"Let Δ be a Cohen-Macaulay complex . . ."

Anders Björner

RS@70

A Stanley joke, circa 1980

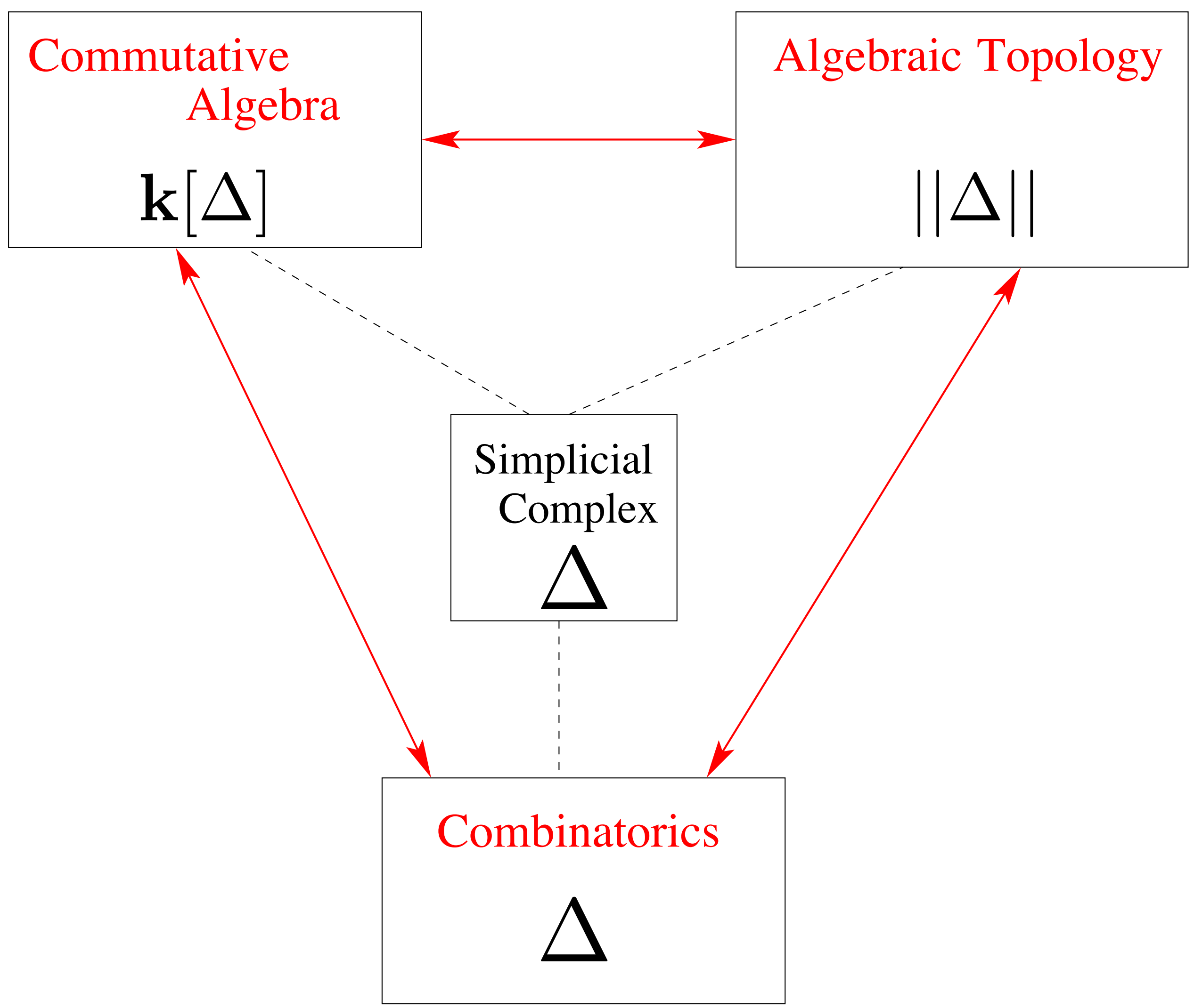
``Imagine that at some time in the future it will be possible to begin a math lecture with

"Let Δ be a Cohen-Macaulay complex ..."

and go on from there without further explanation."

Origins of the theory of C-M complexes (early/mid 1970s)

Baclawski (CM posets, from topological angle)
Hochster (ring theory)
Reisner (Hochster's student)
Stanley — main architect

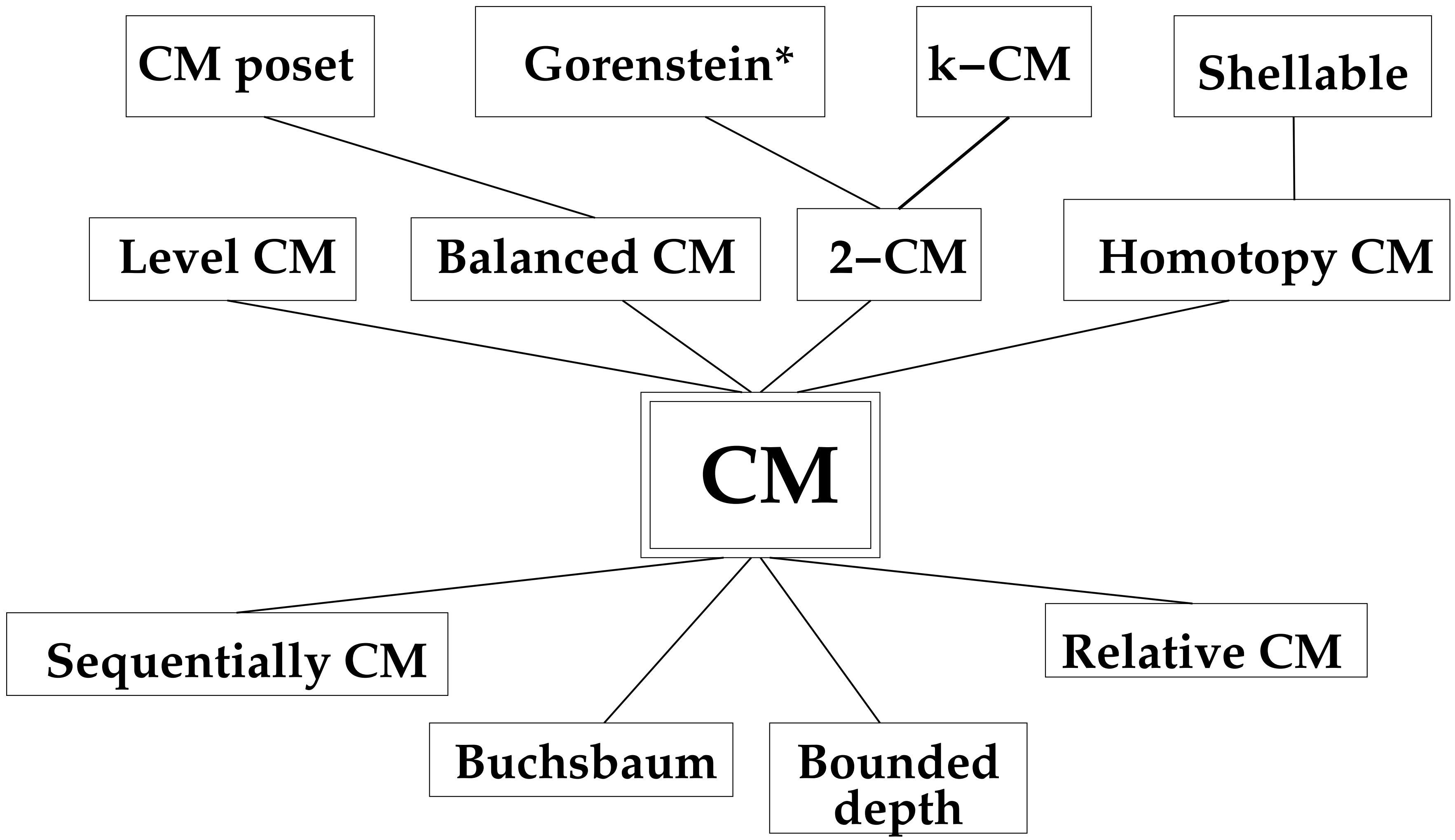


Hochster's formula

$$\beta_{i,j}(\mathbf{k}[\Delta]) = \sum_{E \subseteq V;\, |E|=j} \dim_{\mathbf{k}} \widetilde{H}_{j-i-1}(\Delta|_E; \mathbf{k})$$

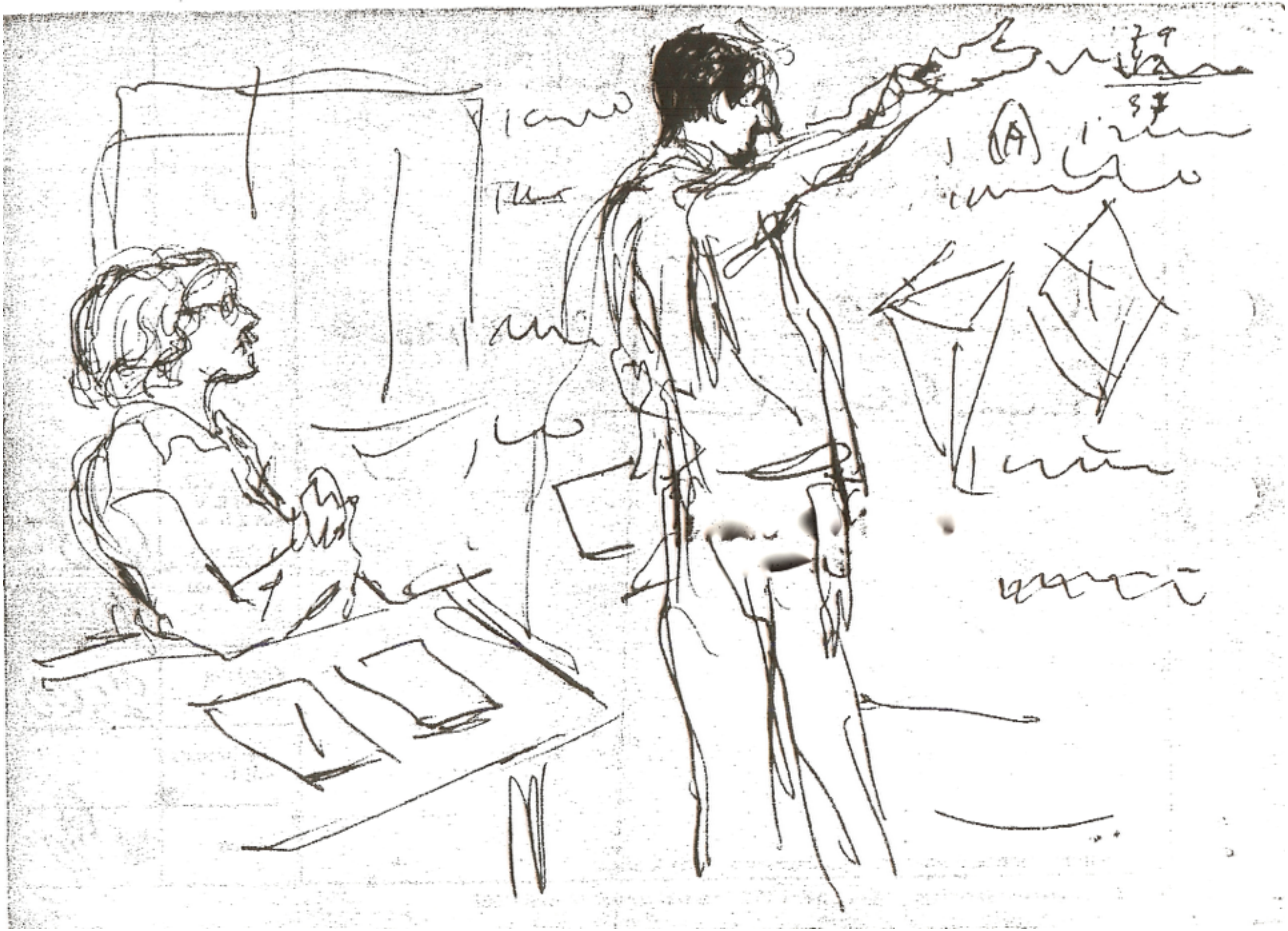
Topological Betti numbers
(Poincaré, homology of spaces, ...)

Ring-theoretic Betti numbers
(Hilbert, minimal free resolutions, syzygy theorem),...


CM poset
Gorenstein*
k−CM
Shellable
Level CM
Balanced CM
2−CM
Homotopy CM
CM
Sequentially CM
Relative CM
Buchsbaum
Bounded depth

Rest of the talk:

1. Some C-M memories 1976-1981
2. A recent C-M result



Institut Mittag-Leffler Spring 1981

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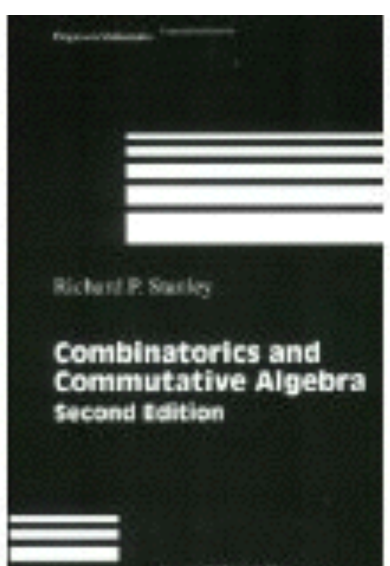
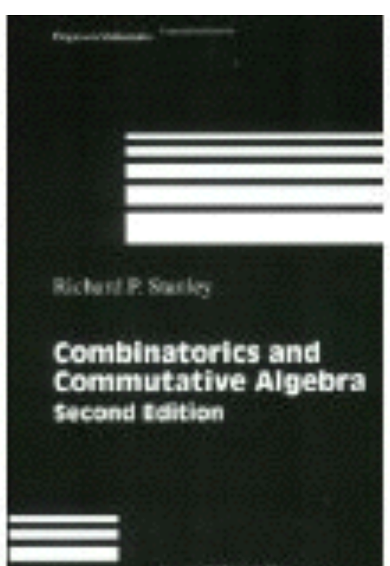


First Nordic Combinatorics Conference
Utstein Abbey, Norway

Institut Mittag-Leffler Spring 1981



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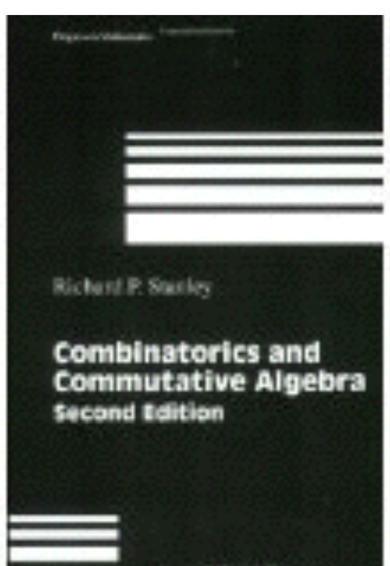

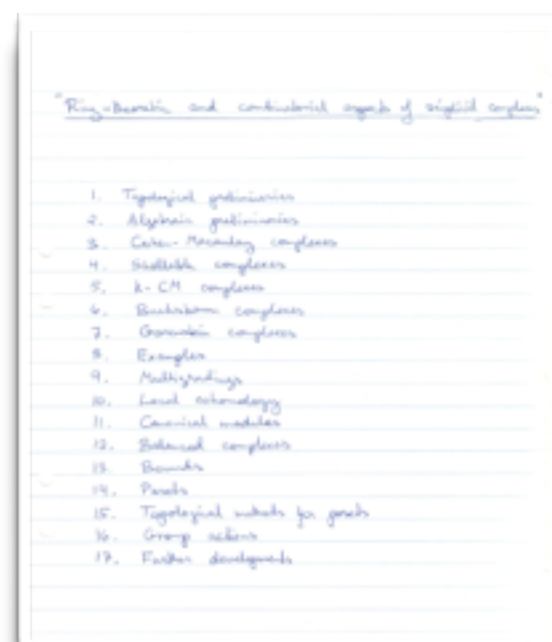

"Ring-theoretic and combinatorial aspects of simplicial complexes"

1. Topological preliminaries
2. Algebraic preliminaries
3. Cohen-Macaulay complexes
4. Shellable complexes
5. k-CM complexes
6. Buchsbaum complexes
7. Gorenstein complexes
8. Examples
9. Multigradings
10. Local cohomology
11. Canonical modules
12. Balanced complexes
13. Bounds
14. Posets
15. Topological methods for posets
16. Group actions
17. Further developments

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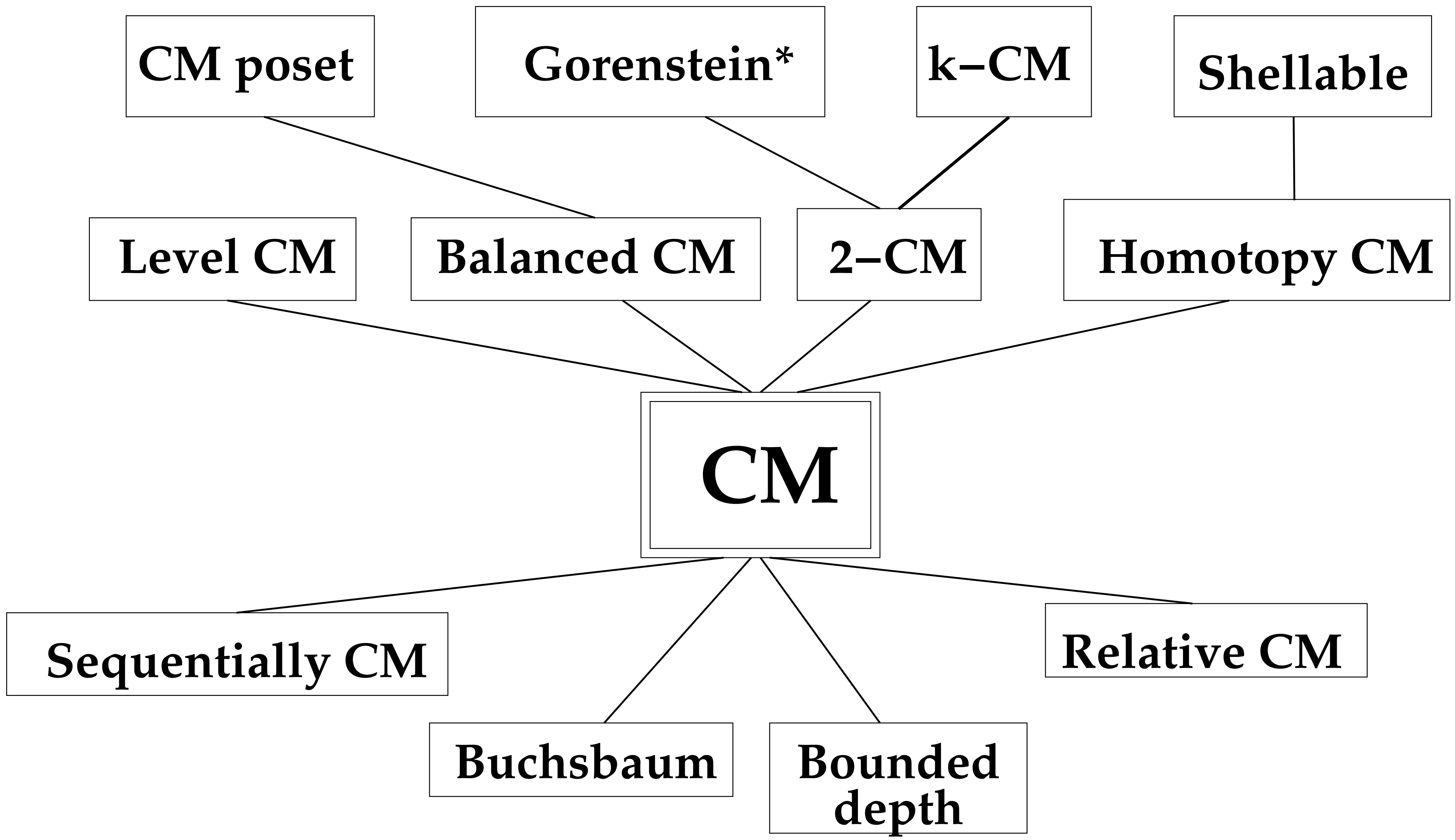
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A recent result (joint work with Karim Adiprasito).
This result was conjectured in 2008 by G. Ziegler and G. Michalkin.
Their motivation was seeking a Lefschetz hyperplane theorem in tropical geometry

Theorem. Let L be a geometric lattice and $w : L_1 \to \mathbb{R}$ a real-valued zero-sum function on its atoms. Extend w linearly to all of L, and assume for all subsets $E \subseteq L_1$ that $w(E) = 0$ if and only if $E = L_1$. Then the subposet

$$L_w^+ = \{x \in L \mid \text{ such that } w(x) > 0\}$$

is homotopy Cohen-Macaulay.

Similarly, L_w^- is homotopy Cohen-Macaulay.

Corollary. The weight function w on the atoms induces a splitting $L = L^+ \oplus L^-$ into two C-M subposets of the same rank..

More generally:

Let $t \le 0$. The “filtered” geometric lattice $L^{>t} = \{x \in L : w(x) > t\}$
is homotopy Cohen-Macaulay.

Previously known in 3 cases:

- Rank=3. [GunterZ & Coworkers, 2008]
  Elementary proof, tricky

- Exactly one atom positive (or negative). [Wachs & Walker, 1994]
  Geometric semilattices

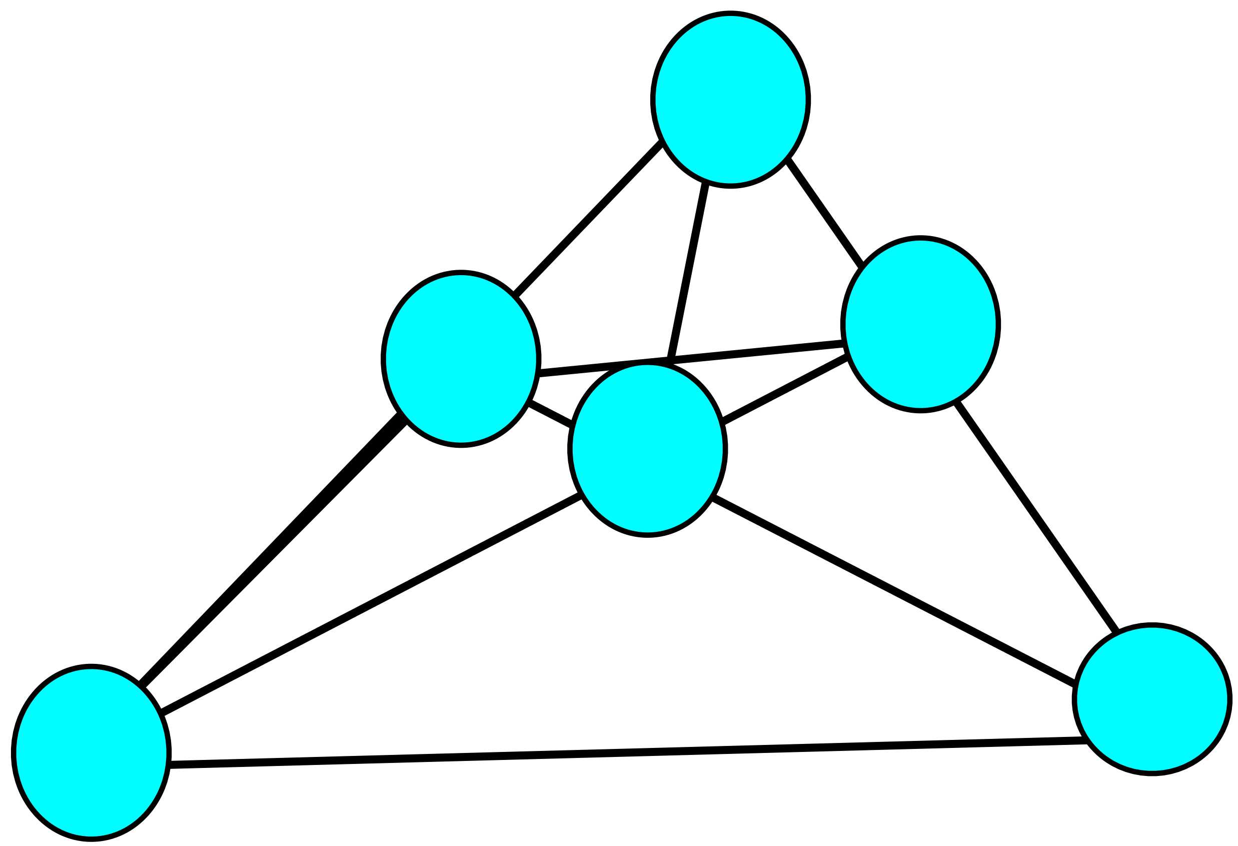
- L Boolean (i.e. case of free matroid)
  L^+ and L^- are isomorphic shellable balls,
  with f-vector the same for all weights w.
  Further: h-polynomials = the classical Eulerian polynomials.

Some combinatorial consequences:

- A characteristic (chromatic) polynomial associated to L^+.
  Coefficients alternate in sign.

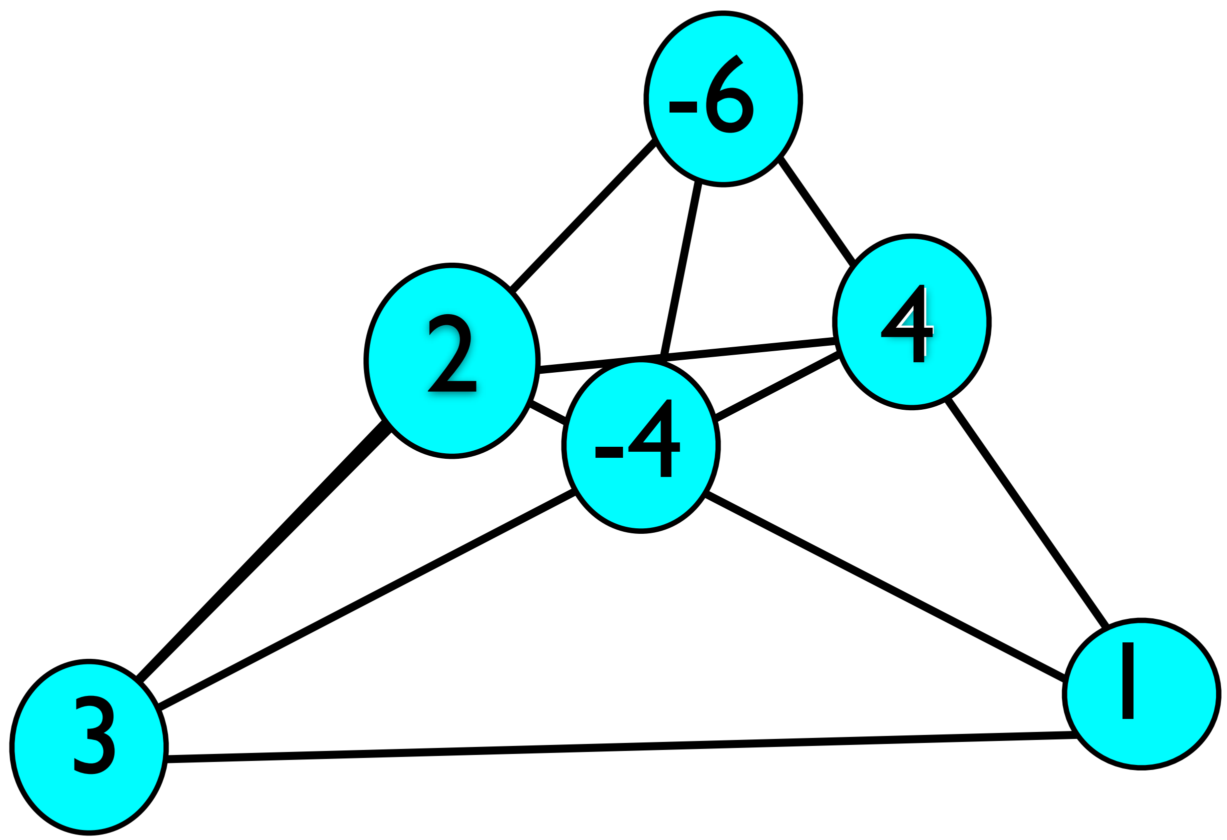
- A “duality” between positive (L^+) and negative (L^-) parts of L.

Illustration of theorem, rank 3 case

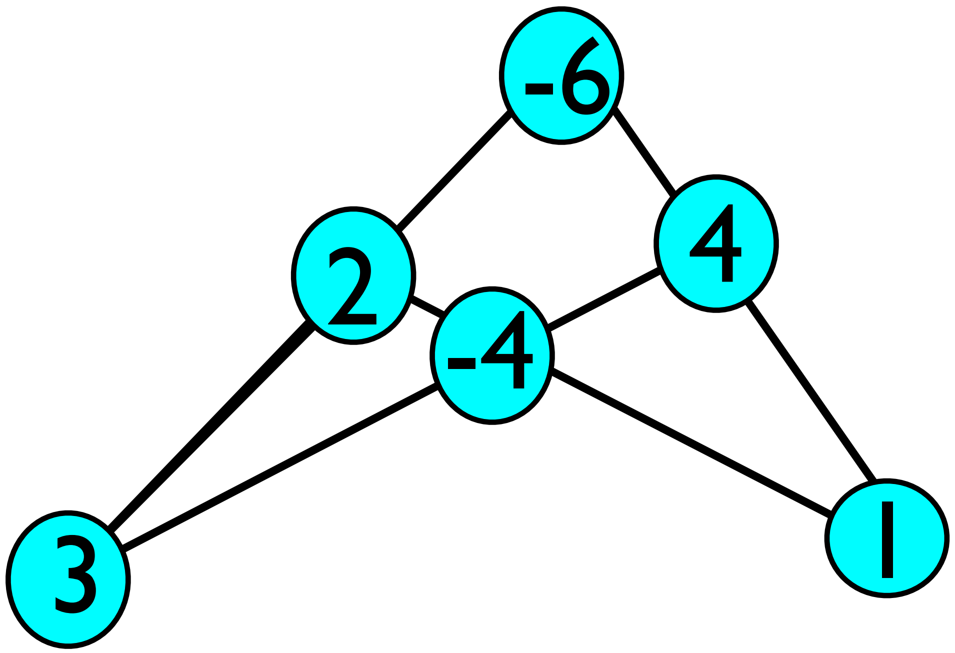


Point configuration in **R**^2
6 points spanning 7 induced lines

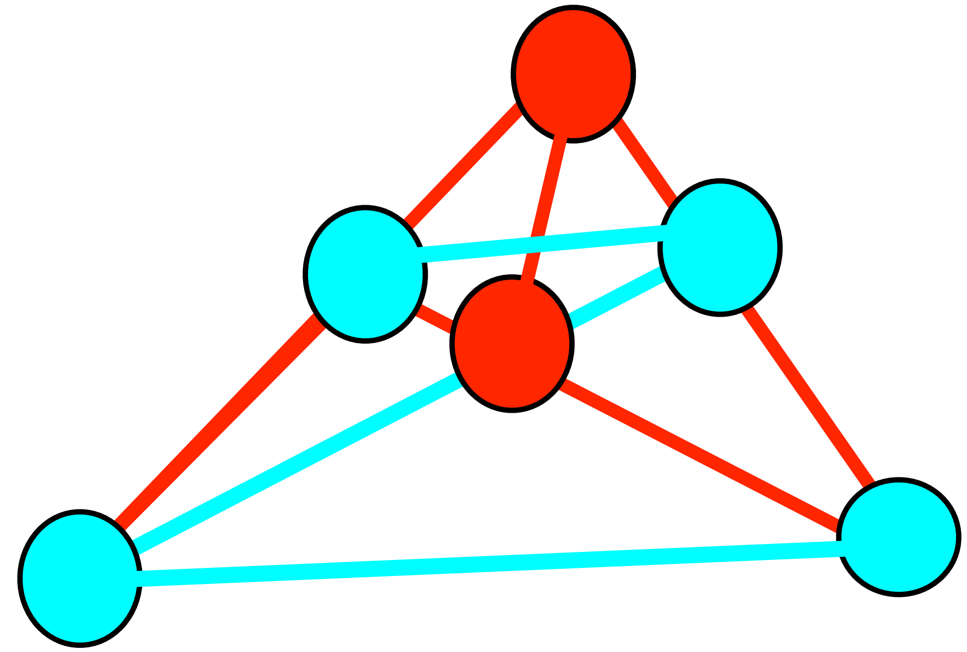
The rank 3 theorem says: Any pair of positive points are connected by a sequence of positive lines



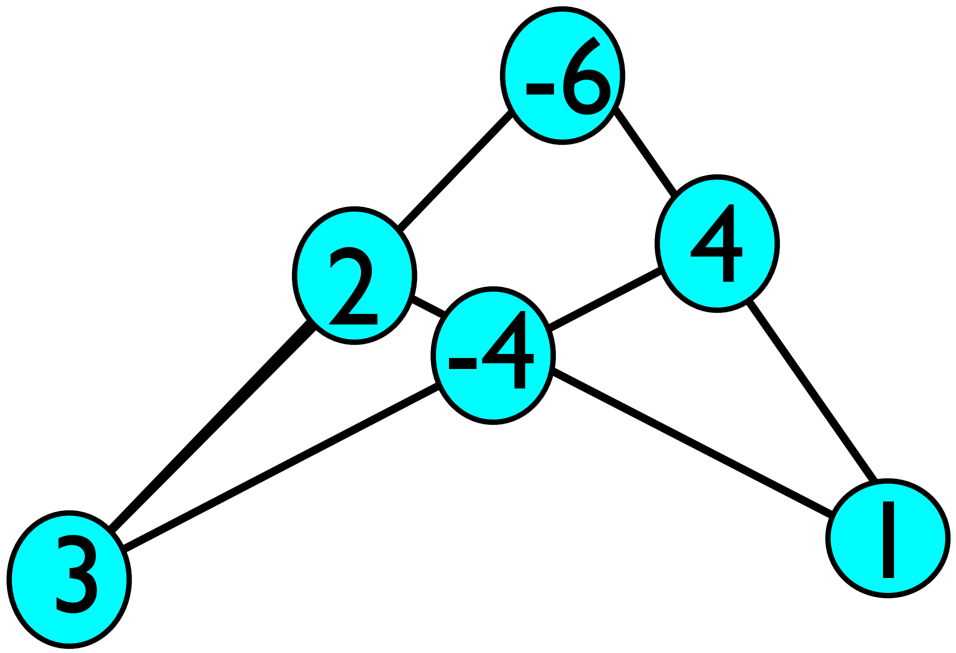
Weighted point configuration
Sum of weights =0



Weighted point configuration



Negative=red points and lines,
throw them away



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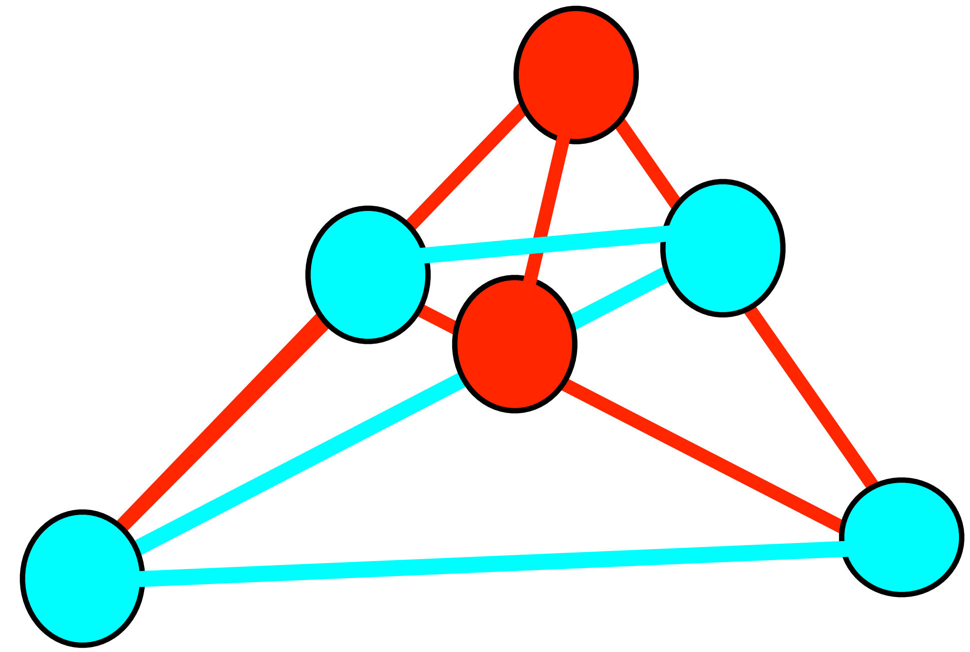

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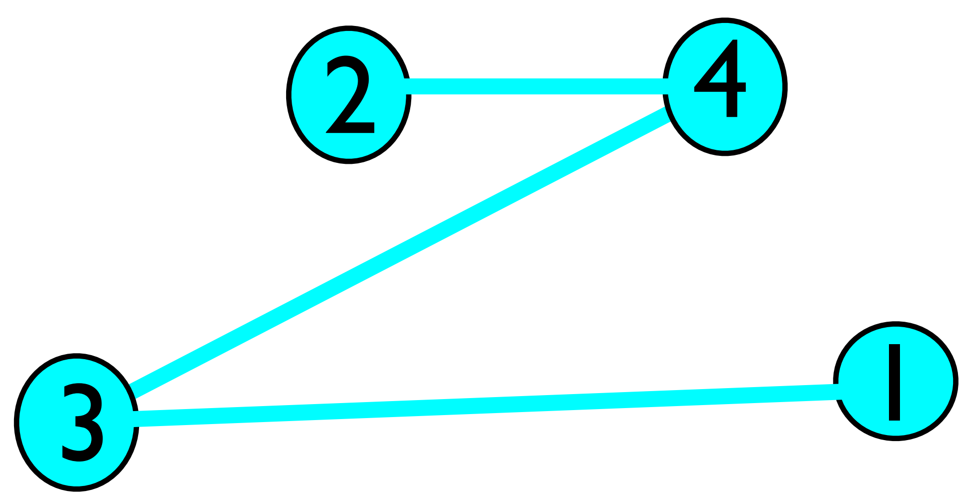
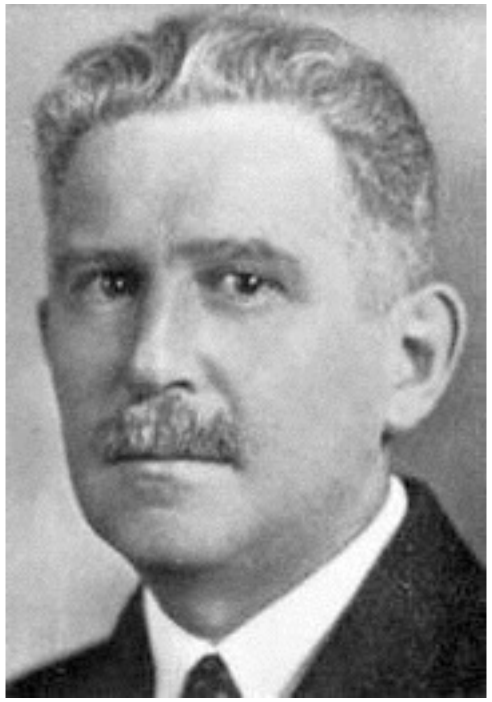


Positive part connected

Tropical geometry

- Several possible approaches
- Tropical Geometry = Algebraic Geometry over the (max, +) semiring
- Tropical Geometry = lim(Algebraic Geometry/$\mathbf{C}$)
- Has been used (Mikhalkin) to derive results in enumerative classical AG.

Lefschetz' hyperplane theorem



The Lefschetz hyperplane theorem

Let X be an n-dimensional projective variety over $\mathbb{C}$ and Y its intersection with a hyperplane. Then,

- $H_i(X) = H_i(Y)$, for all $i \leq n - 2$.
- X can be obtained from Y by attaching n-cells

Idea: Derive information about a variety X from its intersection Y with a hyperplane

Is there a tropical Lefschetz theorem?

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Yes, see our paper (http://arxiv.org/abs/1401.7301).

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"There's no sense in being precise when you don't even know what you're talking about."

J. von Neumann

Richard, thank you for inspiration and support, for long-standing friendship, and for planting the harpoon of Combinatorics into so many interesting whales.