The Pre-History of P-partitions

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Introduction

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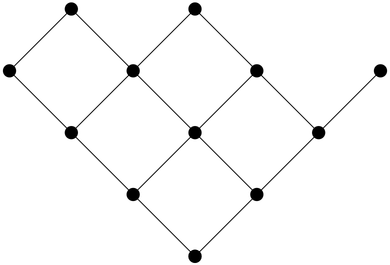
Plane partitions correspond to posets like



The basic idea behind the theory of P-partitions, discovered by Percy MacMahon, is that the set P-partitions can be expressed as a disjoint union of solutions of inequalities like $a \geq b > c > d \geq e$, and the solutions of inequalities like this are easy to count.

Richard Stanley's 1971 Ph.D. thesis was on P-partitions and plane partitions, and the material on P-partitions was published in the AMS Memoir *Ordered Structures and Partitions* in 1972. He was the first to consider P-partitions in full generality, but earlier researchers approached the subject from different points of view, and in this talk I will discuss their work.

MacMahon

The idea behind P-partitions begins with Percy A. MacMahon's work on plane partitions in 1911. The problem that MacMahon considers is that of counting plane partitions of a given shape; that is, arrangements of nonnegative integers with a given sum in a "lattice" such as

$$\begin{array}{cccc} 4 & 4 & 2 & 1 \\ 4 & 3 & 2 & \\ 2 & 1 & & \end{array}$$

in which the entries are weakly decreasing in each row and column.

MacMahon gives a simple example to illustrate his idea. We want to count arrays of nonnegative integers

$$\begin{matrix} p & q \\ r & s \end{matrix}$$

satisfying $p \geq q \geq s$ and $p \geq r \geq s$, and we assign to such an array the weight $x^{p+q+r+s}$. We want to find the sum of these weights.

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MacMahon observes that the set of solutions of these inequalities is the disjoint union of the solution sets of the inequalities

$$\text{(i) } p \geq q \geq r \geq s \quad \text{and} \quad \text{(ii) } p \geq r > q \geq s.$$

To count solutions of the first inequality, $p \geq q \geq r \geq s$, we set $r = s + A$, $q = s + A + B$, and $p = s + A + B + C$, where A, B, and C are arbitrary nonnegative integers, we see that the sum $\sum x^{p+q+r+s}$ is equal to

$$\sum_{A,B,C,s \geq 0} x^{C+2B+3A+4s} = \frac{1}{\mathbf{(1)(2)(3)(4)}},$$

where $\boldsymbol{(n)} = (1 - x^n)$.

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where $\boldsymbol{(n)} = (1 - x^n)$.

Similarly, the generating function for solutions of $p \geq r > q \geq s$ is $x^2/\mathbf{(1)(2)(3)(4)}$, so the generating function for all of the arrays is

$$\frac{1 + x^2}{\mathbf{(1)(2)(3)(4)}}$$

MacMahon explains (but does not prove) that a similar decomposition exists for counting plane partitions of any shape, and moreover, the terms that appear in the numerator have combinatorial interpretations. They correspond to what MacMahon called lattice arrangements, which we now call *standard Young tableaux*. In the example under discussion there are two lattice arrangements,

$$\begin{matrix} 4 & 3 \\ 2 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 4 & 2 \\ 3 & 1 \end{matrix}.$$

They are the plane partitions of the shape under consideration in which the entries are $1, 2, \ldots, n$, where n is the number of entries in the shape.

To each lattice arrangement MacMahon associates a lattice permutation: the ith entry in the lattice permutation corresponding to an arrangement is the row of the arrangement in which $n+1-i$ appears, where the rows are represented by the Greek letters α, β, ... So the lattice permutation associated to $\begin{matrix} 4 & 3 \\ 2 & 1 \end{matrix}$ is $\alpha\alpha\beta\beta$ and to $\begin{matrix} 4 & 2 \\ 3 & 1 \end{matrix}$ is $\alpha\beta\alpha\beta$.

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(A sequence of Greek letters is called a lattice permutation if any initial segment contains at least as many αs as βs, at least as many βs as γs, and so on.)

To each lattice permutation, MacMahon associates an inequality relating p, q, r, and s; the αs are replaced, in left-to-right order with the first-row variables p and q, and the βs are replaced with the second-row variables r and s. A greater than or equals sign is inserted between two Greek letters that are in alphabetical order and a greater than sign is inserted between two Greek letters that are out of alphabetical order. So the lattice permutation $\alpha\alpha\beta\beta$ gives the inequalities $p \geq q \geq r \geq s$ and the lattice permutation $\alpha\beta\alpha\beta$ give the inequalities $p \geq r > q \geq s$. Each lattice permutation contributes one term to the numerator, and the power of x in such a term is the sum of the positions of Greek letters that are followed by a smaller Greek letter.

MacMahon then describes the variation with a bound on the largest part size. The decomposition into disjoint inequalities works exactly as in the unrestricted case, and reduces the problem to counting partitions with a given number of parts and a bound on the largest part.

In a postscript to his 1911 paper, MacMahon considers the analogous situation in which only decreases in the rows are required, not in the columns, and he elaborates on this idea in a 1913 paper. The enumeration of such arrays is not of much interest in itself, since the generating function for an array with $p_1, p_2, \ldots, p_n$ nodes in its n rows is clearly

$$\frac{1}{\mathbf{(1)\cdots(p_1)(1)\cdots(p_2)\cdots\cdots(1)\cdots(p_n)}}.$$

However the same decomposition that is used in the case of plane partitions yields interesting results about permutations.

Given a sequence of elements of a totally ordered set, MacMahon defines a major contact (we now call this a descent) to be a pair of consecutive entries in which the first is greater than the second, and he defines the greater index (now usually called the major index) to be the sum of the positions of the first elements of the major contacts.

(Curiously, MacMahon used the term “major index” for a related concept that does not seem to have been further studied.)

Thus the greater index of $\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma$, where the letters are ordered alphabetically, is $1+6+7=14$. (MacMahon similarly defines the "equal index" and "lesser index" but these do not play much of a role in what follows.) MacMahon's main result in the paper is that the sum $\sum x^p$, where p is the greater index, over all "permutations of the assemblage $\alpha^i\beta^j\gamma^k\cdots$" is

$$\frac{(1)(2)\cdots(i+j+k+\cdots)}{(1)(2)\cdots(i)\cdot(1)(2)\cdots(j)\cdot(1)(2)\cdots(k)\cdots}$$

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Here is MacMahon's example:
We consider the sum of $x^{a_1+a_2+a_3+b_1+b_2}$ over all inequalities $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2$. We see directly that the sum is

$$\frac{1}{\mathbf{(1)(2)(3)\cdot(1)(2)}}.$$

MacMahon breaks up these inequalities just as before into subsets corresponding to all the permutations of $\alpha^3\beta^2$; for example, to the permutation $\alpha\beta\alpha\beta\alpha$ correspond the inequalities $a_1 \geq b_1 > a_2 \geq b_2 > a_3$, where the strict inequalities correspond to the major contacts. The generating function for this set of inequalities is

$$\frac{x^6}{\mathbf{(1)(2)(3)(4)(5)}};$$

here $6 = 2 + 4$ is the the greater index of the permutation $\alpha\beta\alpha\beta\alpha$. Summing the contributions from all ten permutations of $\alpha^3\beta^2$ gives

$$\frac{\sum x^p}{\mathbf{(1)(2)(3)(4)(5)}} = \frac{1}{\mathbf{(1)(2)(3)} \cdot \mathbf{(1)(2)}}.$$

In his book *Combinatory Analysis* (1915–1916) MacMahon elaborates on the analogous result when a bound is imposed on the part sizes. The sum of $x^{a_1+\cdots+a_p}$ over all solutions of $n \geq a_1 \geq \cdots \geq a_p$ is $\boldsymbol{(n+1)\cdots(n+p)/(1)\cdots(p)}$, and MacMahon derives an important, though not well-known, formula that he writes as

$$\sum_{n=0}^{\infty} g^n \frac{\boldsymbol{(n+1)\cdots(n+p_1)\cdots\cdots(n+1)\cdots(n+p_m)}}{\boldsymbol{(1)(2)\cdots(p_1)\cdots\cdots(1)(2)\cdots(p_m)}}$$

$$= \frac{1+g\,\mathrm{PF}_1+g^2\,\mathrm{PF}_2+\cdots+g^{\nu}\,\mathrm{PF}_{\nu}}{(1-g)(1-gx)(1-gx^2)\cdots(1-gx^{p_1+\cdots+p_\nu})}.$$

Here PF_s is the generating function, by greater index, of permutations of the assemblage $\alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_m^{p_m}$ with s major contacts.

This result is sufficiently important that it is worth restating it in more modern notation: Let

$$A_{p_1,\ldots,p_m}(t,q) = \sum_{\pi} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)},$$

where the sum is over all permutations π of the multiset $\{1^{p_1}, 2^{p_2}, \ldots, m^{p_m}\}$, and if $\pi = a_1 \cdots a_p$, where $p = p_1 + \cdots + p_m$ then $\mathrm{des}(\pi)$ is the number of descents of π, that is, the number of indices i for which $a_i > a_{i+1}$, and $\mathrm{maj}(\pi)$ is the sum of the descents of π.

Let $(a;q)_m$ be the q-rising factorial

$$(1-a)(1-aq)\cdots(1-aq^{n-1}),$$

let $(q)_n$ denote $(q;q)_n = (1-q)\cdots(1-q^n)$ and let $\begin{bmatrix} m \\ n \end{bmatrix}$ denote the q-binomial coefficient

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Then

$$\sum_{n=0}^{\infty} t^n \begin{bmatrix} n+p_1 \\ p_1 \end{bmatrix} \begin{bmatrix} n+p_2 \\ p_2 \end{bmatrix} \cdots \begin{bmatrix} n+p_m \\ p_m \end{bmatrix} = \frac{A_{p_1,\ldots,p_m}(t,q)}{(t;q)_{p+1}}.$$

Several specializations are worth mentioning. If we set $q = 1$ then the polynomials $A_{p_1,\ldots,p_m}(t, 1)$ give the solution of Simon Newcomb's problem which MacMahon had solved earlier by a different method. (Curiously, MacMahon does not note the connection with Simon Newcomb's problem.)

In the case $q = 1$, $p_1 = \cdots = p_m = 1$, the polynomials $A_{1^m}(t, 1)$ are the Eulerian polynomials satisfying

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For $p_1 = \cdots = p_m = 1$, we have a formula for q-Eulerian polynomials

$$\sum_{n=0}^{\infty} t^n (1 + q + \cdots + q^n)^m = \frac{A_{1^m}(t, q)}{(t; q)_{m+1}},$$

a result often attributed to Carlitz in 1975.

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MacMahon did not seem to attach any special importance to the case of permutations with distinct entries, and he never used exponential generating functions, which usually give simpler formulas in these cases.

Kreweras

In 1967, Germain Kreweras used an approach similar to MacMahon's, though stated very differently, to solve a common generalization of Simon Newcomb's problem and what he calls "Young's problem". (Kreweras seems to be unaware of MacMahon's work on Simon Newcomb's problem and refers only to Riordan's *Combinatorial Analysis* as a reference on Simon Newcomb's problem.)

In Young's problem, we are given two weakly decreasing sequences $Y = (y_1, \ldots, y_h)$ and $Y' = (y'_1, \ldots, y'_h)$ with $y_i \geq y'_i$ for each i and we ask how many "Young chains" there are from Y' to Y, which are sequences of partitions (weakly decreasing sequences of integers) starting with Y' and ending with Y in which each partition is obtained from the previous one by increasing one part by 1.

In modern terminology, these are standard skew tableaux of shape Y/Y'; that is, fillings of a Young diagram of shape Y with the squares of a Young diagram of shape Y' removed from it, with the integers $1, 2, \ldots, m$ (where m is the total number of squares), so that the entries are increasing in every row and column. For example, if $Y' = (2, 1, 0)$ and $Y = (3, 2, 2)$ then one of the Young chains from Y' to Y is

$$Y' = (2, 1, 0), (3, 1, 0), (3, 1, 1), (3, 2, 1), (3, 2, 2) = Y.$$

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This corresponds to the skew Young tableau

$$\begin{array}{ccc} & & \boxed{1} \\ & \boxed{3} & \\ \boxed{2} & \boxed{4} & \end{array}$$

in which the entry i occurs in row j if the ith step in the chain is an increase by 1 in the jth position.

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In our example

		1
	3	
2	4	

1 and 3 correspond to returns. Kreweras’s returns in Young chains correspond to MacMahon’s major contacts of lattice permutations.

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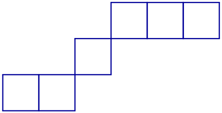
He observes that Simon Newcomb's problem is equivalent to a special case of computing $\theta_r(Y, Y')$; thus the number of permutations of the multiset $\{1^3, 2, 3^2\}$ with r descents is equal to the number of skew Young tableaux of shape



with r returns.

He then gives the solution to this problem in the form

$$\frac{\sum_{r\geq 0}\theta_r(Y, Y')t^r}{(1-t)^{\eta-\eta'+1}} = \sum_{r\geq 0} w_r t^r.$$

Here η is the sum of the entries of Y, η' is the sum of the entries of Y', and w_r is the number of chains

$$Y' \leq Z_1 \leq \cdots \leq Z_r \leq Y;$$

In earlier work, Kreweras had given the formula

$$w_r = \det\left(\binom{y_i - y_j' + r}{i-j+r}\right)_{i,j=1,\ldots,h},$$

where $Y = (y_1, \ldots, y_h)$ and $Y' = (y_1', \ldots, y_h')$. In the case of Simon Newcomb's problem, the determinant is upper triangular, and is therefore a product of binomial coefficients (as can also be seen directly).

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In a later paper, entitled *Polynômes de Stanley et extensions linéaires d'un ordre partiel*, Kreweras studies what is in Stanley's terminology the order polynomial of a naturally labeled poset. Although published in 1981, long after Stanley's 1972 memoir, Kreweras states that Stanley's work was unknown to him when the paper was written.

Knuth

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MacMahon had conjectured that the generating function for solid partitions was $\prod_{i=1}^{\infty}(1-z^i)^{-\binom{i+1}{2}}$. This conjecture had been disproved earlier by Atkin, Bratley, Macdonald, and McKay, but Knuth wanted to compute the number $c(n)$ of solid partitions of n for larger values of n in an (unsuccessful) attempt to find patterns. Knuth realized that MacMahon's approach would work for arbitrary partially ordered sets, not just those corresponding to plane partitions.

Knuth takes a set P (not necessarily finite) partially ordered by the relation $\prec$ and well-ordered by the total order $<$, where $x \prec y$ implies $x < y$. (The main example is $\mathbb{N} \times \mathbb{N}$ with $\prec$ the product order and $<$ the lexicographic order.)

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He defines a P-partition of N to be a function n from P to the set of nonnegative integers satisfying

(i) $x \prec y$ implies $n(x) \geq n(y)$,

(ii) only finite many x have $n(x) > 0$,

(iii) $\sum_{x \in P} n(x) = N$.

Knuth proves that there is a bijection from P-partitions of N to pairs of sequences

$$n_1 \geq n_2 \geq \cdots \geq n_m$$
$$x_1, x_2, \ldots, x_m$$

where $m \geq 0$, the n_i are positive integers with sum N, and the x_i are distinct elements of P satisfying

(S1) For $1 \leq j \leq m$ and $x \in P$, $x \prec x_j$ implies $x = x_i$ for some $i < j$.

(S2) $x_i > x_{i+1}$ implies $n_i > n_{i+1}$ for $1 \leq i < m$.

Knuth is interested primarily in the case in which P is countably infinite, and in this case he uses a modification of the bijection just described to prove that if P is an infinite poset and $s(n)$ is the number of P-partitions of n then

$$\begin{aligned} &1 + s(1)z + s(2)z^2 + \cdots \\ &\qquad = (1 + t(1)z + t(2)z^2 + \cdots)/(1-z)(1-z^2)(1-z^3)\cdots \end{aligned}$$

where $t(k)$ is the number of linear extensions of finite order ideals of P with “index” k, where Knuth’s index is a variant of MacMahon’s greater index.

Thomas

Glânffrwd Thomas's 1977 paper, based on his 1974 Ph.D. thesis appeared after Stanley's memoir, but it was written without knowledge of Stanley's work. The main novelty in Thomas's work is his introduction of quasi-symmetric functions, and the use of Baxter operators in studying them.

Thomas's starting point was the combinatorial definition of Schur functions. If λ is a partition, then a Young tableau of shape λ is a filling of the Young diagram of λ that is weakly increasing in rows and strictly increasing in columns. For example, if λ is the partition $(4, 2, 1)$ then a Young tableau of shape λ is

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 & 1 & 1 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \tag{1}$$

The *Schur function* s_λ is the sum of the weights of all Young tableaux of shape λ, where the weight of a Young tableau is the product of x_i over all of its entries i. (So the weight of the tableau (1) is $x_1^4 x_2 x_4^2$.)

Thomas considers a more general situation, in which we allow as shapes (which Thomas calls “frames”) any subset of $\mathbf{Z} \times \mathbf{Z}$ and he defines a *numbering* of a frame to be filling with positive integers that is weakly increasing in rows and strictly increasing in columns. For example,

$$\begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline & 4 & \\ \hline 2 & & 4 \\ \hline \end{array} \tag{2}$$

is a numbering. To any frame he associates an *index frame* by replacing its entries in increasing order with $1, 2, \ldots, m$, where m is the number of entries, and ties are broken from bottom to top and then left to right. Thus the index numbering corresponding to (2) is

$$\begin{array}{|c|c|c|} \hline 1 & 3 & \\ \hline & 5 & \\ \hline 2 & & 4 \\ \hline \end{array}$$

In fact, there is nothing special about the posets Thomas was studying, and what he did would apply just as well to P-partitions for an arbitrary poset P. From our perspective, the main novelty of his work is in his application of Baxter operators to what we now call quasi-symmetric functions.

As we have seen, the study of P-partitions leads to inequalities like $j_1 \geq j_2 > j_3 \geq j_4$, or equivalently (following Thomas),

$$i_1 \leq i_2 < i_3 \leq i_4.$$

MacMahon was interested in $\sum x^{i_1+\cdots+i_4}$, but Thomas was interested in the multivariable generating function

$$\sum_{i_1 \leq i_2 < i_3 \leq i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

This is an example of what we now call a fundamental quasi-symmetric function; these form a basis for the algebra of quasi-symmetric functions. Thomas used Baxter operators to construct them.

A Baxter operator on a commutative algebra A is linear operator $B : A \to A$ such that for some fixed $\theta \neq 0$,

$$B(aB(b)) + B(bB(a)) = B(a)B(b) + B(\theta ab)$$

for all $a, b \in A$.

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Now let A be the algebra of infinite sequences $(a_1, a_2, \dots)$ with entries in a field, with componentwise operations.

We define two maps $A \to A$:

$$S(a_1, a_2, \dots) = \left(0, a_1, a_1 + a_2, \dots, \sum_{i=1}^{r-1} a_i, \dots\right)$$

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(introduced by Rota and Smith)

$$P(a_1, a_2, \dots) = \left(a_1, a_1 + a_2, \dots, \sum_{i=1}^{r} a_i, \dots\right)$$
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Then S and P are Baxter operators.

An example showing the connection between these operators and quasi-symmetric functions:

Let $x = (x_1, x_2, x_3, \dots)$. Then

$$xS(xS(xP(x))) = \left(\dots, \sum_{1 \le i \le j < k < r} x_i x_j x_k x_r, \dots \right).$$

Happy Birthday
Richard