Oscillating tableaux and a superanalogue of the Robinson-Schensted-Knuth correspondence

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Abstract

In this paper, we investigate the super RSK correspondence, first introduced by Pak and Postnikov (1994), which generalizes both the RSK correspondence and the dual RSK correspondence. The bijection relates objects that are known as oscillating supertableau and (intransitive) supergraphs. We prove a number of theorems that were stated in the original paper of Pak and Postnikov, but whose proofs are missing in the literature. We also prove a generalized Cauchy identity associated to the super RSK correspondence. We believe this idea will lead to the notion of a super Schur process, generalizing the Schur process of Okounkov and Reshetikhin (2003).

1 Introduction

The *Robinson-Schensted-Knuth (RSK) correspondence* is an important bijection between matrices A with nonnegative integer entries and pairs of semistandard Young tableaux (P, Q) which provides bijective proofs of certain symmetric function identities and identities derived from the representation theory of the symmetric group. The RSK correspondence can be considered in terms of a row-insertion algorithm, as it was originally defined by Schensted [13] and Knuth [6] or in terms of the growth diagrams of Fomin [3]. The growth diagram approach naturally gives three more variants of the RSK correspondence, the dual RSK correspondence, the RSK' correspondence, and the Burge correspondence, as explained in [7].

The RSK correspondence and its variants have a number of enumerative applications. Greene's theorem [5] relates the shape of the output tableaux to the length of increasing and decreasing subsequences in a permutation. It plays a role in the combinatorics of plane partitions [15, Sec. 7.20] as well. William Burge used several variants of the RSK correspondence to study the enumeration of undirected multigraphs with certain prescribed vertex degrees [1]. We combine the variants of the RSK correspondence into a *super RSK correspondence* in a way that lets us enumerate a more general class of multigraphs than those studied by Burge, and in doing so fill in enumerative proofs of identities derived by Pak and Postnikov [9] by algebraic means.

The output of the RSK algorithm is a pair of semistandard Young tableaux, but these tableaux can also be viewed as a walk up Young's lattice, where horizontal strips are added at each step. Dual RSK produces a conjugate-semistandard tableau and a semistandard tableau; the conjugate-semistandard tableau can be though of as walking up Young's lattice, adding vertical strips at each step. For the super RSK correspondence, we consider a generalized setting in which either vertical or horizontal strips can be added or subtracted; the specialization to only horizontal strips of the super RSK correspondence, which we call *oscillating RSK*, is well known ([7], [11, Chapter 4]). These bijections, Theorem 4.1 and Theorem 4.2, provide combinatorial proofs of two results of Postnikov and Pak [9, Thms. 7.1 and 8.5].

In addition to giving bijective proofs of Pak and Postnikov's identities, we give an analogue of the Cauchy identity for the super RSK correspondence. The Schur process of Okounkov and Reshitikin [8] is related to a specialization of this Cauchy identity, suggesting that a "super Schur process" could be defined. Using notation that will be defined in Section 5, our super Cauchy identity (Proposition 5.2) is

$$\sum_{\lambda}\left(\prod_{i\in I} s_{(\lambda^i)^{*\epsilon_i}/((\lambda^{i-1})^{*\epsilon_i}}\left(X^i\right)\prod_{j\in J} s_{(\lambda^{j-1})^{*\epsilon_j}/((\lambda^{j})^{*\epsilon_j}}\left(Y^j\right)\right)=\prod_{i\in I, j\in J} H\left(X^i, Y^j, \epsilon^i\epsilon^j\right),$$

where the sum is over all oscillating supertableaux $(\lambda)^{\epsilon}$ of length n starting and ending at $\emptyset$. When $n=1$ and $\epsilon_1 = 1$, this identity specializes to the usual Cauchy identity.

Section 2 covers background about partitions, Young tableaux, and the RSK correspondence. Most of the material is well-known, but Section 2.2 . Section 3 introduces local rules for the RSK correspondence and other related correspondences, laying out the framework in which all four variants can be realized. Section 4 introduces the super RSK correspondence and proves that it is a bijection. Section 5 gives a Cauchy identity for the super RSK correspondence, related to the Schur process. Section 6 gives combinatorial proofs of commutation relations on representation-theoretic operators of Pak and Postnikov, applying our super RSK correspondence and giving an algebraic interpretation of the addititon and deletion of vertical and horizontal strips.

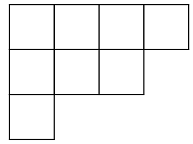

Figure 1: The Young diagram for the partition $\lambda = (4, 3, 1)$.

Acknowledgements

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2 Preliminaries

2.1 Partitions, Young Tableaux, and Symmetric Functions

We recall some basic definitions and introduce the notation we will use. Our terminology and exposition will closely follow that in [15].

We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of natural numbers, and by $\mathbb{P}$ the set of positive integers. For a natural number n, we say an ordered k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ is a *partition* of n, denoted $|\lambda| = n$, if $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. We also write $\lambda \vdash n$, and we write $\ell(\lambda) = k$ to denote the number of parts of λ. When convenient, we consider λ with one or more trailing zeroes, so that if $\ell(\lambda) = k$, λ_k is the last nonzero term in λ. Lastly, denote by $\widehat{\lambda}_k$ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

We define the *Young diagram* corresponding to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ to be the collection of points (i, j) with $i \leq k$ and $j \leq \lambda_i$. We visualize the points in Young diagrams as squares, which may be filled with positive integers. If T is the Young diagram corresponding to λ and T' is the reflection of T in its main diagonal, we say that the partition λ' corresponding to T' is the *conjugate* of λ. The *corners* of a Young diagram are those cells (i, j) for which neither $(i + 1, j)$ nor $(i, j + 1)$ are cells in the diagram. We can also view a Young diagram as an infinite sequence in the alphabet $\{R, U\}$ that starts with infinitely many Us and ends in infinitely many Rs; the portion in between encodes the boundary of the Young diagram as a sequence of right and up moves.

A Young diagram with positive integers in its boxes is called a *Young tableau* if the entries are weakly increasing along rows and columns. The *shape* of a Young tableau T, denoted $\operatorname{sh} T$ is the partition λ corresponding to its Young diagram, while the *type* of a Young tableau is the sequence of multiplicities of the positive integers as entries in the tableau.

A Young tableau is a *semistandard Young tableau* if its entries are strictly increasing along columns and weakly increasing along rows. If $\lambda \vdash n$ and the entries in the semistandard Young tableau T are distinct elements of $\{1, 2, \ldots, n\}$, we say that T is a *standard Young tableau.* Figure 1 shows a Young diagram of shape $(4, 3, 1)$, while Figure 2 shows a Young tableau of shape $(4, 3, 1)$ and type $(2, 2, 4)$ and a standard Young tableau of shape $(4, 3, 1)$. A semistandard Young tableau T can be interpreted as a sequence of partitions $\emptyset = \lambda_0, \lambda_1, \ldots, \lambda_n = \operatorname{sh} T$ in which λ_i is formed from λ_{i-1} by adding a number of i's into the squares of the Young diagram for λ_i that were not in the Young diagram for λ_{i-1}.

We can define a partial order on the set Par of partitions by inclusion of Young diagrams, so that $(5, 3, 2, 2) \geq (4, 3, 2, 1)$, while $(4, 3, 2, 1)$ and $(5, 3, 1, 1)$ are incomparable. Algebraically, $\lambda \subseteq \mu$ if $\lambda_i \geq \mu_i$ for

1	1	2	3
2	3	3	
3			

1	3	5	6
2	4	7	
8			

Figure 2: A semistandard Young tableau and a standard Young tableau of shape λ.

all i. This poset is a lattice, denoted $\mathcal{Y}$, and is known as *Young's lattice.*

The most interesting basis for the algebra of *symmetric functions* (over $\mathbb{Q}$), denoted Λ, is the set of *Schur functions*, which can be defined as

$$s_\lambda = \sum_{\mathrm{sh}(T)=\lambda} x^T,$$

where x^T for a tableau of type $(t_1, t_2, \cdots)$ is defined as $x_1^{t_1} x_2^{t_2} \cdots$.

2.2 Intransitive Graphs and Oscillating Tableaux

For partitions λ, μ with $\lambda \geq \mu$, define the *skew shape* λ/μ as the set-theoretic difference of the Young diagrams for λ and μ. We call those skew shapes with no column containing more than one square *horizontal strips* and those with no row containing more than one square *vertical strips.*

We define an *oscillating tableau* to be a sequence of partitions $(\lambda_0, \lambda_1, \ldots, \lambda_k)$, each differing from the previous by a horizontal strip, and the *weight* of an oscillating tableau to be the vector $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, where $\beta_k = |\lambda_k| - |\lambda_{k+1}|$. Given a vector ϵ of length k whose entries are either -1 or 1, we define an *oscillating supertableau* of weight β^ϵ as a sequence of partitions with exponents $(\lambda_0, \lambda_1^{\epsilon_1}, \ldots, \lambda_k^{\epsilon_k})$, each differing from the previous by a strip, either horizontal or vertical, determined by the ϵ_i. We write λ^{-1} as $\overline{\lambda}$ for clarity.

We interpret λ_i in an oscillating supertableau as stating that λ_i/λ_{i-1} or λ_{i-1}/λ_i, depending on the sign of β_i, is a horizontal strip and $\overline{\lambda}_i$ in an oscillating supertableau as stating that λ_i/λ_{i-1} or λ_{i-1}/λ_i, depending on the sign of β_i, is a vertical strip.

Pak and Postnikov [9] define a representation of the group $S_p \times S_q$ in terms of a group action on a certain type of 2-colored multigraph that we define next.

Definition 2.1. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ be a vector of integers with sum 0 all of whose partial sums are nonpositive, and let ϵ be a vector of length k whose entries are either -1 or 1. An *intransitive supergraph* of type δ^ϵ is a directed mutligraph with k vertices such that

1. If $\epsilon_i = 1$, the vertex i is colored black; if $\epsilon_i = -1$, the vertex i is colored white.
2. If $\delta_i \geq 0$, the indegree of vertex i is δ_i, and the outdegree is 0. If $\delta_i \leq 0$, the outdegree of vertex i is $-\delta_i$, and the indegree is 0.
3. If $i \geq j$, there are no edges from vertex i to vertex j.
4. If $\epsilon_i \neq \epsilon_j$, there is at most one edge from vertex i to vertex j.

An intransitive supergraph with all black vertices is called an *intransitive graph.*

Figure 3 shows an intransitive graph of type $\delta = (-2, 1, -1, 0, -2, 2, 3)$, while Figure 4 shows an intransitive supergraph of type δ^ϵ, where $\delta = (-3, 2, 1, 1, -2, -1, 4)$ and $\epsilon = (-1, -1, 1, -1, 1, 1, 1)$.

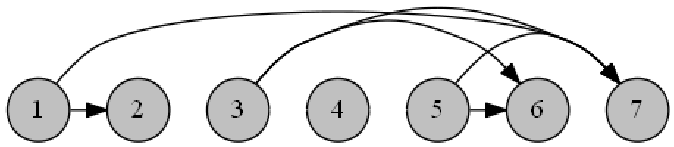

Figure 3: An intransitive graph of type $\delta = (-2, 1, -1, 0, -2, 2, 3)$

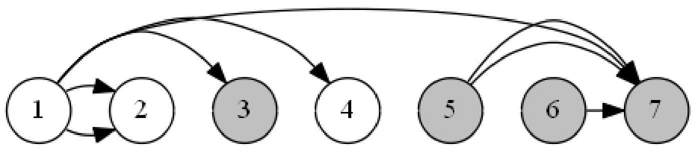

Figure 4: An intransitive supergraph of type δ^ϵ, where $\delta = (-3, 2, 1, 1, -2, -1, 4)$ and $\epsilon = (-1, -1, 1, -1, 1, 1, 1)$

2.3 The RSK Correspondence

The dimension of the irreducible representation of S_n corresponding to the partition λ (i.e. the dimension of the *Specht module* S^λ) is given by f^λ, the number of standard Young tableaux of shape λ (for more about the representation theory of S_n, see [12]). Letting $\operatorname{Reg} S_n$ denote the regular representation of S_n, we have

$$n! = \dim \operatorname{Reg}(S_n) = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2. \qquad (*)$$

This identity can also be proved using a bijection between permutations in S_n and pairs of standard Young tableaux of the same shape; that bijection is due to Robinson [10] and Schensted [13]. The correspondence operates using a row-insertion algorithm that generalizes to take $\mathbb{N}$-matrices as inputs and return pairs of semistandard Young tableaux of the same shape.

Definition 2.2. Given a semistandard Young tableau T and an integer K, define the *row insertion* procedure as follows:

1. Set $i = 1$ and $k = K$.
2. If there is no entry in row i of T that is greater than k, append a cell containing k to the end of row i of T, and return T.
3. Otherwise, find the leftmost entry j in row i of T that is greater than k, set $\ell = j$ and replace the cell containing j with a cell containing k.
4. Set $k = \ell$ and $i = i + 1$.
5. Go to step 2.

We are now in position to define the RSK correspondence. Given an $\mathbb{N}$-matrix A, we form the *two-line array* $\mathcal{A}$ of A by placing in lexicographic order a_{ij} copies of $\substack{i \\ j}$. For example, we have

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 3 & 1 & 2 & 1 & 1 & 2 \end{pmatrix} = \mathcal{A}$$

Definition 2.3. Given an $\mathbb{N}$-matrix A, the *RSK correspondence* (or sometimes *RSK algorithm*) produces a pair of semistandard Young tableaux P and Q of the same shape by the following procedure:

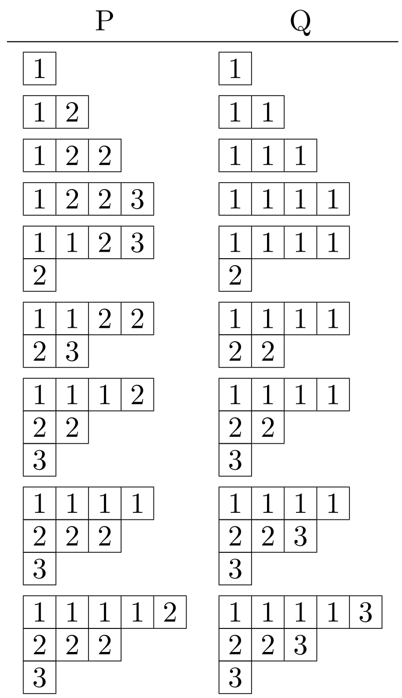

Figure 5: RSK applied to the matrix A.

1. Build the two-line array $\mathcal{A} = \left(\begin{smallmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{smallmatrix}\right)$ from A.
2. Initialize empty tableaux P and Q.
3. For i from 1 to m:
   (a) Insert v_i into P by row insertion. Denote by C the cell in which the insertion stopped.
   (b) Append a cell containing u_i to the cell C of Q.
4. Return (P, Q).

We call the tableau P the *insertion tableau* and the tableau Q the *recording tableau.*

The RSK correspondence is invertible because the row insertion procedure is invertible. If all the rows and columns of A have exactly one 1 each, then A is a permutation matrix, all the u_i are distinct, and all the v_j are distinct, causing P and Q to be standard Young tableaux. This gives us the desired bijection to prove (*).

Observing that

$$\prod_{i,j} \frac{1}{1 - x_i y_j}$$

is the generating function for $\mathbb{N}$-matrices with given row sum vectors $\alpha = (\alpha_1, \alpha_2, \dots)$ given by the exponents on $x_1, x_2, \dots$ and column sum vectors $\beta = (\beta_1, \beta_2, \dots)$ given by the exponents on $y_1, y_2, \dots$, while the right-hand side is the generating function for pairs of semistandard Young tableau of the same shape, the RSK correspondence proves the following:

Proposition 2.1 (Cauchy identity)**.**

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda(y).$$

Schensted's motivation for studying the Robinson-Schensted corrrespondence was the question of the length of the longest increasing and decreasing subsequences of permutations; using the row insertion algorithm given in Section 2.3, he was able to determine the length of the longest increasing and decreasing subsequences in a permutation w [13, Theorems 1 and 2]. This result was extended by Curtis Greene to count the length of the longest k increasing or decreasing subsequences [5]. This theorem is most naturally stated in terms of the RSK correspondence, for which we need to define how we generalize the notions of length and increasing or decreasing subsequences.

Definition 2.4. An *SE-path* in a matrix A is a sequence of entries a_{ij} in which each entry is located weakly to the right of and weakly below the previous entry. An *Se-chain* in A is a sequence of entries in which each entry is located strictly to the right of and weakly below the previous entry; define an *nE-chain* similarly, where the uppercase E denotes weakly right and the lowercase n denotes strictly above. Lastly, define an *ne-chain* as a sequence of entries in which each entry is located strictly right of and strictly above the previous entry.

Definition 2.5. Define the *weight* of a path or chain to be the sum of the entries in the path or chain. Define the *mass* of a path or chain to be the number of nonzero entries in the path or chain. Note that "mass" is not a standard term; in the literature (e.g. [7, sec 4]), the word "weight" is applied to both these notions.

Increasing subsequences in a permutation correspond to SE-paths in its permutation matrix and decreasing subsequences correspond to ne-chains in its permutation matrix.

Theorem 2.1 (Greene, [5])**.** *Let A be an $\mathbb{N}$-matrix, and suppose $A \xrightarrow{RSK} (P, Q)$. Let $\lambda = \operatorname{sh} P = \operatorname{sh} Q$. Then $\widehat{\lambda}_k$ is the maximum sum of the weights of k disjoint SE-paths and $\widehat{\lambda'}_k$ is the maximum sum of the masses of k ne-chains where an entry $a_{ij} = e$ can be in no more than e of the chains.*

One can verify that the example in Figure 5 satisfies this property.

3 The Local Approach to RSK

3.1 Growth Diagrams

Greene's theorem gives us a "global" description of the process of row-insertion. Both this description and the row insertion algorithm obscure the fundamental symmetry property that $A \xrightarrow{RSK} (P, Q)$ implies $A^t \xrightarrow{RSK} (Q, P)$. In this section we lay out a "local" approach to RSK using the growth diagrams of Fomin [3].

To find the output of RSK for a given matrix A, we write it as the entries of a rectangular Young tableau. We label the corners of the squares of the Young tableau with partitions, with corners along the top and left edges of the Young diagram labeled with $\emptyset$ and the remaining corners filled according a set of chosen local rules, applying the rules on squares in a linear extension of the Young diagram. To fix the notation, when considering an individual square the partitions on the corners and the entry inside the square will be denoted as in Figure 6.

The (forward) local rule for RSK is the procedure for finding λ given the other three partitions μ, ν, and ρ at the corners of the square and the entry m inside it. Our presentation will follow that in [7], but not using French notation.

Definition 3.1. The (forward) *RSK local rule* is the following procedure, which takes as input three partitions μ, ν, and ρ (with trailing zeroes as needed) and a natural number m and returns a partition λ:

1. Set $i = 1$ and CARRY $= m$.

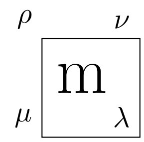

Figure 6: The notational convention for growth diagrams.

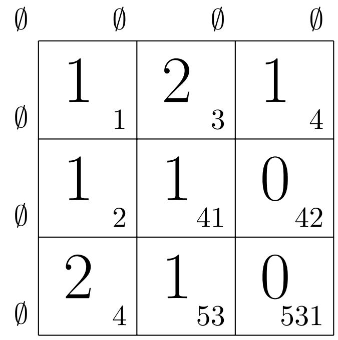

Figure 7: The RSK growth diagram for the matrix A of Section 2.3

2. Do

   (a) Set $\lambda_i = \max(\mu_i, \nu_i) + \text{CARRY}$.

   (b) If $\lambda_i = 0$, return λ. Otherwise, set $\text{CARRY} = \min(\mu_i, \nu_i) - \rho_i$ and $i = i + 1$.

Figure 7 shows the resulting array when the RSK local rule is applied to the matrix A of Section 2.3. Because the partitions $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$ differ by a horizontal strip, the sequences of partitions along the bottom and right edges, $\emptyset \subset 4 \subset 53 \subset 531$ and $\emptyset \subset 4 \subset 42 \subset 531$ can be read as a pair of semistandard tableau

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & & \\ 3 & & & & \end{array} , \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & & \\ 3 & & & & \end{array} \right),$$

which is exactly the image of A under RSK.

Because the RSK local rule is symmetric with respect to transposition, transposing the matrix A transposes the entire RSK growth diagram for A. Reading the recording tableau from the right edge and the insertion tableau gives us a symmetry result about RSK.

Proposition 3.1.

$$A \xrightarrow{RSK} (P, Q) \iff A^t \xrightarrow{RSK} (Q, P).$$

The growth diagram records information for all the submatrices of A, so that we can read off that

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \xrightarrow{RSK} \left(\begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & & & \end{array} , \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \right)$$

3.2 Other RSK-Like Correspondences

If we choose local rules other than those for RSK, we can produce different correspondences between matrices and Young tableaux. The first of these is *dual RSK* (sometimes referred to as *RSK** in the literature, defined by Knuth [6] via the a modified insertion procedure, given below with the differences from Definition 2.2 italicized.

Definition 3.2. Given a *tranpose-semistandard* Young tableau T and an integer K, define the *dual row insertion* procedure as follows:

1. Set $i = 1$ and $k = K$.
2. If there is no entry in row i of T that is *greater than or equal to* k, append a cell containing k to the end of row i of T, and return T.
3. Otherwise, find the leftmost entry j in row i of T that is *greater than or equal to* k, set $\ell = j$ and replace the cell containing j with a cell containing k.
4. Set $k = \ell$ and $i = i + 1$.
5. Go to step 2.

The dual RSK correspondence is a bijection between $0, 1$-matrices and pairs of semistandard Young tableau of conjugate shapes. The restriction to $0, 1$-matrices is necessary to guarantee that the recording tableau Q is semistandard.

Definition 3.3. Given a $0, 1$-matrix A, the *dual RSK correspondence* produces a pair of Young tableaux P and Q by the following procedure:

1. Build the two-line array $\mathcal{A} = \left(\begin{smallmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{smallmatrix} \right)$ from A.
2. Initialize empty tableaux P and Q.
3. For i from 1 to m:
   (a) Insert v_i into P by dual row insertion. Denote by C the cell in which the insertion stopped.
   (b) Append a cell containing u_i to the cell C of Q.
4. Return (P, Q).

Figure 8 gives an example of the dual RSK algorithm applied to the matrix

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The dual RSK correspondence provides a bijective proof of an analogue of Proposition 2.1.

Proposition 3.2 (dual Cauchy identity)**.**

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y).$$

We can define a local rule for the dual RSK correspondence, where we place the $0, 1$-matrix in the growth diagram. In this growth diagram, horizontally-adjacent partitions will differ by a vertical strip and vertically adjacent partitions will differ by a horizontal strip. Our presentation will follow [7].

Definition 3.4. The (forward) *RSK* local rule* is the following procedure, which takes as input three partitions μ, ν, and ρ (with trailing zeroes as needed) and a number $m \in \{0, 1\}$ and returns a partition λ:

1. Set $i = 1$ and CARRY $= m$.

P	Q
1	1
1 3	1 1
1 3 / 1	1 1 / 2
1 2 / 1 3	1 1 / 2 2
1 2 / 1 3 / 1	1 1 / 2 2 / 3

Figure 8: Dual RSK applied to the matrix B.

$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	1 (1)	0 (1)	1 (2)
$\emptyset$	1 (11)	1 (21)	0 (22)
$\emptyset$	1 (111)	0 (31)	0 (32)

Figure 9: The dual RSK growth diagram for the matrix B.

2. Do

   (a) Set $\lambda_i = \max(\mu_i + \text{CARRY}, \nu_i)$.

   (b) If $\lambda_i = 0$, return λ. Otherwise, set $\text{CARRY} = \min(\mu_i + \text{CARRY}, \nu_i) - \rho_i$ and $i = i + 1$.

Figure 9 shows the resulting array when the dual RSK local rule is applied to the matrix B. As before, we can read the transpose-semistandard insertion tableau P from the bottom row and the semistandard recording tableau Q from the right column. While there is no symmetry property for dual RSK easily derived from the local rules, there is an analogue of Greene's theorem for dual RSK.

Theorem 3.1 (dual Greene's)**.** *Let B be an $0,1$-matrix, and suppose $B \xrightarrow{RSK^*} (P, Q)$. Let $\lambda = \operatorname{sh} P = \operatorname{sh} Q$. Then $\widehat{\lambda}_k$ is the maximum sum of the weights (or masses) of k disjoint Se-paths and $\widehat{\lambda'}_k$ is the maximum sum of the masses (or weights) of k nE-chains where an entry $a_{ij} = e$ can be in no more than e of the chains.*

Reflecting the local rule for dual RSK (i.e. swapping the roles of μ and ν) gives another correspondence, the *RSK'* correspondence, in whose growth diagram horizontally-neighboring partitions differ by a horizontal strip and vertically-neighboring partitions differ by a vertical strip.

A fourth local rule, for a correspondence known as *dual RSK'* or the *Burge correspondence* (as the same correspondence was used in [1] to enumerate certain classes of graphs), takes $\mathbb{N}$-matrices as inputs. In growth diagrams for dual RSK', horizontally-neighboring partitions differ by a vertical strip and vertically-neighboring partitions differ by a vertical strip. A direct local rule for the Burge correspondence can be found in [7]. The Burge correspondence can also be computed using the RSK correspondence and the *Schützenberger involution* [16] or by an insertion procedure [4].

In addition to the direct forward and backward local rules for these four correspondences, these correspondences can all be viewed in the same context as Schensted growth diagrams, where the $\mathbb{N}$-matrices or $0,1$-matrices are expanded into permutation matrices, and then the same local rule is applied. In this setting, adjacent partitions can differ in rank by at most 1. This local rule for permutation matrices works as follows:

- If $|\rho| = |\mu| = |\nu|$ and $m = 0$, then $\lambda = \rho$
- If $|\rho| = |\mu| = |\nu|$ and $m = 1$, then $\lambda_1 = \rho_1 + 1$ and $\lambda_i = \rho_i$ for $i > 1$.
- If $|\rho| = |\mu| < |\nu|$, then $\lambda = \nu$. Similarly, if $|\rho| = |\nu| < |\mu|$, then $\lambda = \mu$.
- If $|\rho| < |\mu| = |\nu|$ and $\mu \neq \nu$, then $\lambda = \mu \cup \nu$, the union of the two Young diagrams.
- If $|\rho| < |\mu| = |\nu|$ and $\mu = \nu$, then λ is obtained from μ by appending a 1 to the end of μ.

To get a permutation matrix from an $\mathbb{N}$-matrix or $0,1$-matrix, the nonzero entries in a row or column are expanded into 1s and arranged either from top left to bottom right or from top right to bottom left. The RSK, dual RSK, RSK', and dual RSK' correspondences can be described neatly in this way [7].

- For RSK, arrange entries in each row and column from top left to bottom right. Entries n that are greater than 1 are expanded into $n \times n$ diagonal block matrices of ones.
- For dual RSK, arrange entries in each row from top left to bottom right, but arrange entries in each column from top right to bottom left.
- For RSK', arrange entries in each row from top right to bottom left, but arrange entries in each column from top left to bottom right.
- For dual RSK', arrange entries in each row and column from top right to bottom left. Entries n that are greater than 1 are expanded into $n \times n$ antidiagonal block matrices of ones.

Figure 10 gives examples of these growth diagrams for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

4 The Super RSK Correspondence

In this section we present two bijections

$$\Phi_{\lambda\mu\beta} : OT(\lambda, \mu, \beta) \to \bigcup_{\delta \prec \beta} G(\delta) \times OT\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)\big)$$

and

$$\Phi^{super}_{\lambda\mu\mathrm{b}} : OST(\lambda, \mu, \beta^{\epsilon}) \to \bigcup_{\delta \prec \beta} SG(\delta) \times OST\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)^{\epsilon}\big)$$

which give us combinatorial proof of Theorems 7.1 and 8.7 in the preprint [9].

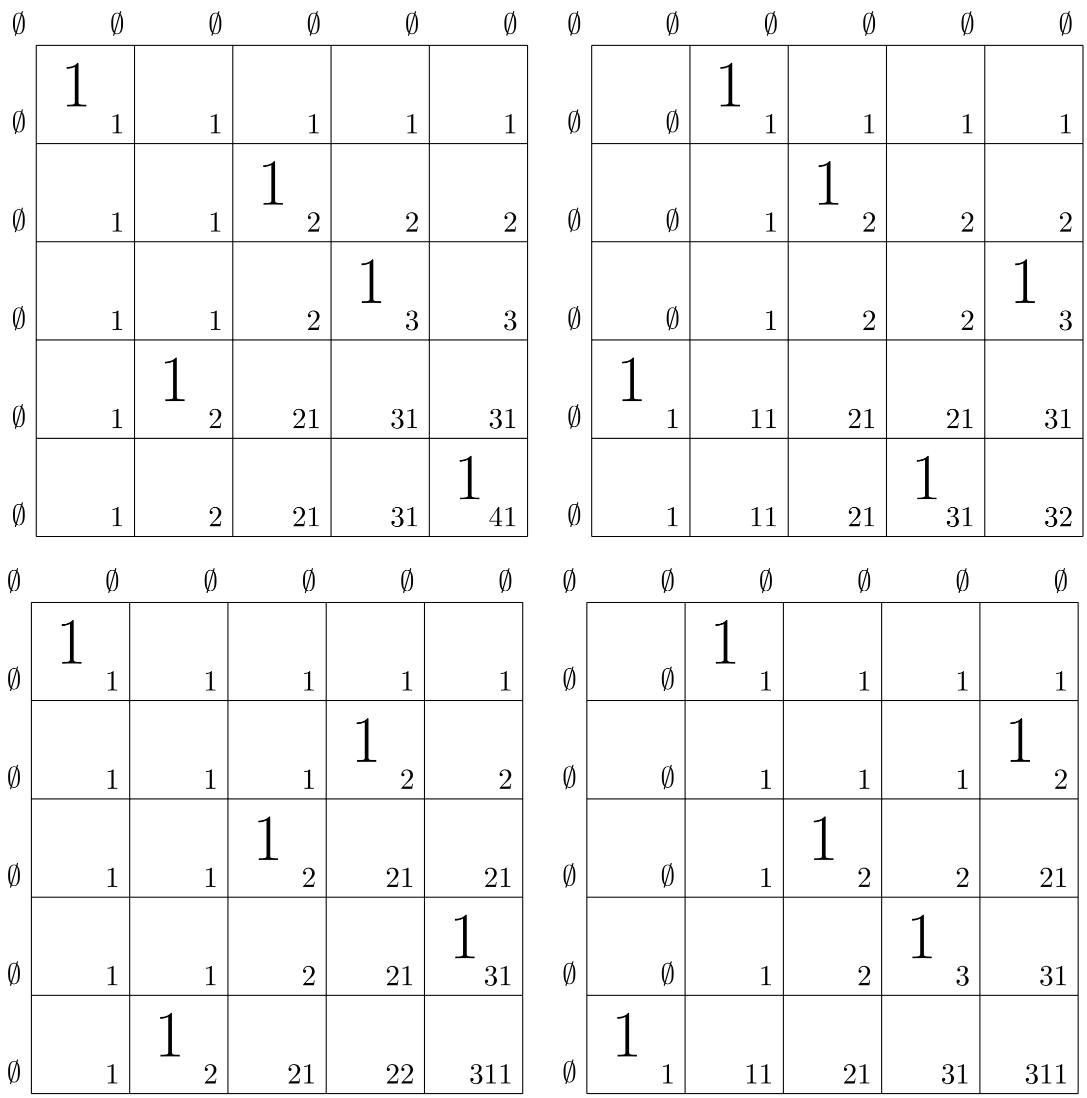

Figure 10: Schensted growth diagrams for RSK, dual RSK, RSK', and dual RSK', in that order, for the matrix A.

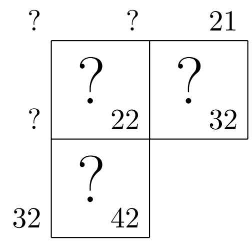

Figure 11: The unfilled growth diagram for the oscillating tableau $32, 42, 22, 32, 21$.

4.1 $\Phi_{\lambda\mu\beta}$

Definition 4.1. Let λ and μ be partitions with $|\lambda| = p$ and $|\mu| = q$, respectively, and let $\beta \in \mathbb{Z}^k$ such that $-p + \beta_1 + \cdots + \beta_i \leq 0$ for all $1 \leq i < k$ and $-p + \sum \beta = -q$. Let G denote the set of all intransitive graphs (of arbitrary type), and let $OT(\lambda, \mu)$ denote the set of all oscillating tableaux from λ to μ (of arbitrary weight). Define the map

$$\widetilde{\Phi}_{\lambda\mu\beta} : OT(\lambda, \mu, \beta) \to G \times OT(\lambda, \mu)$$

by the following procedure:

1. Build the Young shape τ from T based on the signs of the β_i, where we read the signs as an $\{R, U\}$ sequence by having positive signs correspond to Us and negative signs to R, with the entries t_i appearing on the bottom-right corners of the border squares of the tableau.

2. Apply the backward local rule for RSK on the Young shape τ, using the t_i as initial inputs and working in a reverse linear extension of the squares of τ. This will produce a filling $\mathcal{T}$ of τ (read from the entries found to be in the squares of τ by the backward local rule), as well as a sequence S of partitions read from the lower-left corner of τ to the upper right. The sequence S starts with $s_0 = t_0 = \lambda$ and ends with $s_k = t_k = \mu$, so S is an oscillating tableau in $OT(\lambda, \mu)$.

3. Assign distinct integers from $\{1, 2, \ldots, k\}$ to the rows and columns of τ based on where in the $\{R, U\}$ sequence the edge at the end of the row or column falls; for example, a tableau with $\{R, U\}$ sequence $RURU$ would have columns labeled 1 and 3 (from left to right) and rows labeled 2 and 4 (from bottom to top). Represent these numbered rows and columns with a graph whose vertices are labeled by the integers $\{1, 2, \ldots, k\}$, with N directed edges (i, j) if there is an N in the cell corresponding to the intersection of column i and row j; denote this graph by γ. Color the vertex i of γ white if $\epsilon_i = -1$ and black otherwise. The graph γ has type δ, where δ_i is the number of edges entering vertex i (where a negative number of edges entering is a number of edges leaving). Because the partial sums of δ are nonpositive, the sum of the entries of δ is 0, there is no more than one edge between a black vertex and a white vertex, and there are no edges (i, j) for $j < i$, γ meets the requirements of Definition 2.1 and is an intransitive graph.

4. Then let $\widetilde{\Phi}_{\lambda\mu\beta}(T) = (\gamma, S)$.

Given two vectors δ and β in $\mathbb{Z}^k$, we write $\delta \prec \beta$ if $0 \leq \delta_i \leq \beta_i$ or $0 \geq \delta_i \geq \beta_i$ for all $1 \leq i \leq k$. Given a vector of integers, with the zeroes tagged either "row" or "column," let $\operatorname{nor} \alpha$ denote the vector formed by shifting all the positive entries and zeroes tagged "row" in-place (i.e. not changing their relative order) to the beginning, leaving the negative entries and those zeroes tagged "column" in their original order at the end. For example $\operatorname{nor}(-3, 1, -1, 0_{row}, 1, 4, -2, 1) = (1, 0, 1, 4, 1, -3, -1, -2)$. Note that this definition differs from that in [9], but all results they prove with their definition, in particular Lemma 7.2, still hold with this definition.

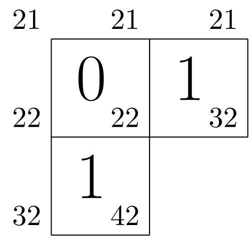

Figure 12: The filled-in growth diagram; the outside contains the oscillating tableau 32, 22, 21, 21, 21.

Figure 13: The graph in $G(\delta)$ corresponding to the oscillating tableau 32, 42, 22, 32, 21.

Example 4.1. The unfilled growth diagram in Figure 11 corresponds to the oscillating tableau 32, 42, 22, 32, 21 of type $\beta = (-1, 2, -1, 2)$. After filling in the growth diagram with the backwards RSK local rule, we get the growth diagram in Figure 12. The interior of the growth diagram contains the adjacency information for the graph in Figure 13, a graph in $G(\delta)$ for $\delta = (-1, 1, -1, 1)$, while the left and top edges give us the oscillating tableau 32, 22, 21, 21, 21, an oscillating tableau of type $\mathrm{nor}(\beta - \delta) = (1, 1, 0, 0)$. In other words, we have

$$\Phi_{(32)(21)(-1,2,-1,2)}\,(32, 42, 22, 32) = \big(\text{Figure 13}, (32, 22, 21, 21, 21)\big).$$

Proposition 4.1. *The image of $\widetilde{\Phi}_{\lambda\mu\beta}$ is a subset of*

$$\bigcup_{\delta \prec \beta} G(\delta) \times OT\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)\big)$$

We will denote by $\Phi_{\lambda\mu\beta}$ the map

$$\Phi_{\lambda\mu\beta} : OT(\lambda, \mu, \beta) \to \bigcup_{\delta \prec \beta} G(\delta) \times OT\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)\big)$$

given by $\Phi_{\lambda\mu\beta}(T) = \widetilde{\Phi}_{\lambda\mu\beta}(T)$.

Proof. Let $T = (\lambda = t_0, t_1, \ldots, t_k = \mu)$ be an oscillating tableau of weight β. Apply the map $\widetilde{\Phi}_{\lambda\mu\beta}$ to T, producing a filled growth diagram $\mathcal{T}$, along whose left and top edges can be read the oscillating tableau $S = (\lambda = s_0, s_1, \ldots, s_k = \mu)$. Let the intransitive graph corresponding to T be $\gamma \in G(\delta)$ for some $\delta \in \mathbb{Z}^k$.

By the local rule for RSK, we know that $|\lambda| - |\mu| = |\nu| - |\rho| + m$. Inductively, applying this fact along a column, we have that if ζ_0 and ξ_0 are two horizontally-adjacent partitions in a column C of an RSK growth diagram and ζ_1 and ξ_1 are two horizontally-adjacent partitions in that column located some number of rows beneath ζ_0 and ξ_0, we have that $(\xi_1 - \zeta_1) - (\xi_0 - \zeta_0)$ is equal to the sum of entries in C located between the rows containing ξ_0 and ξ_1. A similar result holds for vertically-adjacent partitions in row R and sums of entries in R.

Applying this to the filling $\mathcal{T}$ and the oscillating tableau T and S, we have that, if t_i and t_{i+1} are horizontally adjacent, then we have that $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|)$ is the sum of the entries in the column of $\mathcal{T}$ corresponding to vertex $i+1$ of the graph γ. Similarly, for vertically-adjacent t_i and t_{i+1}, $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|)$ is the sum of the entries in the row of $\mathcal{T}$ corresponding to vertex $i+1$ of the graph. But the sum of the entries in those columns (respectively, rows) of $\mathcal{T}$ correspond to the number of edges going out of (resp. into) vertex $i+1$ if it is a column (resp. row); with the convention that outdegree is represented as a negative number, we thus have that $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|) = \delta_{i+1}$.

Now the oscillating tableau S is read along the left and top borders of $\mathcal{T}$, so that all the vertical edges, which correspond to positive entries and whose zero entries we tag with "row," in the weight η of S, occur before any vertical edges, which correspond to negative entries (and zero entries we tag with "column") in η. Because we know that $\eta_j = s_{j+1} - s_j$ corresponding to β_i differs from β_i by δ_i, we know that $\eta = \mathrm{nor}(\beta - \delta)$. So S is of weight $\mathrm{nor}(\beta - \delta)$ as desired, and the map

$$\Phi_{\lambda\mu\beta} : OT(\lambda, \mu, \beta) \to \bigcup_{\delta \prec \beta} G(\delta) \times OT\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)\big)$$

is well-defined. □

Theorem 4.1. *The map $\Phi_{\lambda\mu\beta}$ is a bijection.*

Proof. Let $S = (\lambda = s_0, s_1, \ldots, s_k = \mu)$ be an oscillating tableau of weight $\eta \in \mathbb{Z}^k$ such that η is a normal sequence (in particular, its zeroes are tagged "row" and "column"), and let γ be an intransitive graph of type δ. We want to produce a map $\Psi_{\lambda\mu\eta}$ such that $\Psi_{\lambda\mu\eta}(\gamma, s) \in OT(\lambda, \mu, \beta)$ where $\eta = \mathrm{nor}(\beta - \delta)$.

First, we build a Young diagram τ from γ. We interpret the sequence of signs on terms in δ_i as a $\{R, U\}$ sequence as in the defintion of $\widetilde{\Phi}_{\lambda\mu\beta}$ and build a Young diagram τ. We obtain a filling $\mathcal{T}$ of τ by placing in the square corresponding to vertices a and b the number of edges from a to b.

We have the upper-left edge S and the filling $\mathcal{T}$ of a growth diagram. Applying the forward local rule for RSK, we can read off the lower-right edge a tableau $T = (\lambda = t_0, t_1, \ldots, t_k = \mu)$ of weight $\beta \in \mathbb{Z}^k$. As before, if t_i and t_{i+1} are horizontally adjacent corresponding to horizontally-adjacent entries s_j and s_{j+1} in S, $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|)$ is the sum of the entries in the column $\mathcal{T}$ between t_i and t_{i+1}. By the construction of $\mathcal{T}$, we have that $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|) = -\delta_{i+1}$, and similarly if t_{i+1} and t_i are vertically adjacent, $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|) = -\delta_i$. So, noting that our sign convention has positive outdegree represented as negative, we have $\eta_j = \beta_i - \delta_i$.

Since each $\beta_i + \delta_i$ equals a distinct η_j and increasing i corresponds to farther right or up in τ, so that if $i_2 > i_1$, the corresponding indices for η satisfy $j_2 > j_1$, we have $\eta = \mathrm{nor}(\beta - \delta)$. So the map $\Psi_{\lambda\mu\eta}$ is a well-defined map from $G(\delta) \times OT\big(\lambda, \mu, \mathrm{nor}(\beta - \delta)\big)$ to $G(\lambda, \mu, \beta)$.

The correspondence between pairs of an oscillating tableau of normal weight and an intransitive graph and fillings of Young diagrams is a bijection, as is the local rule for RSK. Since we've shown that $\Psi_{\lambda\mu\eta}$ is well-defined, this tells us that the map $\Psi_{\lambda\mu\eta}$ defined above inverts the portion of $\Phi_{\lambda\mu\beta}$ whose image oscillating tableaux are of weight η. Each possible image tableau weight η has a corresponding $\Psi_{\lambda\mu\eta}$, so the map $\Phi_{\lambda\mu\beta}$ is a bijection as desired. □

We will refer to the procedure taking as input a graph γ and a normal oscillating tableau S, or, equivalently, an $\mathbb{N}$-tableau and a normal oscillating tableau, as the *oscillating RSK correspondence.*

$\Phi^{super}_{\lambda\mu\beta}$

Definition 4.2. Given $\epsilon, \epsilon' \in \{-1, 1\}$, we define an associated super RSK local rule. Let ρ, μ, ν be partitions such that if $\epsilon = -1$, ρ and ν differ by a vertical strip, if $\epsilon = 1$, ρ and ν differ by a horizontal strip, if $\epsilon' = -1$, ρ and μ differ by a vertical strip, and if $\epsilon' = 1$, ρ and μ differ by a horizontal strip. Let $m \in \mathbb{N}$ if $\epsilon\epsilon' = 1$ and $m \in \{0, 1\}$ otherwise. Define the *super RSK local rule* to be:

- if $\epsilon = 1$ and $\epsilon' = 1$, the RSK local rule.

- if $\epsilon = -1$ and $\epsilon' = 1$, the dual RSK local rule.
- if $\epsilon = 1$ and $\epsilon' = -1$, the RSK' local rule.
- if $\epsilon = -1$ and $\epsilon' = -1$, the dual RSK' local rule.

Detailed descriptions of these local rules may be found in [7, Sec. 4].

Pictorially, we let the color white for vertices in oscillating graphs and rows and columns of Young diagrams correspond to an exponent of -1 and the color black correspond to an exponent of 1 in what follows. We denote β^ϵ by b. In the description in terms of Schensted growth diagrams, black rows and columns expand from top left to bottom right, while white rows and columns expand from bottom right to top left.

Given a pair of colors ϵ and ϵ', we can define the backward super RSK local rule by picking the backward local rule corresponding to ϵ and ϵ'. This means that, given a column coloring ϵ and a row coloring ϵ', and southwest and northeast corner partitions (μ, ν), the super RSK local rule is a bijection between pairs (ρ, m) of a partition and either an element of $\{0, 1\}$ or $\mathbb{N}$ and partitions λ.

Definition 4.3. Let λ and μ^ω, $\omega \in \{-1, 1\}$, be partitions with $|\lambda| = p$ and $|\mu| = q$, respectively, let $\beta \in \mathbb{Z}^k$ such that $-p + \beta_1 + \cdots + \beta_i \leq 0$ for $1 \leq i < k$ and $-p + \sum \beta = -q$, and let $\epsilon \in \{-1, 1\}^k$ with $\omega = \epsilon_k$. Let $\beta^\epsilon = \text{b}$. Let SG^ϵ denote the set of all intransitive supergraphs (of arbitrary type δ colored by ϵ) , and let $OST(\lambda, \mu^\omega)$ denote the set of all oscillating supertableaux from λ to μ^ω (of arbitrary weight). Define the map

$$\widetilde{\Phi}^{super}_{\lambda\mu\text{b}} : OST(\lambda, \mu^\omega, \text{b}) \to SG^\epsilon \times OST(\lambda, \mu^\omega)$$

by the following procedure:

1. We interpret the sequence of signs on terms in δ_i as a $\{R, U\}$ sequence as in the defintion of $\widetilde{\Phi}_{\lambda\mu\beta}$ and build a Young diagram τ. Color the columns and rows of τ white if the corresponding ϵ_i is -1 and color them black otherwise.

2. Apply the backward local rule for super RSK on the Young shape τ, using the t_i as initial inputs and working in a reverse linear extension of the squares of τ. This will produce a filling $\mathcal{T}$ of τ (read from the entries found to be in the squares of τ by the backward local rule), as well as a sequence S of partitions read from the lower-left corner of τ to the upper right. The sequence S starts with $s_0 = t_0 = \lambda$ and ends with $s_k = t_k = \mu^\omega$, so S is an oscillating supertableau in $OST(\lambda, \mu)$.

3. Assign distinct integers from $\{1, 2, \ldots, k\}$ to the rows and columns of τ based on where in the $\{R, U\}$ sequence the edge at the end of the row or column falls; for example, a tableau with $\{R, U\}$ sequence $RURU$ would have columns labeled 1 and 3 (from left to right) and rows labeled 2 and 4 (from bottom to top). Represent these numbered rows and columns with a graph whose vertices are labeled by the integers $\{1, 2, \ldots, k\}$, with N directed edges (i, j) if there is an N in the cell corresponding to the intersection of column i and row j; denote this graph by γ. Color the vertex i of γ white if $\epsilon_i = -1$ and black otherwise. The graph γ has type δ, where δ_i is the number of edges entering vertex i (where a negative number of edges entering is a number of edges leaving). Because the partial sums of δ are nonpositive, the sum of the entries of δ is 0, there is no more than one edge between a black vertex and a white vertex, and there are no edges (i, j) for $j < i$, γ meets the requirements of Definition **??** and is an intransitive supergraph.

4. Then let $\widetilde{\Phi}^{super}_{\lambda\mu\text{b}}(T) = (\gamma, S)$.

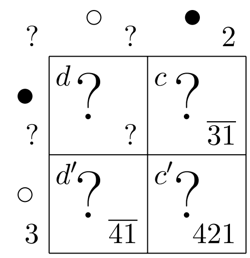

Figure 14: The unfilled growth diagram for the oscillating supertableau $3, \overline{41}, 421, \overline{31}, 2$.

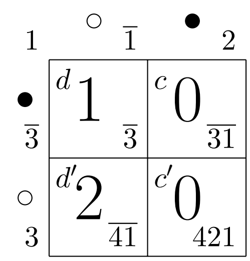

Figure 15: The filled-in growth diagram; the outside contains the oscillating supertableau $3, \overline{3}, 1, \overline{1}, 2$.

Example 4.2. The unfilled colored growth diagram in Figure 14 corresponds to the oscillating supertableau $3, \overline{41}, 421, \overline{31}, 2$ of type $\mathrm{b} = (-2, -\overline{2}, \overline{3}, 2)$; subscripts on entries in squares denote which of the four local moves (classical RSK, dual RSK, RSK', or dual RSK') are to be applied. After filling in the growth diagram with the backwards super RSK local rule, we get the growth diagram in Figure 15. The interior of the growth diagram contains the adjacency information for the graph in Figure 16, a graph in $SG(\delta^\epsilon)$ for $\delta = (-2, -1, 3, 0)$ and $\epsilon = (-1, 1, -1, 1)$, while the left and top edges give us the oscillating supertableau $3, \overline{3}, 1, \overline{1}, 2$, an oscillating supertableau of type $\mathrm{nor}\,((\beta - \delta)^\epsilon) = (\overline{0}, 2, \overline{0}, -1)$. In other words, we have

$$\Phi^{super}_{(3)(2)(-2,-\overline{2},\overline{3},2}\big(3, \overline{41}, 421, \overline{31}, 2\big) = \big(\text{Figure 16}, (3, \overline{3}, 1, \overline{1}, 2)\big).$$

Proposition 4.2. *The image of* $\widetilde{\Phi}^{super}_{\lambda\mu\mathrm{b}}$ *is a subset of*

$$\bigcup_{\delta \prec \beta} SG(\delta) \times OST\big(\lambda, \mu^\omega, \mathrm{nor}\,((\beta - \delta)^\epsilon)\big)$$

Denote by $\Phi^{super}_{\lambda\mu\mathrm{b}}$ *the map*

$$\Phi^{super}_{\lambda\mu\mathrm{b}} : OST(\lambda, \mu^\omega, \mathrm{b}) \to \bigcup_{\delta \prec \beta} SG(\delta^\epsilon) \times OST\big(\lambda, \mu^\omega, \mathrm{nor}\,((\beta - \delta)^\epsilon)\big)$$

given by $\Phi^{super}_{\lambda\mu\mathrm{b}}(T) = \widetilde{\Phi}^{super}_{\lambda\mu\mathrm{b}}(T)$.

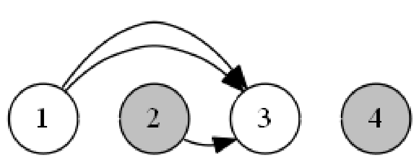

Figure 16: The supergraph in $G\,(\delta^\epsilon)$ corresponding to the oscillating supertableau $3, \overline{41}, 421, \overline{31}, 2$.

Proof. The proof mirrors that of Proposition 4.1, using the super RSK correspondence instead of the oscillating RSK correspondence, with the colors of the rows and columns of the filling $\mathcal{T}$ determined by the colors on the oscillating supertableau T.

□

Theorem 4.2. *The map* $\Phi^{super}_{\lambda\mu \mathrm{b}}$ *is a bijection.*

Proof. Let $S = (\lambda = s_0, s_1, \ldots, s_k = \mu^\omega)$ be an oscillating supertableau of weight $\mathrm{e} = \eta^{\epsilon'}$ where $\eta \in \mathbb{Z}^k$ and $\epsilon \in \{-1, 1\}^k$ such that η is a normal sequence (in particular, its zeros are tagged "row" and "column"), and let γ be an intransitive supergraph of type δ^ϵ. We want to produce a map $\Psi^{super}_{\lambda\mu \mathrm{e}}$ such that $\Psi^{super}_{\lambda\mu \mathrm{e}}(S, \gamma) \in OST(\lambda, \mu^\omega, \mathrm{b})$ where $\mathrm{b} = \beta^\epsilon$ and $e = \mathrm{nor}\left((\beta - \delta)^\epsilon\right)$.

First, we build a colored Young diagram τ from γ. If $\delta_i < 0$, we add a column on the right in the Young diagram, and if $\delta_i > 0$, we add a row on the top in the Young diagram; δ_i of zero add rows or columns based on the tag on the zero. The square corresponding to column vertex a and row vertex b is in τ if $a < b$. We fill τ with a filling $\mathcal{T}$ by placing in the square corresponding to vertices a and b the number of edges from a to b, and we color the rows and columns of τ white or black based on the colors ϵ appearing in ϵ

We have the upper-left edge S and the filling $\mathcal{T}$ of a growth diagram. Applying the super RSK local rule using the coloring of the tableau, we can read off the lower-right edge a tableau $T = (\lambda = t_0, t_1, \ldots, t_k = \mu^\omega)$ of weight $\mathrm{b} = \beta^\epsilon$ for $\beta \in \mathbb{Z}^k$ and $\epsilon \in \{-1, 1\}^k$. As before, if t_i and t_{i+1} are horizontally adjacent corresponding to horizontally-adjacent entries s_j and s_{j+1} in S, $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|)$ is the sum of the entries in the column $\mathcal{T}$ between t_i and t_{i+1}. By the construction of $\mathcal{T}$, we have that $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|) = -\delta_{i+1}$, and similarly if t_{i+1} and t_i are vertically adjacent, $(|t_{i+1}| - |t_i|) - (|s_{j+1}| - |s_j|) = -\delta_i$. So, noting that our sign convention has positive outdegree represented as negative, we have $\eta_j = \beta_i - \delta_i$.

Each $\beta_i + \delta_i$ equals a distinct η_j and increasing i corresponds to farther right or up in τ, so that if $i_2 > i_1$, the corresponding indices for η satisfy $j_2 > j_1$, and thus $\eta = \mathrm{nor}(\beta - \delta)$. Checking the colorings ϵ and ϵ', $\epsilon'_j = \epsilon_i$ as well, so that $\mathrm{e} = \mathrm{nor}\left((\beta - \delta)^\epsilon\right)$. So the map $\psi^{super}_{\lambda\mu \mathrm{e}}$ is a well-defined map from $SG(\delta^\epsilon) \times OST\big(\lambda, \mu^\omega, \mathrm{nor}\left((\beta - \delta)^\epsilon\right)\big)$ to $OST(\lambda, \mu^\omega, \mathrm{b})$.

The correspondence between pairs of an oscillating supertableau of normal weight and an intransitive supergraph and fillings of colored Young diagrams is a bijection, as is the local rule for RSK. Since we've shown that $\Psi^{super}_{\lambda\mu \mathrm{e}}$ is well-defined, this tells us that the map $\Psi^{super}_{\lambda\mu \mathrm{e}}$ defined above inverts the portion of $\Phi^{super}_{\lambda\mu \mathrm{b}}$ whose image oscillating tableaux are of weight η. Each possible image tableau weight η has a corresponding $\Psi^{super}_{\lambda\mu \mathrm{e}}$, so the map $\Phi^{super}_{\lambda\mu \mathrm{b}}$ is a bijection as desired. □

We will refer to the procedure taking as input a supergraph γ^ϵ and a normal oscillating supertableau $S^{\epsilon'}$, or, equivalently, a colored $\mathbb{N}$-tableau and a normal oscillating supertableau, as the *super RSK correspondence.*

5 A Cauchy Identity for Super RSK

The Cauchy identity 2.1 allows us to define a probability measure on $\mathcal{Y}$ by setting

$$\mathbf{SM}(\lambda) = \frac{s_\lambda(X) s_\lambda(Y)}{\prod_{i,j} (1 - x_i y_j)^{-1}},$$

and specializing the symmetric functions suitably. The probability measure $\mathbf{SM}$ is known as the *Schur Measure.*

Okounkov and Reshetikhin generalized the Schur measure to a probability measure on the space of oscillating tableau tableau known as the *Schur process* [8]. The fact that it is a probability measure is connected to the natural Cauchy identity for oscillating RSK, as the transition weight in the Schur process from λ_{i-1} to λ_i is proportional to a specialization of either $s_{\lambda_{i-1}/\lambda_i}(X^i)$, where X^i is a vector of variables, or $s_{\lambda_i/\lambda_{i-1}}(X^i)$.

Let $\lambda = (\lambda^0 = \emptyset, \lambda^1, \lambda^2, \cdots, \lambda^n = \emptyset)$ be an oscillating tableau of length n. Define the set I to be the set of indices i such that $\lambda^i \geq \lambda^{i-1}$ and J to be the set of indices j such that $\lambda^j \leq \lambda^{j-1}$. Letting X^i and Y^j denote vectors of variables for $i \in I$ and $j \in J$, define

$$F(X,Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \qquad \mathcal{W}(\lambda) = \prod_{i \in I} s_{\lambda^i/\lambda^{i-1}}\left(X^i\right) \prod_{j \in J} s_{\lambda^{j-1}/\lambda^j}(Y^j).$$

Proposition 5.1.

$$\sum_{\lambda} \mathcal{W}(\lambda) = \prod_{i \in I, j \in J} F(X^i, Y^j),$$

where the sum is over all oscillating tableau λ of length n.

We can define a *dual Schur measure* by setting

$$\mathbf{dSM}(\lambda) = \frac{s_\lambda(x) s_{\lambda'}(y)}{\prod_{i,j}(1 + x_i y_j)}$$

and specializing the symmetric functions suitably. The proof that this gives a probability measure relies on the Cauchy identity for dual RSK (Proposition 3.2).

Let ϵ be a vector in $\{-1,1\}^n$ and let $\lambda^\epsilon = (\lambda^0 = \emptyset, \lambda^1, \lambda^2, \cdots, \lambda^n = \emptyset)^\epsilon$ be an oscillating supertableau of length n. Define the set I to be the set of indices i such that $\lambda^i \geq \lambda^{i-1}$ and J to be the set of indices j such that $\lambda^j \leq \lambda^{j-1}$. From ϵ, we know that for i in I, λ^i differs from λ^{i-1} by a seqeunce of horizontal strips if $\epsilon_i = 1$ and a sequence of vertical strips if $\epsilon_i = -1$ and, similarly, for j in J, λ^{j-1} differs from λ^j by a sequence of horizontal strips if $\epsilon_j = 1$ and a sequence of vertical strips if $\epsilon_j = -1$. Let $\{X^i | i \in I\}$ and $\{Y^j | j \in J\}$ be two sets of sets of indeterminates.

Letting $(\lambda)^{*i} = \lambda$ if $i = 1$ and $(\lambda)^{*i} = -1$ if $i = -1$, define

$$G(X,Y) = \prod_{i \in I, j \in J} (1 + x_i y_j), \qquad \mathcal{SW}(\lambda^\epsilon) = \prod_{i \in I} s_{(\lambda^i)^{*\epsilon_i}/((\lambda^{i-1})^{*\epsilon_i}}\left(X^i\right) \prod_{j \in J} s_{(\lambda^{j-1})^{*\epsilon_j}/((\lambda^j)^{*\epsilon_j}}(Y^j).$$

Define

$$H(X,Y,\epsilon') = \begin{cases} F(X,Y) & \text{if } \epsilon' = 1 \\ G(X,Y) & \text{if } \epsilon' = -1 \end{cases}$$

We can now state the super Cauchy identity.

Proposition 5.2. *Fix $\epsilon \in \{-1,1\}^n$. We have*

$$\sum_{\lambda} \mathcal{SW}(\lambda^\epsilon) = \prod_{i \in I, j \in J} H\left(X^i, Y^j, \epsilon^i \epsilon^j\right),$$

where the sum is over all oscillating supertableaux $(\lambda)^\epsilon$ of length n starting and ending at $\emptyset$.

Proof. Each term $H(X^i, Y^i, \epsilon^i \epsilon^j)$ on the right hand side is the generating function in the variables X^i, Y^j. If $\epsilon^i \epsilon^j = 1$, it is the generating function of $\mathbb{N}$-matrices of finite support, while if $\epsilon^i \epsilon^j = -1$, it is the generating function for $0,1$-matrices of finite support. Considering an individual term of the product on the

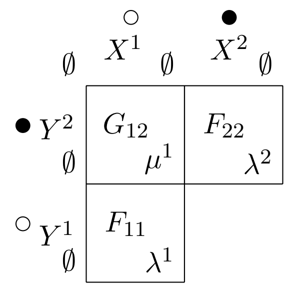

Figure 17: The correspondence in the superCauchy identity; F_{ij} (respectively, G_{ij}) denotes that $F(X^i, Y^j)$ (respectively, $G(X^i, Y^j)$) is the generating function for fillings of that block.

right-hand side, we have $|I||J|$ matrices of finite support. If we extend all of these matrices to be square matrices of the same size, we can glue the matrices together such that all the matrices coming from terms with X^i for fixed i are vertically adjacent, ordered in decreasing order of the index j on Y^j and all the matrices coming from terms with Y^j for fixed j are horizontally adjacent, ordered in increasing order of the index i on X^i. The result of this gluing is a Young tableau with $|J|$ rows and $|I|$ columns. We color column i with ϵ^i and row j with ϵ^j, so we have a colored Young tableau $\mathcal{T}$.

By Theorem 4.2, we know that the super RSK correspondence gives a bijection between pairs (γ, S) of an intransitive supergraph of type β^η and an oscillating tableau in $OST\big(\emptyset, \emptyset, (0, 0, \ldots, 0)^\eta\big)$ and oscillating supertableaux in $OST\left(\emptyset, \emptyset, \beta^\eta\right)$. Since there is only one oscillating supertableau in $OST\big(\emptyset, \emptyset, (0, 0, \ldots, 0)\big)$, and since the adjacency and vertex coloring information of γ can be encoded as a colored Young tableau, we can consider the super RSK correspondence as giving a bijection between colored Young tableau with column sum vector corresponding to the opposite of the sums of the negative portions of β and row sum vector corresponding to the sums of the positive portions of β (zeros are allocated to either rows or columns as needed) and oscillating supertableau in $OST\left(\emptyset, \emptyset, \beta^\eta\right)$. A picture is shown in Figure 17.

Call the block of matrices corresponding to a set of variables X^i or Y^j a supercolumn or a superrow, respectively, and let m be the number of rows in a superrow. Applying the super RSK correspondence to the colored Young tableau $\mathcal{T}$ (after writing $\emptyset$ along the left and top edges), we build an oscillating supertableau T of type β^η, where η consists of blocks of m copies of ϵ^k for $1 \leq k \leq n$.

If $i \in I$ and $\epsilon_i = 1$, the skew shape λ^i/λ^{i-1} one gets from supercolumn i can be viewed as a skew-semistandard Young tableau by considering the sequence from λ^{i-1} to λ^i in T as being a sequence of insertions of rows of 1s, then 2s, and so on, where the number of ks to insert is the sum of column k in supercolumn i, the column corresponding to X^i. This means that the generating function that encodes the transition from λ^{i-1} to λ^i is $s_{\lambda^i/\lambda^{i-1}}(X^i)$.

If $i \in I$ and $\epsilon_i = -1$, the skew shape λ^i/μ^{i-1} one gets from supercolumn i can be viewed as a skew-conjugate-semistandard Young tableau by considering the sequence from λ^{i-1} to λ^i in T as being a sequence of insertions of columns of 1s, then 2s, and so on, where the number of ks to insert is the sum of column k in supercolumn i, the column corresponding to X^i. This means that the generating function that encodes the transition from λ^{i-1} to λ^i is $s_{(\lambda^i)'/(\lambda^{i-1})'}(X^i)$.

Similarly, if $j \in J$, the skew shape λ^{j-1}/λ^j one gets from a superrow i can be viewed as a skew-semistandard or skew-conjugate-semistandard (if $\epsilon_j = 1$ or $\epsilon_j = -1$, respectively) Young tableaux by considering the backwards sequence from λ^j to λ^{j-1} in T as a sequence of insertions of rows or columns, respectively, of 1s, 2s, and so on, where the number of ks to insert is the sum of row k in superrow i, the superrow corresponding to Y^j. This means that the generating function that encodes the transition from

λ^{j-1} to λ^j is $s_{(\lambda^{j-1})^{*\epsilon_j}/(\lambda^j)^{*\epsilon_j}}(Y^j)$.

Multiplying the generating functions for the transitions in our extended refined oscillating supertableau, we get that the generating function for the entire refined semistandard tableau (λ, μ) is $\mathcal{SW}(\lambda^\epsilon)$. Since there is a bijection between the colored tableau enumerated by the right-hand side of the equation and refined oscillating supertableau whose generating functions are the left-hand side, we have

$$\sum_\lambda \mathcal{SW}(\lambda^\epsilon) = \prod_{i\in I, j\in J} H\left(X^i, Y^j, \epsilon^i\epsilon^j\right)$$

as desired.

□

If ϵ is a vector of all 1s, we obtain a Cauchy identity whose terms resemble transition probabilities in the Schur process; we get Proposition 5.1 as a corollary. Note that if we set $n = 1$, $\epsilon = (1, 1)$, the above proof is identical to the proof of the Cauchy identity (Proposition 2.1), and if we set $n = 1$, $\epsilon = (-1, 1)$, the above proof is identical to that of the dual Cauchy identity.

6 Increasing and Decreasing Operators

In this section, we apply the super RSK correspondence to fill in a proof of a statement due to Pak and Postnikov [9] giving a set of commutation relations among certain operators on group representations. These operators can be viewed as algebraic realizations of the addition and deletion of horizontal and vertical strips.

Definition 6.1. Define the *increasing operators* $I(n)$ and $I(\overline{n})$ and the *decreasing operators* $D(n)$ and $D(\overline{n})$ as follows, where $\mathrm{inv}_n(V)$ denotes the space of S_n-invariants in the vector space V and $\mathrm{skew}_n(V)$ denotes the space of skew invariants of S_n in V (i.e. those $v \in V$ with $\sigma v = -v$ for all $\sigma \in S_n$):

$$\begin{aligned} I(n)\cdot V &= \mathrm{ind}_{S_p}^{S_{p+n}}(V) \\ I(\overline{n})\cdot V &= \mathrm{ind}_{S_p\times S_n}^{S_{p+n}}(V\otimes \mathrm{sgn}_n) \\ D(n)\cdot V &= \mathrm{inv}_n\left(\mathrm{res}_{S_p\times S_n}^{S_{p+n}} V\right) \\ D(\overline{n})\cdot V &= \mathrm{skew}_n\left(\mathrm{res}_{S_p\times S_n}^{S_{p+n}} V\right) \end{aligned}$$

Our main result of this section is the following set of commutation relations for these increasing and decreasing operators:

Theorem 6.1 ([9, Thm. 9.1])**.** *Let $m, n \in \mathbb{N}$. Then if $[A, B]$ denotes the commutator $AB - BA$, we have*

1. $[I(m), I(n)] = [I(\overline{m}), I(\overline{n})] = [D(m), D(n)] = [D(\overline{m}), D(\overline{n})] = 0.$
2. $[I(m), I(\overline{n})] = [D(m), D(\overline{n})] = 0.$
3. $[I(m+1), D(n+1)] = I(m)D(n)$ *and* $[I(\overline{m+1}), D(\overline{n+1})] = I(\overline{m})D(\overline{n}).$
4. $[I(m+1), D(\overline{n+1})] = D(\overline{n})I(m)$ *and* $[I(\overline{m+1}), D(n+1)] = D(n)I(\overline{m}).$

We will give a combinatorial proof of these relations using the super RSK correspondence.

6.1 The Combinatorial Realization

Let $\text{Rep}(G)$ denote the set of equivalence classes of complex finite-dimensional representations of the finite group G. Denote by R the direct sum $R = \text{Rep}(S_0) \oplus \text{Rep}(S_1) \oplus \text{Rep}(S_2) \oplus \ldots$. Let $\mathcal{M}$ be the category whose objects are finite groups and whose morphisms from G to H are equivalence classes of finite-dimensional representations of $G \times H$; we can view $\text{Rep}(G)$ as the set of morphisms from the trivial group to G in this category. For a representation W of $G \times H$, define $\langle W \rangle$ as the operator which sends $V \in \text{Rep}(G)$ to $V \circ W$, which is in $\text{Rep}(H)$.

For $\beta \in \mathbb{Z}^k$, $\epsilon \in \{-1,1\}^{p+k+q}$, the group $S_p \times S_q$ acts on the set of intransitive supergraphs of type δ, where

$$\delta_i = \begin{cases} -1^{\epsilon_i} & \text{if } 1 \le i \le p \\ \beta_{i-p}^{\epsilon_i} & \text{if } p < i \le p+k \\ 1^{\epsilon_i} & \text{if } p+k < i \le p+k+q \end{cases} \tag{**}$$

by having S_p permute the first p vertices and S_q permute the last q. This action gives a complex finite-dimensional representation of $S_p \times S_q$; letting $\mathrm{b} = \left(\beta_1^{\epsilon_{p+1}}, \beta_2^{\epsilon_{p+2}}, \ldots, \beta_k^{\epsilon_{p+k}}\right)$, we denote this representation by $M(p, \mathrm{b}, q)$. More properties of this representation are discussed in [9, Sec. 8]; Sections 5 and 6 of that paper discuss the restriction of this representation to the case that ϵ is a vector of all 1s.

In [9, Sec. 9], Pak and Postnikov define this pair of increasing operators and a pair of decreasing operators on R by considering the operators $\sum_{-p+\beta+q=0}\langle M(p,\mathrm{b},q)\rangle$ where b is a vector of length 1. Because there is a natural composition operation on compatible intransitive graphs, the operator $\sum_{-p+\beta+q=0}\langle M(p,\mathrm{b},q)\rangle$ for general b can be expressed as the composition, for $\langle \mathrm{b} \rangle$ defined to be the sum $\sum_{-p+\beta+q=0}\langle M(p,\mathrm{b},q)\rangle$, $\langle \mathrm{b}_1 \rangle \cdot \langle \mathrm{b}_2 \rangle \cdots \langle \mathrm{b}_k \rangle$, so it is enough to just consider the following four operators:

$$I(n) = \langle -n^1 \rangle, \quad D(n) = \langle n^1 \rangle, \quad I(\overline{n}) = \langle -n^{-1} \rangle, \quad D(\overline{n}) = \langle n^{-1} \rangle.$$

Because R consists of all equivalence classes of representations of all finite symmetric groups, it has a basis consisting of the set of all irreducible representations of all symmetric groups. Those irreducible representations are the representations π_λ for all the partitions $\lambda \in \mathcal{Y}$. We identify the representation π_λ with the partition λ, so that our increasing and decreasing operators can be seen as acting on finite linear combinations with complex coefficients of partitions λ.

Consider the operator $\langle M(p, -n^1, q)\rangle$. As a representation, $M(p, -n^1, q)$ is spanned by all the intransitive graphs of the form $(**)$ for $\beta_1 = -1$ and $\epsilon_{p+1} = 1$. By Theorem 4.2, there is a bijection between these intransitive graphs and oscillating supertableaux given by setting the upper-left boundary of a $q \times (p+1)$ colored Young shape equal to $\emptyset$ and applying the super RSK correspondence. The last column of the filled colored Young tableau, corresponding to the vertex with outdegree n, has column sum n, and, since $\epsilon_{p+1} = 1$, it is a black column. This means that the partition μ at the lower-left corner of that column and the partition λ at the lower-right corner of that column differ by a horizontal strip of size n.

This means that if $|\mu| = p$ and $|\lambda| = p + n$, $M(p, -n^1, q)$ can take μ to λ if and only if μ and λ differ by a horizontal strip of size n. Viewed in terms of the operation of $I(n)$ on partitions, this says that $I(n) \cdot \mu = \sum_\lambda \lambda$, where the sum is over all λ differing from μ by a horizontal strip of size n. So $I(n)$ corresponds to adding a horizontal strip of size n. By a similar argument, $I(\overline{n})$ corresponds to adding a vertical strip of size n, $D(n)$ corresponds to removing a horizontal strip of size n, and $D(\overline{n})$ corresponds to removing a vertical strip of size n.

6.2 Proof of Theorem 6.1

Proof of Theorem 6.1. Part 1 of the theorem is proved in [9, Sec. 10] using a bijection ψ_1 between semistandard Young tableaux of shape λ/ν and weight (m,n) and semistandard Young tableaux of shape λ/ν

and weight (n, m) given by a process of "toggling" the shape of the intermediate partition μ between ν and λ. The toggling operation is equivalent to the toggling done in the local rule for RSK. This bijection proves that $I(m)$ and $I(n)$ commute, its reverse proves that $D(m)$ and $D(n)$ commute, and transposing all the tableaux proves that $I(\overline{m})$ and $I(\overline{n})$ commute and that $D(\overline{m})$ and $D(\overline{n})$ commute.

For proving part 2 of the theorem, view the operator $I(m)$ as inserting a row of m 1s to get a skew tableau T of shape μ/ν and the operator $I(\overline{n})$ as inserting a row of n 2s to get a skew tableau T' of shape λ/ν. To get a skew tableau of shape λ/ν where the column of n 1s was inserted and then a row of m 2s was inserted, simply subtract all the entries of T' from 3 and reverse the order of every row. This shows that $I(m)$ and $I(\overline{n})$ commute, and reversing this bijection shows that $D(m)$ and $D(\overline{n})$ commute.

For part 3, the operator $I(m+1)D(n+1)$ corresponds on a super RSK growth diagram to a corner square at the intersection of a black row with sum $n+1$ and a black column with sum $m+1$; travelling along the lower-right edge of this square adds a horizontal strip of size $m+1$ and then deletes a horizontal strip of size $n+1$. This is an oscillating tableau of weight $\big(-(m+1)^1, (n+1)^1\big)$. By the super RSK correspondence (or just the normal oscillating RSK correspondence), such a lower-right edge corresponds to a pair of an oscillating tableau of weight $\big((n+1-k)^1, (-m-1+k)^1\big)$ and an intransitive graph of type $(-k^1, k)$. We can view this as the corner square containing the entry k and having its upper-left boundary of weight $\big((n+1-k)^1, (-m-1+k)^1\big)$. Given a fixed partition μ at the lower-left corner of this corner square containing k, the possible values of λ are those formed from applying $D(j)I(i)$ to μ for $0 \leq i, j \leq \min(m+1, n+1)$. Writing this algebraically, we have

$$I(m+1)D(n+1) = \sum_{k=-1}^{\min(m,n)} D(n-k)I(m-k).$$

From here, we'll proceed by induction. For $m = 0$ or $n = 0$, $I(m)$ and $D(m)$ or $I(n)$ and $D(n)$ are equal to the identity operator, while for m or n less than 0, $I(m)$ and $D(m)$ or $I(n)$ and $D(n)$ are equal to the 0 operator, so we have a base case. Suppose that $[I(i), D(j)] = I(i-1)D(j-1)$ for all $i \leq m$ and $j \leq n$. We have

$$\begin{aligned}
I(m+1)D(n+1) &= \sum_{k=-1}^{\min(m,n)} D(n-k)I(m-k) \\
I(m+1)D(n+1) - D(n+1)I(m+1) &= \sum_{k=0}^{\min(m,n)} D(n-k)I(m-k) \\
[I(m+1)D(n+1)] &= D(n)I(m) + \sum_{k=1}^{\min(m,n)} D(n-k)I(m-k) \\
&= I(m)D(n) - I(m-1)D(n-1) + \sum_{k=1}^{\min(m,n)} D(n-k)I(m-k)
\end{aligned}$$

by our inductive assumption. Now by our oscillating RSK argument above, we know

$$I(m-1)D(n-1) = \sum_{k=-1}^{\min(m-2,n-2)} D(n-k)I(m-k),$$

which, upon reindexing the sum and substituting, tells us that

$$[I(m+1)D(n+1)] = I(m)D(n)$$

as desired. Using the super RSK local rule in the case of a white row and a white column will similarly prove $[I(\overline{m+1}), D(\overline{n+1})] = I(\overline{m})D(\overline{n})$.

For part 4, the operator $I(\overline{m+1})D(n+1)$ corresponds on a super RSK growth diagram to a corner square at the intersection of a black row with sum $n+1$ and a white column with sum $m+1$; travelling along the lower-right edge of this square adds a vertical strip of size $m+1$ and then deletes a horizontal strip of size $n+1$. This is an oscillating supertableau of weight $\left(-(m+1)^{-1}, (n+1)^{1}\right)$. By the super RSK correspondence, such a lower-right edge corresponds to either a 0 or a 1 in the square and an oscillating supertableau of weight either $\left((n+1)^{1}, -(m+1)^{-1}\right)$ or $\left(n^{1}, -m^{-1}\right)$. Writing this algebraically, we have $I(\overline{m+1})D(n+1) = D(n+1)I(\overline{m+1}) + D(n)I(\overline{m})$, or

$$[I(\overline{m+1}, D(n+1)] = D(n)I(\overline{m})$$

as desired. The other case in part 4 is similar. □

7 Further Questions

Question 1. Fomin's growth diagram construction can be generalized to a larger class of posets than just Young's lattice. We define an *r-differential poset* to be a poset $\mathcal{P}$ such that

1. $\mathcal{P}$ has an element $\widehat{0}$ such that $\lambda \geq \widehat{0}$ for all $\lambda \in \mathcal{P}$.
2. If λ and μ are two distinct elements in $\mathcal{P}$, then if there are exactly k elements of $\mathcal{P}$ covered by both λ and μ, then there are exactly k elements of $\mathcal{P}$ that cover both λ and μ.
3. If $\lambda \in \mathcal{P}$ is covers exactly k elements of $\mathcal{P}$, it is covered by exactly $k+r$ elements of $\mathcal{P}$.

For example, Young's lattice is a 1-differential poset. Differential posets were introduced by Richard Stanley, who used increasing and decreasing operators on them to translate certain enumerative problems into partial differential equations [14]. Fomin's construction of dual graded graphs [3] is essentially equivalent.

The other interesting example of a 1-differential poset is the *Young-Fibonacci lattice* $Z(1)$, whose elements are words in the alphabet $\{1, 2\}$ with rank function given by the sum of the letters in the word and the cover relation $a \leq b$ if either $a = 2^i v$ and $b = 2^i 1v$ or $a = 2^i 1v$ and $b = 2^{i+1}v$. A Robinson-Schensted algorithm, described in terms of insertion and in terms of growths, can be given for the Fibonacci lattice [11, Chapter 5]. The automorphism group of the Fibonacci lattice $Z(1)$ is S_2 [2], with the nontrivial automorphism being given by $\omega(w11) = w2$, $\omega(w2) = w11$, and $\omega(w) = w$ otherwise.

Since the horizontal and vertical strips inserted in the RSK correspondence are related by transpostion, the nontrivial automorphism in the automorphism group of Young's lattice, it would be interesting to see how the super RSK correspondence generalizes to $Z(1)$. In particular, the proper notion of a vertical and horizontal strip might be found by studying the growth diagrams for the super RSK correspondence on $Z(1)$; the main difficulty appears to be finding a good way to represent the elements of $Z(1)$ the way that Young diagrams represent $\mathcal{Y}$.

Question 2. Okounkov and Reshetikhin [8] calculate the correlation function for the Schur process, giving a determinantal formula for the correlation function of the Schur process. It would be interesting to try to find such a formula for the super Schur process. In particular, if one could express the correlation function of the Schur process in terms of Schur functions in some indeterminates, one could get the correlation function of the super Schur process by applying the involution ω on Λ which sends e_n to h_n. This would be analogous to the algebraic proof of the dual Cauchy identity from the Cauchy identity; the skew Schur functions in the variables on which the involution is applied see their corresponding partitions transposed, so that $\omega_{X^i} s_{\lambda/\mu}\left(X^i\right) = s_{\lambda'/\mu'}\left(X^i\right)$. Other results about the Schur process should also be able to be lifted to the super Schur process by means of appropriate applications of the involution ω.

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