The combinatorics of CAT(0) cube complexes

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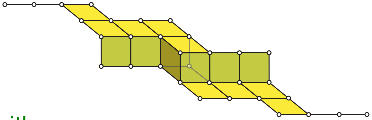
Richard Stanley's Birthday Conference
MIT
June 26, 2014

Summary.

1. There are **many** (locally/globally) CAT(0) cube complexes “in nature".
2. Globally CAT(0) cube complexes have an elegant, useful structure.

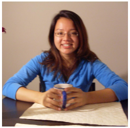
Problem 1. Find and study globally CAT(0) cube complexes in combinatorics.

Problem 2. Describe combinatorial structure of locally CAT(0) cube complexes.



Based on joint work with:

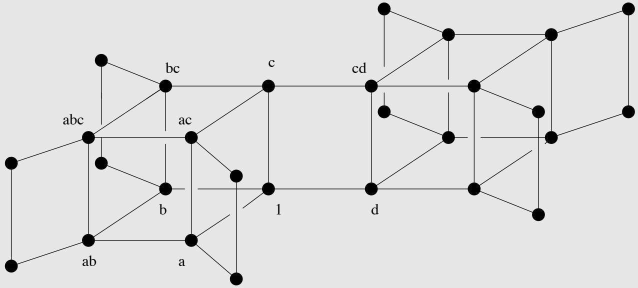
- Megan Owen (Waterloo), Seth Sullivant (NCSU)
- Rika Yatchak (SFSU/NCSU), Tia Baker (SFSU)
- Diego Cifuentes (Andes/MIT), Steven Collazos (SFSU/Minnesota)

1. CAT(0) CUBE COMPLEXES IN NATURE.

A. Geometric Group Theory.

The Cayley graph of a right-angled Coxeter group. (Davis)



$$a^2 = b^2 = c^2 = d^2 = 1 \qquad (ab)^2 = (ac)^2 = (bc)^2 = (cd)^2 = 1$$

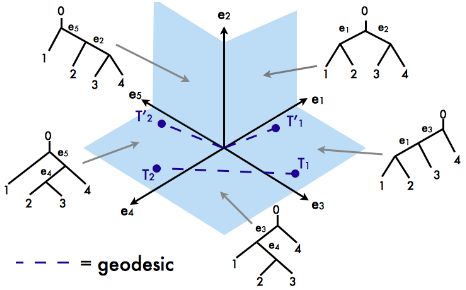
Study a RACG by its geometric action on the Davis complex.

(Rick Scott. Right-angled mock reflection and mock Artin groups.)

1. CAT(0) CUBE COMPLEXES IN NATURE.

B. Phylogenetic trees

The space of phylogenetic trees. (Billera, Holmes, Vogtmann):

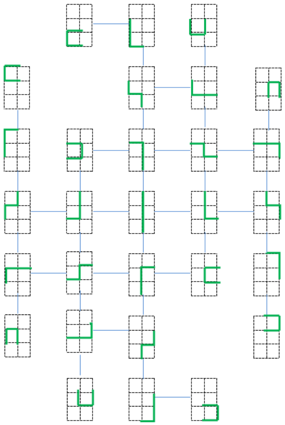


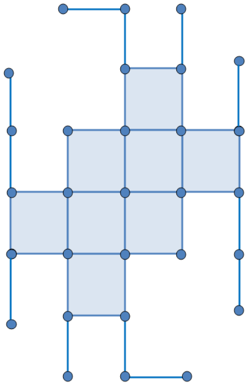
Build and navigate the space of all possible evolutionary trees.

(Megan Owen. Computing Geodesic Distances in Tree Space.)

1. CAT(0) CUBE COMPLEXES IN NATURE.

C. Moving (some) robots. (Abrams, Ghrist)



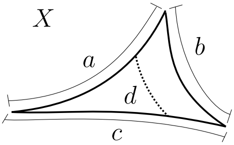


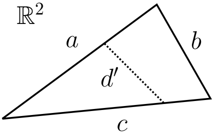
Build and navigate the space of all positions of the robot.

(F.A., Tia Baker, Rika Yatchak. Moving robots efficiently...CAT(0) complexes)

2. DEFINITIONS.

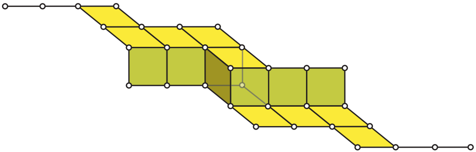
• A metric space X is CAT(0) if it has non-positive curvature everywhere; i.e., triangles are “thinner" than flat triangles. Roughly, X is “saddle shaped".





$d \leq d'$

• A cube complex is obtained by gluing cubes face-to-face.



(Like a simplicial complex, but the building blocks are cubes.)

3. WHICH CUBE COMPLEXES ARE CAT(0)?

Gromov 1987: Combinatorial / topological characterization.

A.-Owen-Sullivant 2008: Purely combinatorial characterization.
(Also Roller–Sageev.)

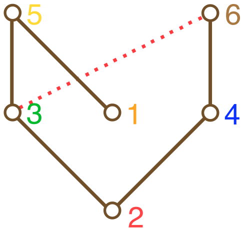
Theorem. There is an explicit bijection
(Pointed) CAT(0) cube complexes $\leftrightarrow$ posets with inconsistent pairs.

Poset with inconsistent pairs:

- a poset P, and
- a set of “inconsistent pairs"

such that

$$x, y \text{ inconsistent}, \quad y < z$$
$$\downarrow$$
$$x, z \text{ inconsistent.}$$



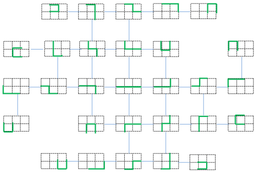
Applications: diameter, enumeration, Hopf algebra structure

4. CAT(0) CUBE COMPLEXES IN COMBINATORICS.

Reconfiguration system: a discrete system that changes according to local rules (satisfying some simple conditions)

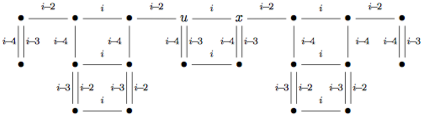
Theorem. (Ghrist – Peterson, 2004)
Any reconfiguration system gives a locally CAT(0) cube complex.
Many give globally CAT(0) cube complexes, which we understand.

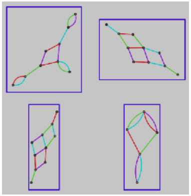
Fact.
Combinatorics is **full** of reconfiguration systems.

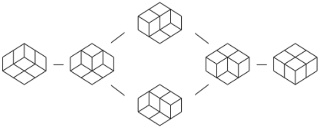


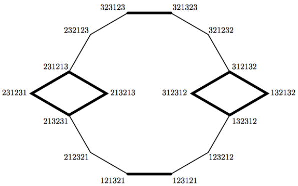
4. CAT(0) CUBE COMPLEXES IN COMBINATORICS.

Fact.
Combinatorics is **full** of reconfiguration systems.
They all give rise to locally CAT(0) cube complexes.









J. Propp. Lattice structure for orientations of graphs.
S. Corteel, L. Williams. Tableaux combinatorics for the ASEP.
V. Reiner and Y. Roichman. Diameter of graphs of reduced words and galleries.
S. Assaf. Dual equivalence graphs and a combinatorial proof of LLT and Macdonald positivity.
S. Billey, Z. Hamaker, A. Roberts, B. Young. Coxeter-Knuth graphs and a signed Little bijection.

CAT(0) CUBE COMPLEXES IN COMBINATORICS.

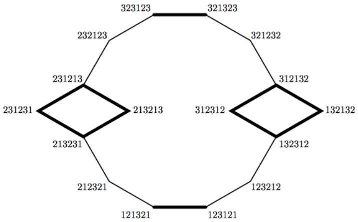
Fact. Combinatorics is **full** of reconfiguration systems.
They all give rise to locally CAT(0) cube complexes.

Question.
Which ones give rise to globally CAT(0) cube complexes?

Example. $G(w)$ = graph of reduced words of a permutation w.
Vertices: reduced words $i_1 \dots i_k$ where $s_{i_1} \dots s_{i_k} = w$
Edges: braid relations

$$...i(i+i)i... \leftrightarrow ...(i+1)i(i+1)...$$
$$...ij... \leftrightarrow ...ji... \qquad (|i-j| > 2)$$

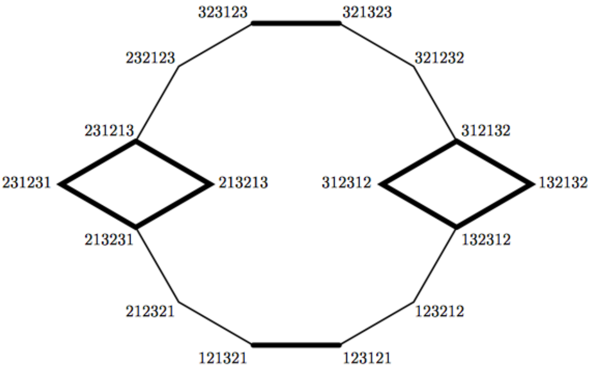
Question. For which $w \in S_n$ is $G(w)$ the skeleton of a globally CAT(0) cube complex?



- Coxeter-Knuth graphs?
- dual equivalence graphs?

LOCALLY CAT(0) CUBE COMPLEXES IN COMBINATORICS.

Problem 2.
What if our cube complex is only locally CAT(0)?
Give a combinatorial model for CAT(0) cube complexes.



many thanks

The papers (and more detailed slides) are available at:

http://math.sfsu.edu/federico
http://arxiv.org

F. Ardila, M. Owen, S. Sullivant. Geodesics in CAT(0) cubical complexes. Advances in Applied Mathematics **48** (2012) 142–163.

F. Ardila, T. Baker, R. Yatchak. Moving robots efficiently using the combinatorics of CAT(0) cubical complexes. SIAM J. Discrete Math. **28** (2014) 986 – 1007