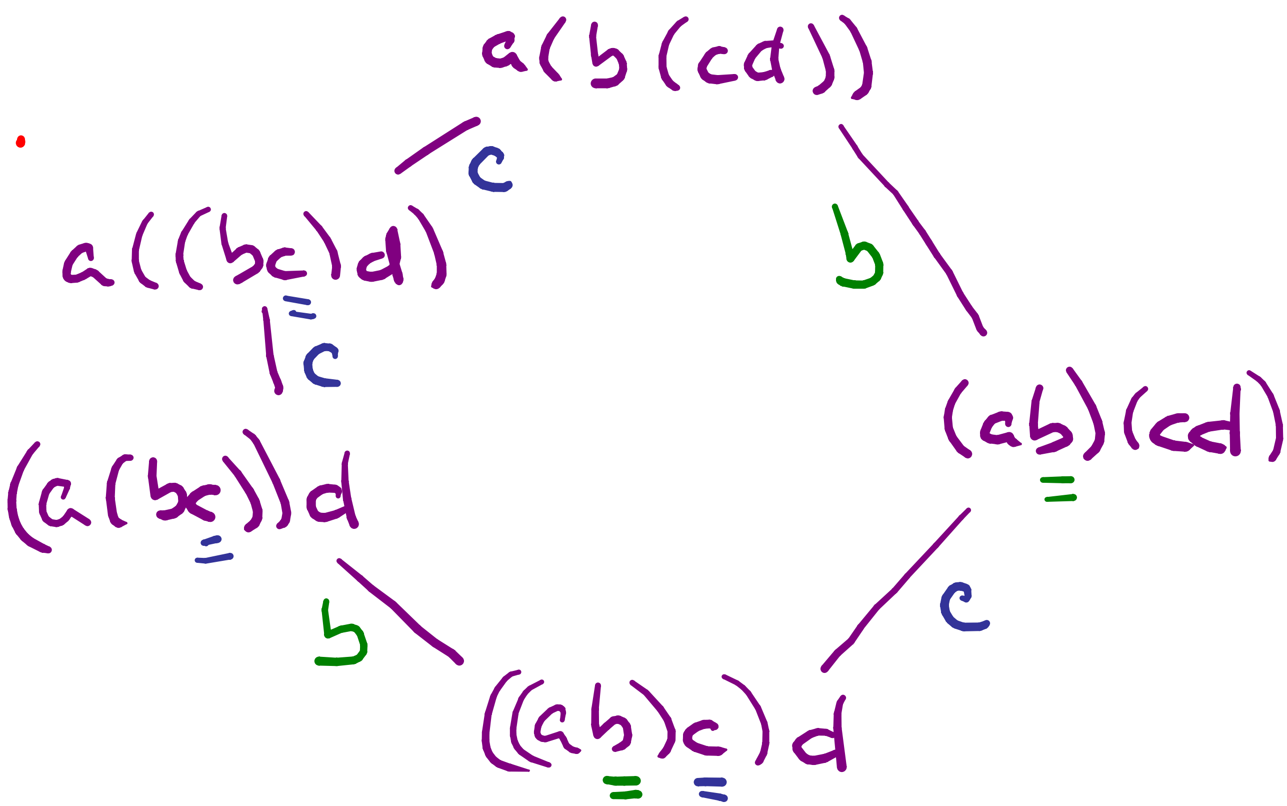
3rd Example: Tamari lattice

Idea: Poset of binary trees with

cover relations:  $((a,b),c) \lessdot (a,(b,c))$

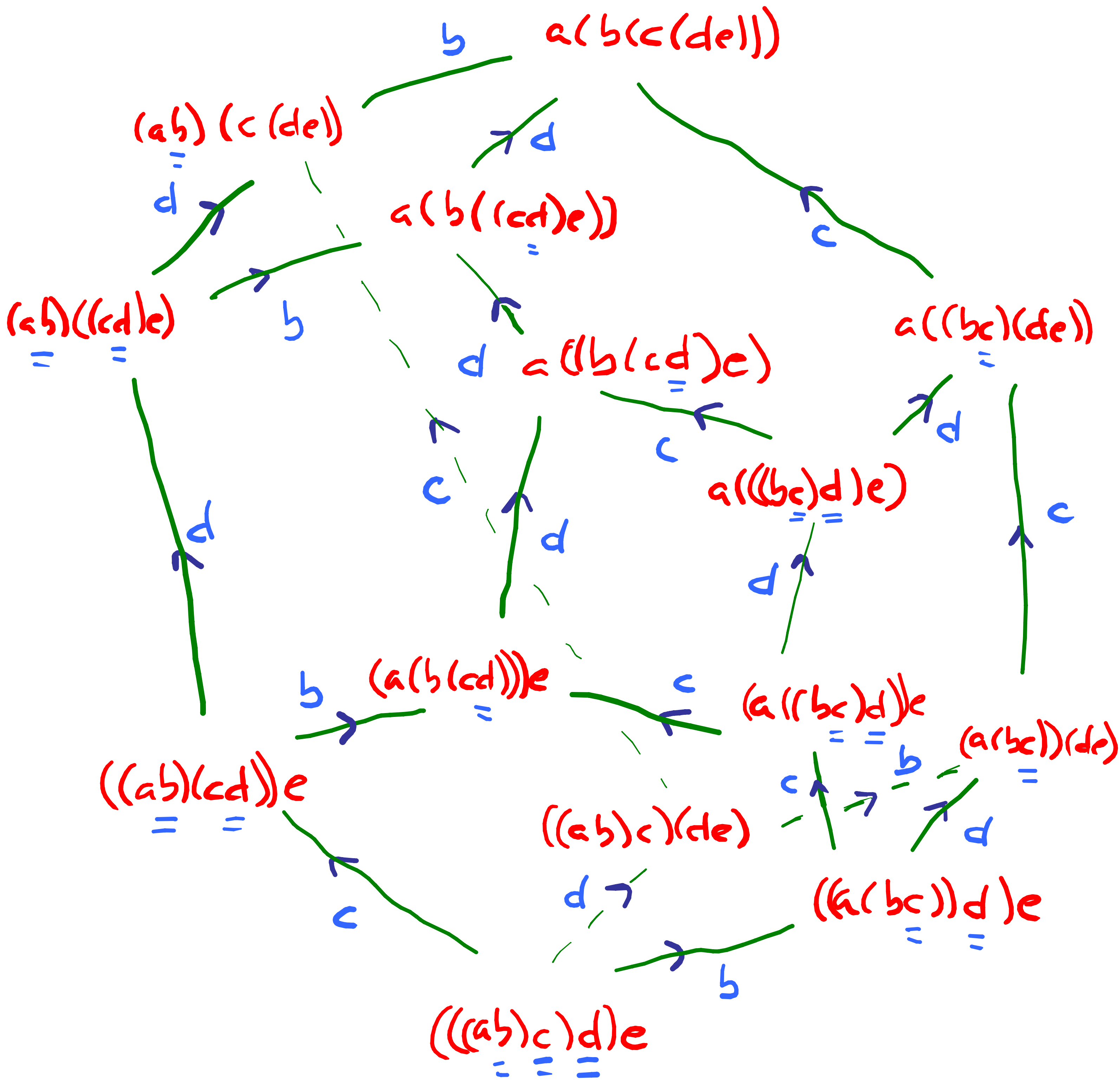
- label $u \lessdot v$ with letter to immediate left of right parenthesis being moved in binary bracketing

e.g.



Homotopy type 1st due to Björner-Wachs via nonpure lexicographic shellability & by Pallo

SB-Labeling for Tamari Lattice

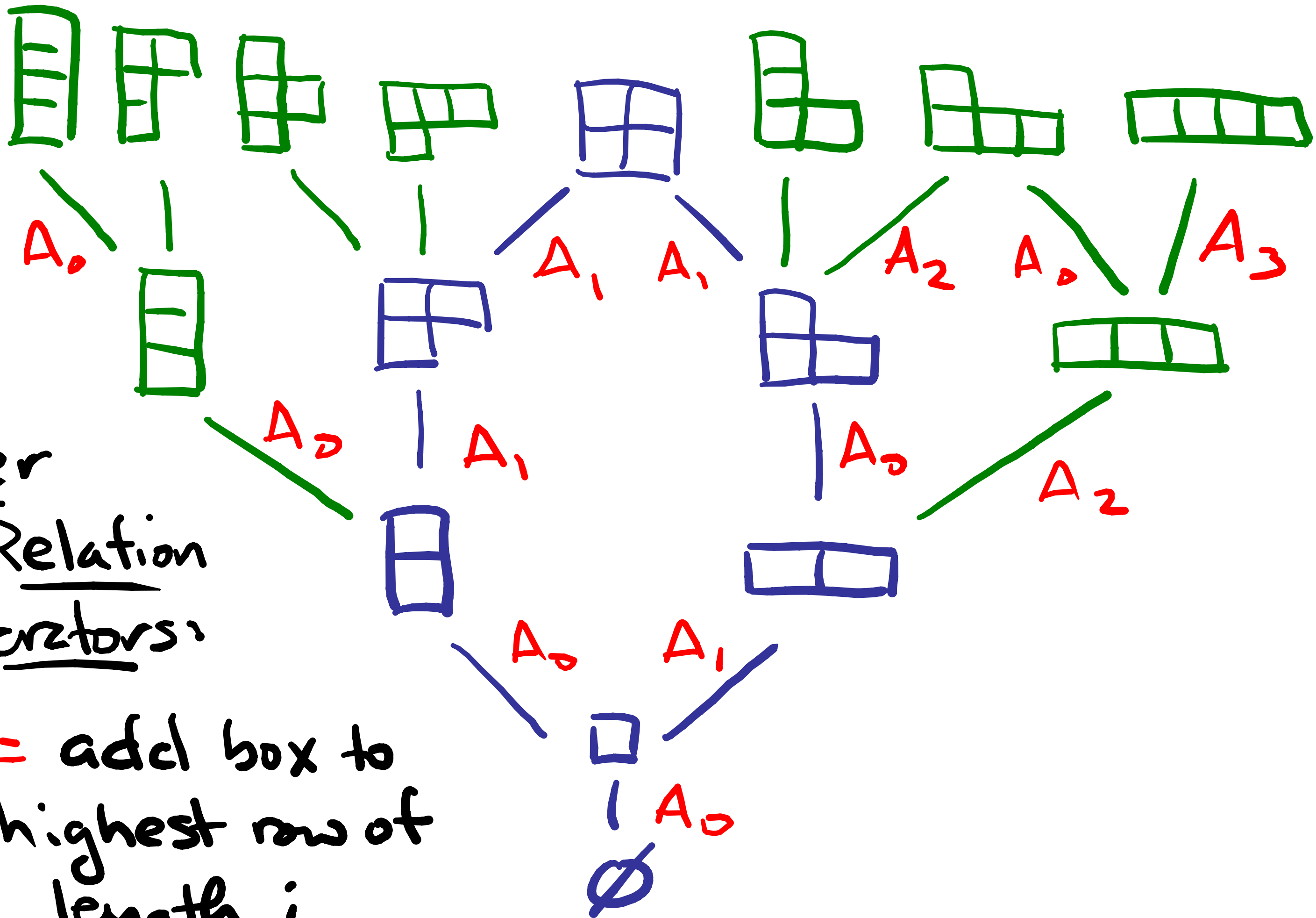


Relations: $A_i A_j(u) = A_j A_i(u)$

& $A_j A_j A_i(u) = A_i A_j(u)$ for $i < j$

4th Example: Pieri rule poset for noncommutative symmetric functions

(analogue of Young's lattice for compositions introduced by Hoglund - Luoto - van Willigenburg)



Cover Relation Operators:

$A_i :=$ add box to highest row of length i

$A_0 :=$ add new row of length 1 at top of shape

Relations: $A_{i+1}A_iA_{i+1}(u) = A_{i+1}A_{i+1}A_i(u)$

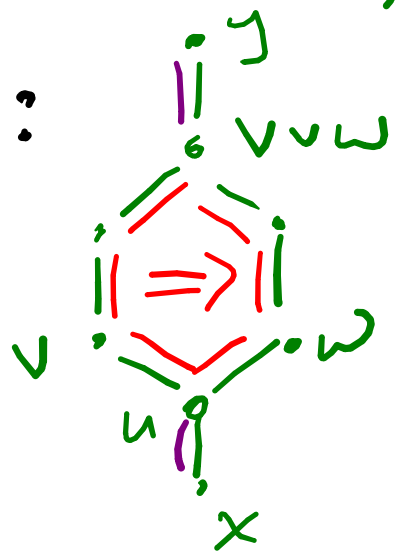
or $A_iA_j(u) = A_jA_i(u)$ for each $u \in L$.

(outgrowth of work w/ Dye, Mészáros & Tenner)

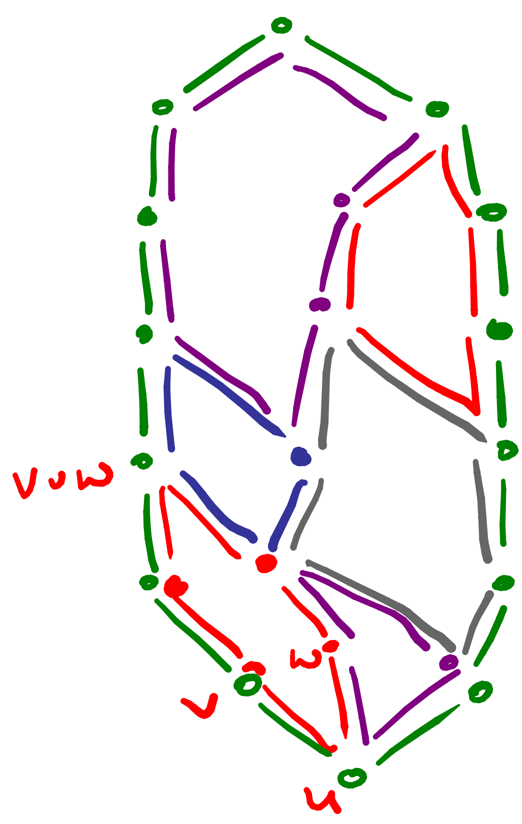
Thm 1. SB labeling (Index 2 Formulation) $\Leftrightarrow$ SB labeling (General Index Formulation)

Proof Ingredients:

1. any two saturated chains on $[x, y]$ connected by "basic moves":

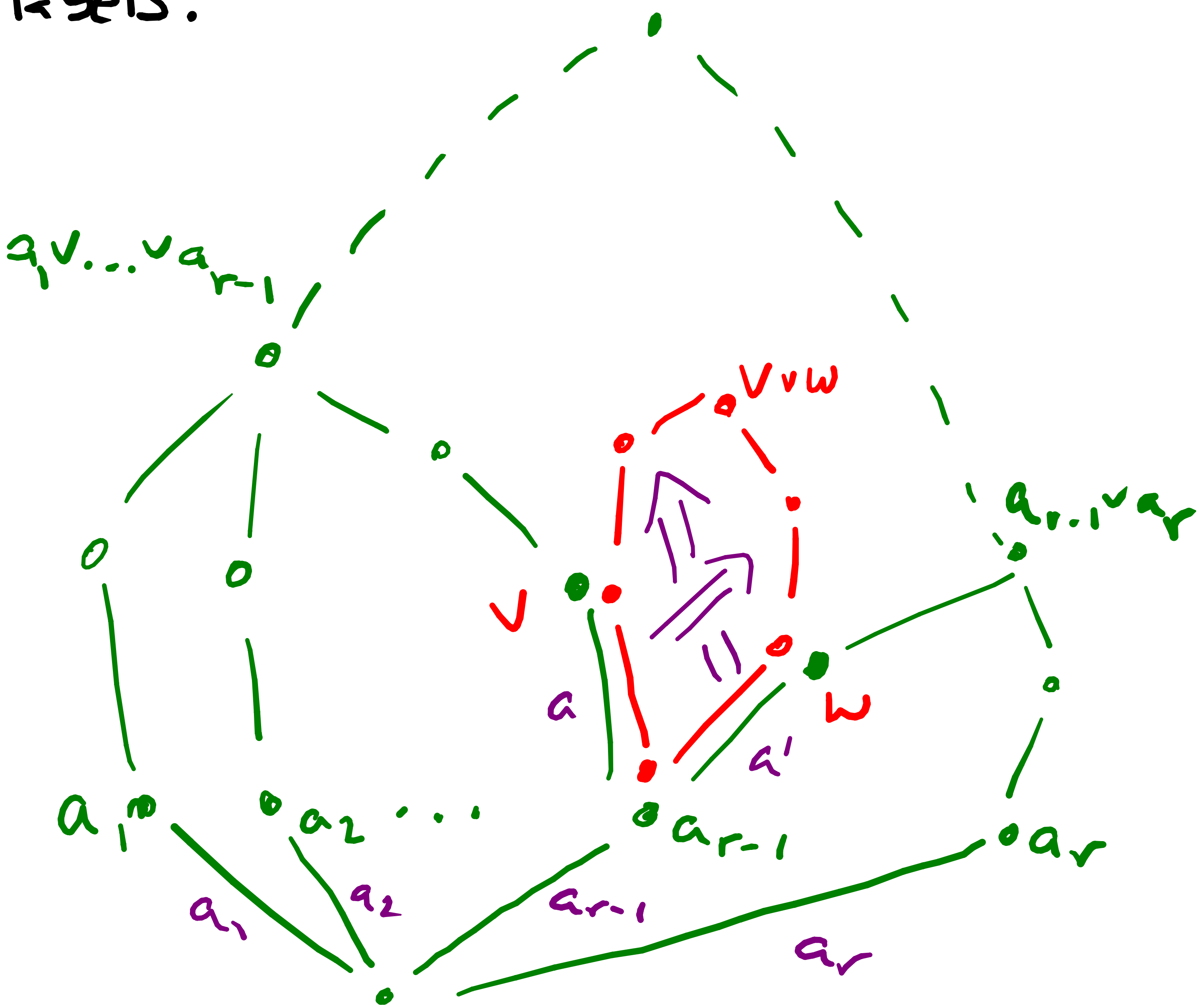


2. preservation of set of labels under basic moves



3. induction on # atoms

4. upward propagation of atom labels:



Thm 2: Let L be a finite lattice admitting an SB-labeling λ. Then each open interval is homotopy equivalent to a ball or sphere.

Proof Idea: SB-labeling guarantees:

$$\bigvee_{i \in S} a_i = \bigvee_{i \in T} a_i \Rightarrow S = T$$

sets of atoms

so subposet Q of joins of atoms in L is Boolean algebra B_n, implying:

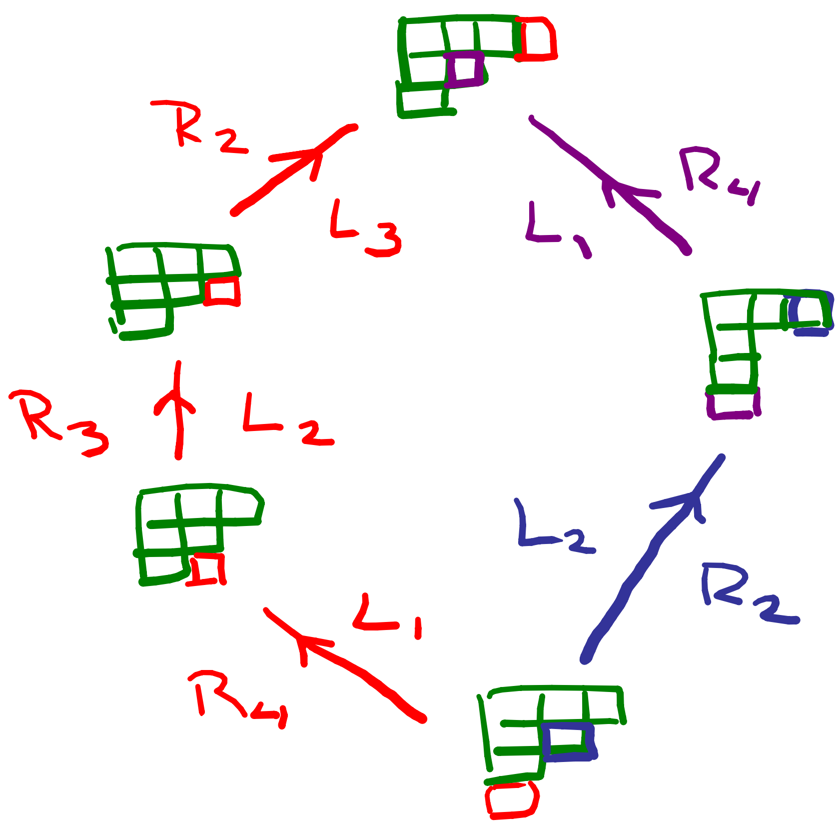
$$\Delta(L \setminus \{\hat{0},\hat{1}\}) \simeq \Delta(Q \setminus \{\hat{0},\hat{1}\}) \simeq S^{n-2}$$

Walker dual closure map result using

$$x \longmapsto \bigvee_{a \leq x} a$$

Open Questions

Qn 1: Does dominance order on integer partitions admit an "SB"-labeling?



Qn 2: SB-labeling for Cambrian lattices?

Concluding Remarks:

"SB-labeling" = ...

Seventieth Birthday Labeling?

Happy Birthday, Richard!