Weighted walks around dissected polygons – Conway-Coxeter friezes and beyond

Christine Bessenrodt

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		1	1
	1	0	2
1	0	0	4

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		1	1
	1	0	2
1	0	0	4
			7

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		1	1^2
	1	0	2^2
1	0	0	4^2
			7^2

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		1	1^2
	1	0	2^2
1	0	0	4^2
			7^2
			70

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$$\begin{array}{|ccc|c} \hline & & 1 & 1^2 \\ & 1 & 0 & 2^2 \\ 1 & 0 & 0 & 4^2 \\ \hline & & & 7^2 \end{array}$$

Stanley@70

Happy Birthday, Richard!

Conway, Coxeter (1973)

$$
\begin{array}{ccccccccccccccccccc}
\dots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \dots \\
\dots & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \dots \\
\dots & & \cdot & & \cdot & & \cdot & & \mathbf{b} & & \cdot & & \cdot & & \cdot & & \cdot & & \dots \\
\dots & \cdot & & \cdot & & \cdot & & \mathbf{a} & & \mathbf{d} & & \cdot & & \cdot & & \cdot & & \cdot & \dots \\
\dots & & \cdot & & \cdot & & \cdot & & \mathbf{c} & & \cdot & & \cdot & & \cdot & & \cdot & & \dots \\
\dots & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \dots \\
\dots & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \dots
\end{array}
$$

$$\text{a,b,c,d} \in \mathbb{N}, \quad \mathbf{ad - bc = 1}$$

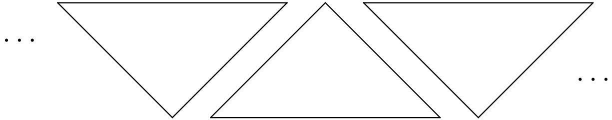
Conway-Coxeter friezes

A frieze pattern of height 4:

…		1		1		1		1		1		1		1		1		1	…
…	1		3		1		2		2		1		3		1		2		…
…		2		2		1		3		1		2		2		1		3	…
…	1		1		1		1		1		1		1		1		1		…

A frieze pattern of height 4:

…		1		1		1		1		1		1		1		1		1	…
…	1		3		1		2		2		1		3		1		2		…
…		2		2		1		3		1		2		2		1		3	…
…	1		1		1		1		1		1		1		1		1		…



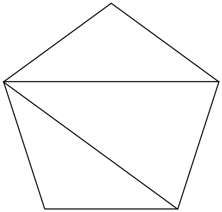
Classification of friezes via triangulated polygons

$$
\begin{array}{cccccccccccccccccccc}
\dots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & \dots \\
\dots & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots
\end{array}
$$

Classification of friezes via triangulated polygons

... 1 1 1 1 1 1 1 1 1 ...
... 1 3 1 2 2 1 3 1 2 ...
... 2 2 1 3 1 2 2 1 3 ...
... 1 1 1 1 1 1 1 1 1 ...

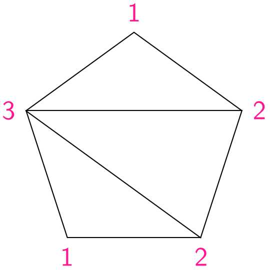
$\updownarrow$



Classification of friezes via triangulated polygons

$$
\begin{array}{cccccccccccccccccccccc}
\dots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & \dots \\
\dots & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots
\end{array}
$$

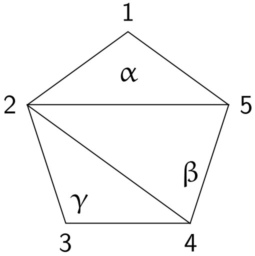
$$\updownarrow$$



Count number of triangles at each vertex!

Arcs

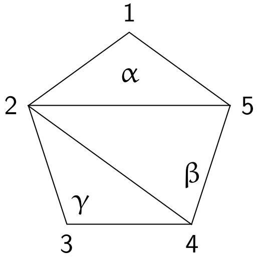
Broline, Crowe, Isaacs (1974)



An **arc** from vertex i to vertex j is a sequence of **different** triangles $(t_{i+1}, t_{i+2}, \ldots, t_{j-1})$ such that t_k is incident to vertex k, for all k.

arcs	1	2	3	4	5
from 1 to	-	$\emptyset$	(α), (β), (γ)	(α,γ), (β,γ)	(α,γ,β)

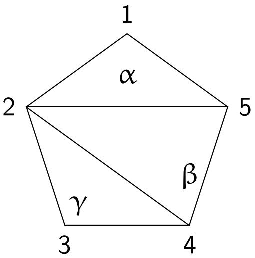
Broline, Crowe, Isaacs (1974)



An **arc** from vertex i to vertex j is a sequence of **different** triangles $(t_{i+1}, t_{i+2}, \dots, t_{j-1})$ such that t_k is incident to vertex k, for all k.

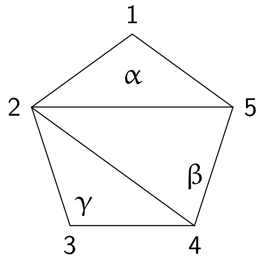
arcs	1	2	3	4	5
from 1 to	-	$\emptyset$	(α), (β), (γ)	(α, γ), (β, γ)	(α, γ, β)
count!	0	1	3	2	1

Arc enumeration



$$W = \begin{pmatrix} 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

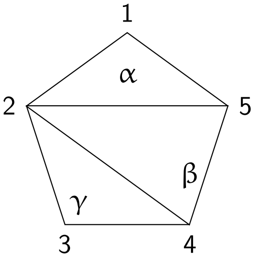
Arc enumeration – and back to the frieze



$$W = \begin{pmatrix} 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

...		1		1		1		1		1		1		1		1		1	...
...	1		3		1		2		2		1		3		1		2		...
...		2		2		1		3		1		2		2		1		3	...
...	1		1		1		1		1		1		1		1		1		...

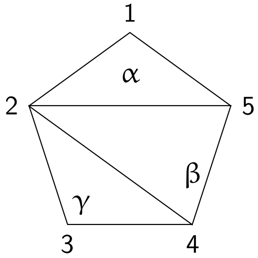
Arc enumeration – and back to the frieze



$$W = \begin{pmatrix} 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

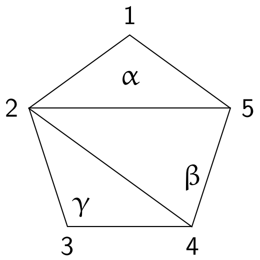
$$\begin{array}{cccccccccccccccccccc}
\dots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
\dots & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & \dots \\
\dots & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots
\end{array}$$

The arc enumeration matrix



$$W = \begin{pmatrix} 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix} \qquad \det W = 8$$

The arc enumeration matrix – the frieze table



$$W = \begin{pmatrix} 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix} \qquad \det W = 8$$

Theorem (Broline, Crowe, Isaacs 1974)
Let W be the arc enumeration matrix to a triangulated n-gon.

(i) W is a symmetric matrix, with its upper/lower part equal to the fundamental domain of the frieze to the triangulation.

(ii) $\det W = -(-2)^{n-2}$.

Renewed interest: recent generalizations and refinements

Remarks

1. Frieze patterns in the context of cluster algebras of type A! Caldero, Chapoton; Propp; Assem, Dupont, Reutenauer, Schiffler, Smith; Baur, Marsh; Morier-Genoud, Ovsienko, Tabachnikov; Holm, Jørgensen, ...

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Remarks

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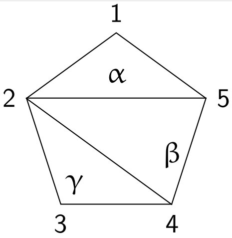
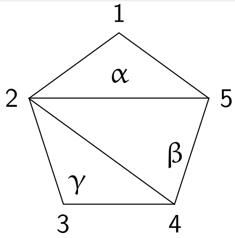
2. Generalization to **d**-angulations and a refinement giving the **Smith normal form** of the corresponding "frieze table". In this context, a **generalized frieze pattern** is associated to the d-angulation where the local 2×2 determinants are 0 or 1. (Joint work with Thorsten Holm and Peter Jørgensen, JCTA 2014.)

Weighted arcs



arcs	1	2	3	4	5
from 1 to	-	$\emptyset$	(α), (β), (γ)	(α, γ), (β, γ)	(α, γ, β)

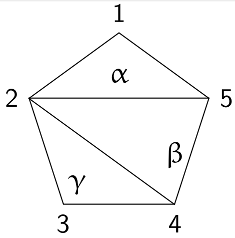
Weighted arcs



arcs	1	2	3	4	5
from 1 to	-	∅	(α), (β), (γ)	(α, γ), (β, γ)	(α, γ, β)
weights!	0	1	a+b+c	ac+bc	abc

$$W = \begin{pmatrix} 0 & 1 & a+b+c & ac+bc & abc \\ abc & 0 & 1 & c & bc \\ ab+ac+bc & abc & 0 & 1 & b+c \\ a+b & ab & abc & 0 & 1 \\ 1 & a & ab+ac & abc & 0 \end{pmatrix}$$

Weighted arcs

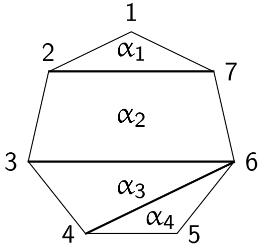


arcs	1	2	3	4	5
from 1 to	-	$\emptyset$	(α), (β), (γ)	(α,γ), (β,γ)	(α,γ,β)
weights!	0	1	a+b+c	ac+bc	abc

$$W = \begin{pmatrix} 0 & 1 & a+b+c & ac+bc & abc \\ abc & 0 & 1 & c & bc \\ ab+ac+bc & abc & 0 & 1 & b+c \\ a+b & ab & abc & 0 & 1 \\ 1 & a & ab+ac & abc & 0 \end{pmatrix}$$

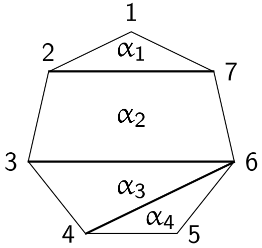
$$\det W = a^5b^5c^5 + a^4b^2c^4 + a^4b^4c^2 + a^2b^4c^4 + abc^3 + ab^3c + a^3bc + 1$$

Walks around dissected polygons



Let $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ be a dissection of a polygon, where the piece α_k is a $\mathbf{d_k}$-gon, $k = 1, \dots, m$.

Walks around dissected polygons



Let $\mathcal{D} = \{\alpha_1, \ldots, \alpha_m\}$ be a dissection of a polygon, where the piece α_k is a $\mathbf{d_k}$-gon, $k = 1, \ldots, m$.
A (counterclockwise) **walk** from vertex i to vertex j is a sequence of pieces $s = (p_{i+1}, p_{i+2}, \ldots, p_{j-1})$ such that
(i) p_k is incident to vertex k, and
(ii) α_r appears at most $\mathbf{d_r - 2}$ times in s, for any r.

The weight matrix (without edge weights)

Let $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ be a dissection of an n-gon.
Weight of a piece α_k : $w(\alpha_k) = x_k \in \mathbb{Z}[x_1, \dots, x_m] = \mathbb{Z}[x]$.
Weight of a walk $s = (p_{i+1}, \dots, p_{j-1})$:

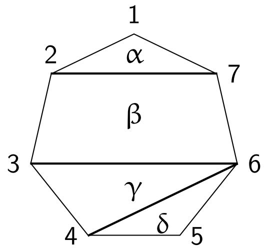
$$x^s = \prod_{k=i+1}^{j-1} w(p_k) \in \mathbb{Z}[x] .$$

For vertices i and j we set

$$w_{i,j} = \sum_{s\text{: walk from } i \text{ to } j} x^s \in \mathbb{Z}[x] .$$

Weight matrix associated to $\mathcal{D}$:

$$\mathbf{W}_{\mathcal{D}}(\mathbf{x}) = (w_{i,j})_{1 \le i,j \le n} .$$



$$\begin{pmatrix}
0 & 1 & a+b & \begin{matrix} ab+ac \\ +b^2+bc \end{matrix} & \begin{matrix} (a+b)(b+c)d \\ +(a+b)bc \end{matrix} & (a+b)bcd & ab^2cd \\
ab^2cd & 0 & 1 & b+c & b(c+d)+cd & bcd & b^2cd \\
(a+b)bcd & ab^2cd & 0 & 1 & c+d & cd & bcd \\
(a+b)(b+c)d & ab(b+c)d & ab^2cd & 0 & 1 & d & (b+c)d \\
\begin{matrix} ab+ac+ad \\ +b^2+bc+bd \end{matrix} & \begin{matrix} ab^2+abc \\ +abd \end{matrix} & ab^2(c+d) & ab^2cd & 0 & 1 & b+c+d \\
a+b & ab & ab^2 & ab^2c & ab^2cd & 0 & 1 \\
1 & a & ab & ab(b+c) & ab(b+c)d+ab^2c & ab^2cd & 0
\end{pmatrix}$$

The weight matrix $W_{\mathcal{D}}$ is **not** symmetric!

Complementary symmetry

Let $\mathcal{D}$ be a polygon dissection with pieces of degree $d_1, \ldots, d_m$.

Define a **complementing map** $\phi_{\mathcal{D}}$ on weights by giving it on walk weights $x^s = \prod_{i=1}^m x_i^{s_i}$ (and linear extension):

$$\phi_{\mathcal{D}}(x^s) = \prod_{i=1}^{m} x_i^{d_i - 2 - s_i} .$$

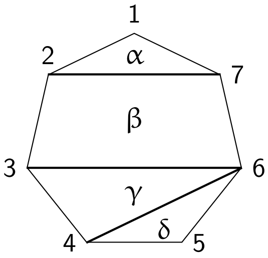
Theorem

Let $W_{\mathcal{D}} = (w_{i,j})$ be the weight matrix associated to $\mathcal{D}$. Then

$$w_{j,i} = \phi_{\mathcal{D}}(w_{i,j}) \qquad \text{for all } i, j ,$$

i.e., $W_{\mathcal{D}}$ is a **complementary symmetric** matrix.

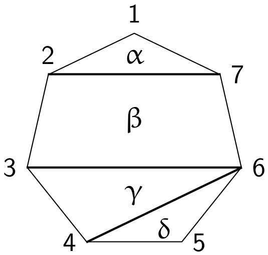
The determinant of the weight matrix



det $W_{\mathcal{D}}$

$$\begin{aligned}
= \quad & 1 + a^5b^{10}c^3d^3 + a^2b^8c^2d^2 + a^5b^8c^3d^5 + a^2b^6c^2d^4 \\
& + a^6b^{12}c^4d^6 + a^3b^{10}c^3d^5 + a^4b^6c^2d^2 + ab^2c^3d + a^5b^8c^5d^3 \\
& + a^2b^6c^4d^2 + a^6b^{12}c^6d^4 + a^3b^{10}c^5d^3 + a^5b^6c^5d^5 + a^2b^4c^4d^4 \\
& + a^6b^{10}c^6d^6 + a^3b^8c^5d^5 + a^4b^{12}c^6d^6 + a^7b^{14}c^7d^7 \\
& + a^4b^4c^4d^2 + ab^4cd + a^3b^2cd + a^4b^4c^2d^4 + ab^2cd^3
\end{aligned}$$

The determinant of the weight matrix



det $W_{\mathcal{D}}$

$$
\begin{aligned}
&= 1 + a^5b^{10}c^3d^3 + a^2b^8c^2d^2 + a^5b^8c^3d^5 + a^2b^6c^2d^4 \\
&\quad + a^6b^{12}c^4d^6 + a^3b^{10}c^3d^5 + a^4b^6c^2d^2 + ab^2c^3d + a^5b^8c^5d^3 \\
&\quad + a^2b^6c^4d^2 + a^6b^{12}c^6d^4 + a^3b^{10}c^5d^3 + a^5b^6c^5d^5 + a^2b^4c^4d^4 \\
&\quad + a^6b^{10}c^6d^6 + a^3b^8c^5d^5 + a^4b^{12}c^6d^6 + a^7b^{14}c^7d^7 \\
&\quad + a^4b^4c^4d^2 + ab^4cd + a^3b^2cd + a^4b^4c^2d^4 + ab^2cd^3 \\
&= (1 + a^3b^2cd) \cdot (1 + ab^2c^3d) \cdot (1 + ab^2cd^3) \cdot (1 + ab^4cd + (ab^4cd)^2)
\end{aligned}
$$

Determinant formula for the weight matrix

Theorem

Let $\mathcal{D}$ be a polygon dissection with polygon pieces of degree $d_1, \ldots, d_m$. Set $c = \prod_{k=1}^{m} x_k^{d_k - 2}$. Then

$$\det W_{\mathcal{D}}(x_1, \ldots, x_m) \quad = \quad (-1)^{n-1} \prod_{k=1}^{m} \sum_{j=0}^{d_k - 2} (c\, x_k^2)^j \,.$$

The weight matrix with edge weights

Let $\mathcal{D} = \{\alpha_1, \ldots, \alpha_m\}$ be a dissection of an n-gon.

Weight of the edge e_k between vertex k and $k+1 \pmod{n}$:

$w(e_k) = q_k \in \mathbb{Z}[q_1, \ldots, q_n] = \mathbb{Z}[q] \subset \mathbb{Z}[x; q]$, for $k = 1, \ldots, n$.

Weight of a walk $s = (p_{i+1}, \ldots, p_{j-1})$:

$$x^s q^s = x^s \prod_{k=i}^{j-1} q_k \in \mathbb{Z}[x; q] .$$

For vertices i and j define

$$v_{i,j} = \sum_{s\text{: walk from } i \text{ to } j} x^s q^s \in \mathbb{Z}[x; q] .$$

Weight matrix associated to $\mathcal{D}$:

$$\mathbf{W}_{\mathcal{D}}(\mathbf{x}; \mathbf{q}) = (v_{i,j})_{1 \le i,j \le n} .$$

The weight matrix with edge weights

For an n-gon dissection $\mathcal{D}$ with degrees $d_1, \ldots, d_m$, define a **complementing map $\psi_{\mathcal{D}}$** on walk weights by complementing with respect to both

$$c = \prod_{i=1}^{m} x_i^{d_i - 2} \quad \text{and} \quad \varepsilon = \prod_{j=1}^{n} q_j$$

(and linear extension).

Theorem
The weight matrix $W_{\mathcal{D}}(x; q) = (v_{i,j})$ is complementary symmetric with respect to $\psi_{\mathcal{D}}$:

$$v_{j,i} = \psi_{\mathcal{D}}(v_{i,j})\,.$$

Determinant formula and diagonal form

Theorem

Let $\mathcal{D}$ be a dissection of an n-gon, with pieces of degree $d_1, \ldots, d_m$. Set $c = \prod_{i=1}^{l} x_i^{d_i-2}$, $\varepsilon = \prod_{i=1}^{n} q_i$, $R = \mathbb{Z}[x_1^{\pm}, ..., x_m^{\pm}\,; q_1^{\pm}, ..., q_m^{\pm}]$. Then

$$\det W_{\mathcal{D}}(x; q) = (-1)^{n-1}\, \varepsilon \prod_{i=1}^{m} \sum_{j=0}^{d_i-2} (\varepsilon c x_i^2)^j \,.$$

Determinant formula and diagonal form

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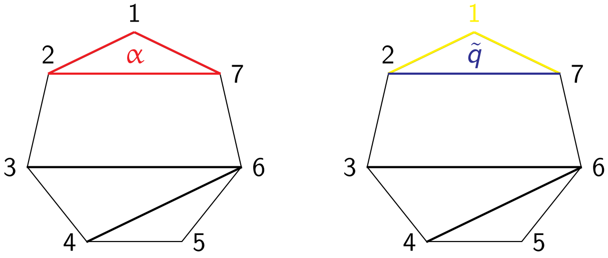
$$\det W_{\mathcal{D}}(x;q) = (-1)^{n-1}\,\varepsilon \prod_{i=1}^{m} \sum_{j=0}^{d_i-2} (\varepsilon c x_i^2)^j \,.$$

Furthermore, there are matrices $P, Q \in \mathrm{GL}(n, R)$ such that

$$P \cdot W_{\mathcal{D}}(x;q) \cdot Q = \mathrm{diag}\big(\sum_{j=0}^{d_1-2} (\varepsilon c x_1^2)^j, \ldots, \sum_{j=0}^{d_m-2} (\varepsilon c x_l^2)^j, 1, \ldots, 1 \big)\,.$$

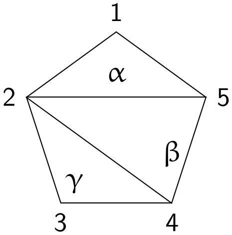
A crucial reduction tool



A non-trivially dissected polygon always has an "ear", a piece with only one interior edge.

Cut off ear, and put weight from cut-off ear on edge!

Generalized polynomial friezes



1		1		1		1		1		1		. . .	
. . .	a		a+b+c		c		b+c		a+b		a		
ab		a(b+c)		(a+b)c		bc		a(b+c)+bc		ab		. . .	
. . .	abc		abc		abc		abc		abc		abc		

Let $\mathcal{D}$ be a dissection of P_n with weight matrix $W_{\mathcal{D}} = (v_{i,j})$. Take a pair of boundary edges of P_n, say $e = (i, i+1)$ and $f = (j, j+1)$. The corresponding 2×2 minor has the form

$$d(e,f) := \det \begin{pmatrix} v_{i,j} & v_{i,j+1} \\ v_{i+1,j} & v_{i+1,j+1} \end{pmatrix}.$$

Theorem

1. $d(e,e) = -\varepsilon c$.
2. When $e \neq f$, $d(e,f)$ is 0 or a monomial.

Local determinants in generalized friezes, II

Theorem

For boundary edges $e = (i, i+1) \neq f = (j, j+1)$, we have $d(e, f) \neq 0$ if and only if there exists a ("zig-zag") sequence

$$e = z_0, z_1, \ldots, z_{s-1}, z_s = f \ ,$$

where $z_1, \ldots, z_{s-1}$ are diagonals of the dissection, s.t. for all k:

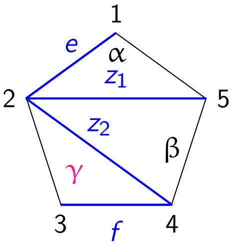
- z_k and z_{k+1} are incident;
- z_k and z_{k+1} belong to a common piece $p_k \in \mathcal{D}$;
- the pieces $p_0, \ldots, p_{s-1}$ are pairwise different.

If this is the case, then

$$d(e, f) = q_i q_j \prod_{k=i+1}^{j-1} q_k^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)} \ ,$$

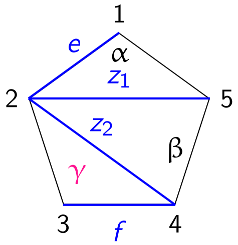
where β runs over all ("zig") pieces of $\mathcal{D}$ which have at most one vertex between $j+1$ and i (counterclockwise).

Example



$$W = \begin{pmatrix} 0 & 1 & \mathbf{a+b+c} & \mathbf{ac+bc} & abc \\ abc & 0 & \mathbf{1} & \mathbf{c} & bc \\ ab+ac+bc & abc & 0 & 1 & b+c \\ a+b & ab & abc & 0 & 1 \\ 1 & a & ab+ac & abc & 0 \end{pmatrix}$$

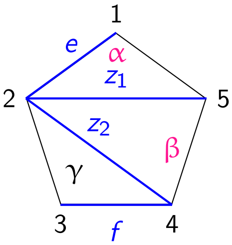
Example



$$W = \begin{pmatrix} 0 & 1 & \mathbf{a+b+c} & \mathbf{ac+bc} & abc \\ abc & 0 & \mathbf{1} & \mathbf{c} & bc \\ ab+ac+bc & abc & 0 & 1 & b+c \\ a+b & ab & abc & 0 & 1 \\ 1 & a & ab+ac & abc & 0 \end{pmatrix}$$

$$d(e,f) \quad = \quad c^2$$

Example



$$W = \begin{pmatrix} 0 & 1 & a+b+c & ac+bc & abc \\ abc & 0 & 1 & c & bc \\ \mathbf{ab+ac+bc} & \mathbf{abc} & 0 & 1 & b+c \\ \mathbf{a+b} & \mathbf{ab} & abc & 0 & 1 \\ 1 & a & ab+ac & abc & 0 \end{pmatrix}$$

$$\begin{aligned} d(e,f) &= c^2 \\ d(f,e) &= a^2 b^2 \end{aligned}$$