On constant terms of Eisenstein series
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Abstract
In this paper, we analyze the general properties of the Jacquet-Shalika method. This method has applications to topics such as a different proof of the Prime Number Theorem. Specifically, we look at the constant terms of Eisenstein series twisted by Maass forms over arbitrary proper parabolic subgroups. We then discuss a general framework for applying the Jacquet-Shalika method to the non-vanishing of certain functions, and we specialize to certain examples.

1 Introduction
The Prime Number Theorem (PNT) is a theorem in analytic number theory that states the number of primes less than \( n \) is asymptotically equal to \( \frac{n}{\log n} \) as \( n \to \infty \) [7]. In the Appendix we will prove the PNT can be deduced from the following fact.

Proposition 1. The Riemann zeta function \( \zeta(s) \) does not vanish on the line \( \Re(s) = 1 \).

There are several known proofs of Proposition 1 including one involving Merten’s lemma [7, Lemma 16.4]. There is also a very interesting proof using the tools of modular forms and Eisenstein series. This proof was first outlined by Jacquet and Shalika in [4] and relies on the computation of the constant term of an Eisenstein series over \( SL_2 \). In this paper, we will explain how to generalize this method to \( SL_n \) for \( n \geq 2 \).

1.1 Organization of the Paper
Section 2 will discuss preliminaries, including notation and definitions to be used throughout the paper. Section 3 will carry out the actual computation of the constant terms in a general fashion. Finally, Section 4 will give some special examples and applications of our formula.
1.2 Acknowledgments

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2 Preliminaries

2.1 L-functions

The zeta function is an example of an L-function.

**Definition 1.** An L-function is a series \( L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) that admits:

- An Euler product, i.e. a factorization \( \prod_p f_p(p^{-s}) \) for some functions \( f_p \)
- A meromorphic continuation to the entire complex plane
- A functional equation \( \Lambda(s) = \varepsilon \Lambda(1-s) \) for some \( \varepsilon \) and \( \Lambda(s) = F(s)L(s) \) for some function \( F \).

The zeta function is just the L-function when we take \( a_n = 1 \) for all \( n \).

2.2 Eisenstein series

**Definition 2.** Let \( \mathfrak{H} \) be the upper half plane, i.e. those complex numbers \( x + yi \) with \( y > 0 \).

**Definition 3.** The standard Eisenstein series over \( \mathfrak{H} \) is

\[
E_s(z) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|mz + n|^{2s}}.
\]  

(1)

The following expansion is well-known, see e.g. [3].

**Proposition 2.** \( E_s(z) \) has a Fourier expansion

\[
E_s(z) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s \sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n \geq 1} \sigma_{s - \frac{1}{2}}(n) K_{s - \frac{1}{2}}(2\pi n|y|) e^{2\pi i x}.
\]  

(2)

where \( \phi(s) = \sqrt{\frac{2\pi}{\Gamma(s-\frac{1}{2})\zeta(2s-1)}} \).
2.3 Proof of Proposition 1

We will now prove Proposition 1. First, a general criterion that is stated in [2].

**Proposition 3.** Let \( \langle f, g \rangle = \int_{G_k \setminus G_A} f(x)g(x) \, dx \) be the Petersson inner product. With this inner product, cuspidal Maass forms are orthogonal to Eisenstein series.

**Proof of Proposition 1.** Suppose \( \zeta(1 + iy) = 0 \). Since \( \zeta(z) = \overline{\zeta(\overline{z})} \) we have \( \zeta(1 - iy) = 0 \). Then \( E_s(z)\zeta(2s) \) has constant Fourier term equal to zero for \( s = \frac{1+iy}{2} \), which means \( E_s \) is a cuspidal form. Since cuspidal forms are orthogonal to Eisenstein series, we must have \( \langle E_s(z), E_s(z) \rangle = 0 \). This forces \( E_s(z) = 0 \), which is a contradiction since the non-constant terms do not vanish for sufficiently large \( y \). \( \square \)

This proof can generalize and forms the basis of the Jacquet-Shalika method [3]. In the rest of the paper we will outline how to apply the Jacquet-Shalika method in general.

2.4 Adeles

We will be using the adelic framework of modular forms.

**Definition 4.** The adele group \( \mathbb{A} \) over a field \( k = \mathbb{Q} \) is the set of all sequences \((x_\infty; x_2, x_3, x_5, x_7, \ldots)\) where \( x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p \) for all primes \( p \), and \( x_p \in \mathbb{Z}_p \) for all but finitely many primes \( p \).

The rationals are diagonally embedded into \( \mathbb{A} \) by \( x = (x; x, x, x, \ldots) \).

2.5 Definition of Maass form

Let \( K \) be the maximal compact subgroup of \( G_A \), and let \( k = \mathbb{Q} \).

**Definition 5.** An **adelic Maass form** is a function \( f : G_k \backslash G_A / K \rightarrow \mathbb{C} \) such that \( f \) is an eigenfunction of the invariant differential operators.

We will be considering adelic Maass forms that are also eigenfunctions of the spherical Hecke algebra \( \mathcal{H}_p^\circ \).

**Definition 6.** For a prime \( p \), the **spherical Hecke algebra** \( \mathcal{H}_p^\circ \) is the space of locally constant \( \mathbb{C} \)-valued functions on \( GL(2, \mathbb{Q}) \) that are \( K_p \) bi-invariant,
2.6 Weyl group

Definition 7. The Weyl group $W$ of $G = \text{SL}_n$ is the set of equivalence classes of matrices in $\text{SL}_n$ with entries $+1, 0, -1$ such that there is exactly one nonzero entry in each row and each column. Two matrices are equivalent if by replacing each element by its absolute value we get the same matrix. It is clear that $W$ is in bijective correspondence with permutations of $[n] = \{1, 2, \ldots, n\}$ and in fact we will associate the permutation $\sigma$ with the equivalence class in $W$ with the property that $a_{ij}$ is nonzero exactly when $j = \sigma(i)$. The following lemma is easy to verify.

Lemma 1. If $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are related by $B = w^{-1}Aw$, then $b_{\sigma(i)\sigma(j)} = \pm a_{ij}$, with $+$ sign if $i = j$.

2.7 Parabolic subgroups

Definition 8. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an ordered partition of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. Then for $1 \leq k \leq n$, we define $\text{gr}_\lambda(k)$ as the smallest integer $s \geq 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s \geq k$. In that case, we say $k$ is in group $s$ of $\lambda$.

Definition 9. Call a subgroup $H$ of $G$ free at position $(i, j)$ if some element of $H$ is nonzero at $(i, j)$. Otherwise, we say $H$ is zero at $(i, j)$.

Definition 10. The (proper) parabolic subgroup $P_\lambda$ of rank $r$ associated to an ordered partition $\lambda$ of $n$ is defined as the group of invertible matrices that is free at $(i, j)$ if and only if $\text{gr}_\lambda(i) \leq \text{gr}_\lambda(j)$. Also define $\text{gr}_{P_\lambda}(k) = \text{gr}_\lambda(k)$, and say $k$ is in the $s$-th $P$-group where $s = \text{gr}_{P_\lambda}(k)$. When the context is clear, we omit the subscript $\lambda$. A proper parabolic subgroup is a parabolic subgroup not equal to $G$. 

i.e. $\Phi(kgk') = \Phi(g)$ for $k, k' \in K_p$, $g \in G(\mathbb{Q}_p)$. The spherical Hecke algebra is the restricted direct product

\[ H^\circ = \bigotimes_{p < \infty} H^\circ_p. \] 

which is the unit for all but finitely many primes $p$. The action of $H^\circ$ on a function $f$

is

\[ T \ast f(g) = \int_{\text{GL}_A} T(h)f(gh)dh. \]
A special parabolic subgroup is the Borel subgroup $B$ associated to $(1, 1, \ldots, 1)$, consisting of all upper triangular invertible matrices. Also the whole group $G$ is just a parabolic subgroup associated to $(n)$.

**Definition 11.** The unipotent radical $N(P)$ of a parabolic subgroup $P$ is those matrices in $P$ whose $(i, j)$ entry is $\delta_{i,j}$ if $\text{gr}_P(i) = \text{gr}_P(j)$.

**Definition 12.** The block form $M(P)$ of a parabolic subgroup $P$ is defined as the group of invertible matrices that is free if and only if $\text{gr}_P(i) = \text{gr}_P(j)$. Let $M_k$ be the $\lambda_k \times \lambda_k$ submatrix formed from the rows and columns with indices in the $k$-th $P$-group.

**Definition 13.** We define the compact subgroup $K(P)$ to be the embedding of $K(\lambda_1) \times K(\lambda_2) \times \cdots \times K(\lambda_r)$ in $M(P)$.

We have the Levi decomposition $P = N(P)M(P)$ for any parabolic subgroup $P$, and in general we have the Langlands decomposition $G = N(P)M(P)K(P)$. Note also that $N(P)$ normalizes $M(P)$, i.e. $mN(P)m^{-1} = N(P)$ for any $m \in M(P)$.

**Definition 14.** A cuspidal Maass form is a Maass form $f$ such that

$$I_P = \int_{N_k \backslash N} f(ng)dn = 0 \quad (5)$$

for all unipotent radicals $N(P)$ of proper parabolic subgroups of $G$.

In fact, it only suffices to verify this for a special type of parabolic subgroup.

**Definition 15.** A maximal parabolic subgroup is a parabolic subgroup of rank 2.

**Proposition 4.** If $I_P = 0$ for all maximal parabolic subgroups $P$ then $f$ is cuspidal.

**Proof.** For any parabolic $P$, there is a maximal parabolic $Q$ that contains $P$. This means $N = N(P) \supset N(Q)$. Let $N'$ be the subgroup of $N$ that is unity along the diagonal and zero at the free (non-diagonal) indices of $N(Q)$. (In a sense, $N'$ is the complement of $N(Q)$ with respect to $N$). Then

$$\int_{N_k \backslash N_k} \phi(ng)dn = \int_{N_k' \backslash N_k} \int_{N_k(Q) \backslash N_k(Q)} \phi(n_1n_2g)dn_1dn_2 = 0 \quad (6)$$

since the inner integral vanishes. $\square$
Now we are ready to define Maass forms on parabolic subgroups.

**Definition 16.** A Maass form on $P_\lambda$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a function $f : P_k \setminus P_\lambda$ such that $f(p) = \prod_{i=1}^r \phi_i(M_i)$, where each $\phi_i$ is a Maass form on $SL(\lambda_i)$.

### 2.8 Parabolic maps

The terminology in this section is non-standard but will simplify the exposition.

**Definition 17.** A parabolic subgroup map, or **group map**, is a permutation of $[n]$. Let $P, Q$ be arbitrary parabolic subgroups. A **proper group map** from $P$ to $Q$ is a group map $w$ such that $i > j$ and $\text{gr}_P(i) = \text{gr}_P(j)$ implies $w(i) > w(j)$, and also $w(i) > w(j)$ and $\text{gr}_Q(w(i)) = \text{gr}_Q(w(j))$ implies $i > j$.

**Definition 18.** A $P$-preserving group map is a group map from $P$ to $P$ such that $\text{gr}_P(i) = \text{gr}_P(j)$ implies $\text{gr}_P(w(i)) = \text{gr}_P(w(j))$.

### 2.9 Characters

Let $Z$ be scalar multiples of the identity in $G$. For a sequence of complex numbers $s = (s_1, s_2, \ldots, s_r)$ satisfying $p_1s_1 + p_2s_2 + \cdots + p_rs_r = 0$, define the character $\chi_s : N_kM_kZ\setminus P_\lambda$ by $\chi_s(m) = \prod_{i=1}^r |\det(M_i)|^{s_i}$ for $m \in M_k$.

### 2.10 Twisted Eisenstein series

**Definition 19.** An Eisenstein series twisted by a Maass form $\phi$ on parabolic subgroup $P$ is given by

$$E_P(g, \phi, s) = \sum_{P_k \setminus G_k} \phi(\gamma g)\chi_s(\gamma g).$$

### 2.11 Modular function

When integrating over a region $R$ controlled by a variable $r$, we would sometimes like to make the substitution $r \to grg^{-1}$, for some $g$. The extra multiplicative factor this substitution creates is called the modular function, $\delta(R, g)$. 

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[6]
2.12 Previous results

Jacquet and Shalika were the first to obtain a non-vanishing result via looking at constant terms of Eisenstein series. Langlands [5] developed this method further and computed the constant term where $P$ and $Q$ are two maximal parabolics. In this paper, we will look at general parabolics $P, Q$ and provide a formula for the constant term. Specifically, we will prove:

**Theorem 1.** Let $W' = W(P, Q)$ be defined as in Section 3.2. Then for each choice of $w$, there exist subgroups $M^+$ of $M = M(Q)$ and $N^-$ of $N = N(Q)$ such that for some functions $M_\phi(w, s)$ that depend only on $\phi, w, s$ and for all $g \in B$, we have

$$
\int_{N_k \backslash N_k} E_P(g, \phi, s) dn = \sum_{w \in W'} M_\phi(w, s) \sum_{\beta \in M^+_k \backslash M_k} \delta(N^-, \beta g) \chi_s(w \beta g^{-1}) \phi(w \beta g^{-1}).
$$

(8)

**Remark 1.** Because the LHS is right $K$-invariant, knowing the values for $g \in B$ allows us to determine the values for $g \in P$.

3 Constant Term Calculations

Let $P, Q$ be parabolic subgroups of $G$, and let $f(g) = \phi(g) \chi_s(g)$, where $\phi$ is a Maass form on $P$. We will evaluate the constant term of $E_P(g, \phi, s)$ over $Q$. Recall $E_P(g, \phi, s) = \sum_{\gamma \in P_k \backslash G_k} f(\gamma g)$. Let $N$ be the unipotent radical of $Q$. Let $W_P = W \cap P$ and $W_Q = W \cap Q$.

$$
\int_{N_k \backslash N_k} \sum_{\gamma \in P_k \backslash G_k} f(\gamma g) dn = \int_{N_k \backslash N_k} \sum_{w \in W_P \backslash W / W_Q} \sum_{\gamma \in (w^{-1} P_k w \cap Q_k) \backslash Q_k} f(w \gamma g) dn
$$

(9)

$$
= \sum_{w \in W_P \backslash W / W_Q} \int_{N_k \backslash N_k} \sum_{\gamma \in (w^{-1} P_k w \cap Q_k) \backslash Q_k} f(w \gamma g) dn
$$

(10)

$$
= \sum_{w \in W_P \backslash W / W_Q} \int_{N_k \backslash N_k} \sum_{\gamma \in (w^{-1} P_k w \cap Q_k) \backslash Q_k} f(w \gamma g) dn.
$$

(11)

An explanation of why is as follows.

Recall the Bruhat decomposition $G_k = B_k W B_k$ where $W$ is the Weyl group. Since $B_k \subset P_k$ and $B_k \subset Q_k$, we have $G_k = P_k W Q_k$, hence $G_k$ can
be decomposed into double cosets $P_k w Q_k$ for $w \in W$. In fact, we have the disjoint union

$$G = \bigsqcup_{w \in W \backslash W \backslash W} P_k w Q_k.$$  

Thus, the summation over $P_k \setminus G$ can be written as a double summation, one over $P_k \setminus G / Q_k$ and the other over $P_k \setminus P_k w Q_k$. The set $P_k \setminus G / Q_k$ is isomorphic to $W P_k \setminus W / Q_k$, while the second set $P_k \setminus P_k w Q_k$ is isomorphic to $w \cdot (w^{-1} P_k w \cap Q_k) \setminus Q_k$.

In the second step, we change the order of summation. We are allowed to do so if we assume the choice of $w$ and the integration $N_k \setminus N_k$ are independent.

In the third step, we replace $n$ by $\gamma^{-1} n \gamma$. Since $\gamma \in Q_k$, the change of variables is uni-modular, i.e. no extra multiplicative factors are introduced.

### 3.1 Weyl group double cosets

Now we provide a new characterization of $W P_k \setminus W / W Q_k$. Let $P,Q$ be associated to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ respectively.

**Proposition 5.** (a) The double cosets of $W P_k \setminus W / W Q_k$ are in bijective correspondence with $r \times s$ non-negative integer matrices whose $i$-th row sums to $\lambda_i$ and $j$-th column sums to $\mu_j$ via the map

$$\rho : W P_k W Q_k \to \{a_{ij} = \text{number of } i\text{-th } P\text{-group elements mapped to } j\text{-th } Q\text{-group elements under } w\}$$  \hspace{1cm} (12)$$

(b) Each double coset of $W P_k \setminus W / W Q_k$ contains an element that is a proper group map from $P$ to $Q$.

**Proof.** (a) Well-defined: The elements in $W P$ are $P$-preserving and the elements in $W Q$ are $Q$-preserving. These operations clearly leave $a_{ij}$ unchanged.

Injective: let $w_1, w_2$ be such that $\rho(w_1) = \rho(w_2)$. For a fixed $i$ and $w$ let $g_{ij}(w)$ be the image of the $i$-th $P$-block under $w$ intersected with the $j$-th $Q$-block. Then since $g_{ij}(w_1)$ has the same cardinality as $g_{ij}(w_2)$, there is a bijection between $g_{ij}(w_1)$ and $g_{ij}(w_2)$. Since this analysis works for all $i,j$, we see there exists an element $w$ of $W Q$ such that $g_{ij}(w_1) = g_{ij}(w_2 w)$. Now perform a similar process to find an element $w'$ of $W P$ such that $w_1 = w' w_2 w$.

Surjective: assign $w(1), w(2), \ldots, w(n)$ in order. The choice for each $w(i)$ is unique because $w$ is proper and $\rho(w)$ is given. Note that this construction also proves (b).  \hfill \Box
3.2 Evaluating Constant Terms

Let $\rho$ be the bijection defined in (12). Since $\phi$ is a cusp form, we are able to cancel many terms in the expansion (9). Let

$$I(w) = \int_{N_k \setminus N_\lambda} \sum_{\beta \in (w^{-1}P_k w \cap Q_k) \setminus Q_k} f(wn\beta g)dn. \quad (13)$$

$$= \sum_{\beta \in (w^{-1}P_k w \cap Q_k) \setminus Q_k} \int_{N_k \setminus N_\lambda} f(wn\beta g)dn \quad (14)$$

Here we change the order of integration and summation, assuming that the integration and summation are independent.

**Proposition 6.** We have $I(w) = 0$ whenever some row in $\rho(w)$ contains two (or more) nonzero elements.

**Proof.** Let row $\iota$ contain two or more nonzero elements. First, by Proposition 5 we may assume $w$ is a proper group map. Then

$$\int_{N_k \setminus N_\lambda} f(wn\beta g)dn = \int_{N_k \setminus N_\lambda} f(wn^{-1}w\beta g)dn. \quad (15)$$

Let $\rho(w) = \{a_{ij}\}$, and let $N'$ be the subgroup of $G$ with 1's on the diagonal and zeroes everywhere else except $M(P)_i$, which is the unipotent subgroup of the parabolic associated to $\lambda = (a_{i1}, a_{i2}, \ldots, a_{is})$. If $(i, j)$ represents a free index of $N'$, then $\gr Q(w(i)) < \gr Q(w(j))$, so $(i, j)$ is a free index of $wNw^{-1}$. Hence, $N'$ is a subgroup of $wNw^{-1}$. Let $N''$ be a subgroup of $wNw^{-1}$ that is unity on the main diagonal and zero at all the non-diagonal free indices of $N'$. We know $\chi_s$ is left-invariant on $N'$. Hence we can factor the integral

$$\int_{N_k \setminus N_\lambda} f(wn^{-1}w\beta g)dn$$

$$= \int_{N'' \setminus N'} \chi_s(n_1 w \beta g) \int_{N_k \setminus N'_\lambda} \phi(n_2 n_1 w \beta g)dn_2dn_1 = 0. \quad (16)$$

Hence, since $\phi_i$ is cuspidal, the inner integral must vanish, so we have $I(w) = 0$. \hfill $\square$

**Corollary 1.** We have $I(w) = 0$ unless the partition associated to $P$ refines the partition associated to $Q$, i.e. $P$ is formed from $Q$ by successively merging two numbers in the partition. (For instance, $(1, 1, 2, 2, 3)$ refines $(3, 3, 3)$.)
3.3 Nonzero terms

Proof. If $I(w) \neq 0$, then every row in $\rho(w)$ contains exactly one nonzero element, which implies every $\mu_j$ is the sum of one or more of the $\lambda_i$. 

Let $W(P, Q)$ be the set of $w \in W$ such that every row in $\rho(w)$ contains exactly one nonzero element.

3.3 Nonzero terms

We will now assume $I(w) \neq 0$. The computation we want is

$$I(w) = \sum_{\gamma \in N_k \setminus N_k} \sum_{\beta \in M_k \setminus M_k} \int_{N_k \setminus N_k} f(wn\gamma\beta g)dn \quad (17)$$

$$= \sum_{\beta \in M_k \setminus M_k} \int_{N_k \setminus N_k} f(wn\beta g)dn \quad (18)$$

$$= \sum_{\beta \in M_k \setminus M_k} \int_{N_k} f(wn\beta g)dn \quad (19)$$

$$= \sum_{\beta \in M_k \setminus M_k} J_w(\beta g), \quad (20)$$

where

$$J_w(g) = \int_{N_k} f(wn\gamma g)dn \quad (21)$$

$$= \int_{N_k} \delta(N^-, g)f(wn\gamma n)dn \quad (22)$$

$$= \int_{N_k} \delta(N^-, g)f((w w^{-1})\gamma n)dn \quad (23)$$

$$= \delta(N^-, g)\chi_s(w g w^{-1}) \int_{N_k} \phi((w g w^{-1})\gamma n)\chi_s(w n)dn. \quad (24)$$

We will justify each step. First, we may assume $w$ is a proper group map.

**Definition 20.** Call $(i, j)$ positive if $i < j$, $w^{-1}(i) < w^{-1}(j)$, and $gr_Q(i) \neq gr_Q(j)$. Call $(i, j)$ negative if $i > j$, $w^{-1}(i) < w^{-1}(j)$, and $gr_Q(i) \neq gr_Q(j)$. Let $N^+$ be the subgroup of $N$ that is free on the positive elements (and zero on the negative elements), and $N^-$ be the subgroup of $N$ that is free on the negative elements (and zero on the positive elements). Clearly, $N = N^+ N^-$. 10
3.3 Nonzero terms

3 CONSTANT TERM CALCULATIONS

Definition 21. Let $M^+$ be the subgroup of $M$ that is free at the positions $(i, j)$ such that $\text{gr}_P(w^{-1}(i)) \leq \text{gr}_P(w^{-1}(j))$.

Lemma 2. We have $w^{-1}P_kw \cap Q_k = N_k^+M_k^+$.

Proof. Let $(i, j)$ be a position that is free in both $w^{-1}P_kw$ and $Q_k$. Then $(w^{-1}(i), w^{-1}(j))$ is free in $P_k$, so $\text{gr}_P(w^{-1}(i)) \leq \text{gr}_P(w^{-1}(j))$. Also $(i, j)$ is free in $Q_k$, so $\text{gr}_Q(i) \leq \text{gr}_Q(j)$. Now if $\text{gr}_Q(i) = \text{gr}_Q(j)$ then $(i, j)$ is free in $M^+$. If $\text{gr}_Q(i) < \text{gr}_Q(j)$ then $i < j$, and the fact that $\text{gr}_P(w^{-1}(i)) \leq \text{gr}_P(w^{-1}(j))$ and $w$ being proper implies $w^{-1}(i) < w^{-1}(j)$. This means $(i, j)$ is free in $N^+$. Hence $(i, j)$ is free in $N_k^+M_k^+$. This proves $w^{-1}P_kw \cap Q_k \subseteq N_k^+M_k^+$.

For the reverse inclusion, $N_k^+M_k^+ \subset N_kM_k = Q_k$ is obvious. If $(i, j)$ is free in $N_k^+$, then $w^{-1}(i) < w^{-1}(j)$, so $(i, j)$ is free in $w^{-1}P_kw$. If $(i, j)$ is free in $M_k^+$, then $\text{gr}_P(w^{-1}(i)) \leq \text{gr}_P(w^{-1}(j))$, which also implies $(i, j)$ is free in $w^{-1}P_kw$. This completes the proof of the lemma.

Equation (18) depends on the following Lemma.

Lemma 3. Every element in $(w^{-1}P_kw \cap Q_k) \setminus Q_k$ is the semidirect product of $N_k^+ \setminus N_k$ and $M_k^+ \setminus M_k$.

Proof. First, since $Q = NM$, every $g \in (w^{-1}P_kw \cap Q_k) \setminus Q_k$ is expressible as $nm$ for some $n \in N_k^+$ and $m \in M_k^+$. To prove uniqueness, suppose $N_k^+n_1M_k^+m_1 = N_k^+n_2M_k^+m_2$ for some $n_1, n_2 \in N_k$ and $m_1, m_2 \in M_k$. Then $n_2^{-1}n_1^{-1}m''m_2^{-1}m''^{-1}$ is free in $M_k^+$, so since $N_k \cap M_k = \{I\}$, both sides are the identity. This immediately proves $N_k^+n_1 = N_k^+n_2$ and $M_k^+m_1 = M_k^+m_2$, as desired.

Note that (19) is a standard unfolding argument. Equation (20) uses the fact that $f(\text{eng})$ is constant on $N_k^+ \setminus N_k$ (since $wnw^{-1} \in N(P)$ for $n \in N_k$) to factor the integral $\int_{N_k^+ \setminus N_k} 1\,dn$ out, which we may assume equals 1. Equation (23) uses the fact that $ng \in Q$, so $ng = g_1g_2n'$ for some $g_1 \in N_k^+, g_2 \in M_k(Q)$, and $n' \in N_k^-$. The change of variables $n \to n'$ introduces an extra factor $\delta(N^-, g)$, and also $\phi(g_1g_2n) = \phi(g_2n)$.

Equation (24) depends on the multiplicativity of $\chi_s$ and the fact that $wgw^{-1} \in P$, which follows from $g$ being upper triangular and in $M(Q)$. To simplify further, we need the following Lemma.

Lemma 4. Let $g \in P$. The function $F(g, \phi, s) = \int_{N_k^-} \phi(\text{gwn})\chi_s(\text{wn})\,dn$ can be expressed as $\phi(g)M_{\phi}(w, s)$ for some function $M$ depending on $\phi$, $w$, and $s$.}

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3.3 Nonzero terms

Proof. The analysis presented here generalizes Section 5 of [5]. Since $\phi$ is left-$N(P)$ invariant, we may assume $g \in M(P)$. Use the Langlands decomposition to write $gwn = n(gwn)m(gwn)k(gwn)$, where $n \in N(P), m \in M(P), k \in K(P)$. Since $g \in M(P)$ and since both $\phi$ and $\chi_s$ are left $N(P)$ and right-$K$ invariant, we get

$$F(g, \phi, s) = \int_{N_k^-}^\phi \phi(gm(wn))\chi_s(m(wn))dn$$

$$= \int_{GL_A(\lambda_1) \times GL_A(\lambda_2) \times \ldots \times GL_A(\lambda_r)} \phi(gm)G(w, s, \phi, m)dm$$

$$= \int_{GL_A(\lambda_1)} \int_{GL_A(\lambda_2)} \ldots \int_{GL_A(\lambda_r)} \prod_{i=1}^r \phi_i(g_i m_i) \cdot G(w, s, \phi, m_1, m_2, \ldots, m_r)dm_1 dm_2 \ldots dm_r,$$

where we used the map $n \rightarrow m(wn)$ to effect a change of variables, which introduces a new bi-$K$-invariant function $G$.

Consider the integral over $GL_A(\lambda_1)$, holding $m_2, m_3, \ldots, m_r$ fixed. We immediately recognize

$$\int_{GL_A(\lambda_1)} \phi_1(g_1 m_1)G(m_1, m_2, \ldots, m_r)dm_1$$

as the action of a Hecke operator $T_1$ on $\phi_1$. Since $\phi_1$ is an eigenfunction of $T_1$, we can replace (29) with $G_1(w, s, \phi, m_2, m_3, \ldots, m_r)\phi_1(g_1)$ for some function $G_1$ that notably does not depend on $g$. We can repeat this process with $GL_A(\lambda_2)$, using $G_1$ instead of $G$. This process obviously being iterable, we finally end up with

$$F(g, \phi, s) = G_r(w, s, \phi)\phi_1(g_1)\phi_2(g_2) \ldots \phi_r(g_r).$$

The conclusion follows immediately. \qed

Remark 2. Take $g = 1$ to obtain

$$M_\phi(w, s) = \frac{1}{\phi(1)} \int_{N_k^-} \phi(wn)\chi_s(wn).$$

Putting all of our contributions together we get

$$I_P = \sum_{w \in W'} M_\phi(s) \sum_{\beta \in M_k^+ \setminus M_k} \delta(\beta g)\chi_s(w/\beta gw^{-1})\phi(w/\beta gw^{-1})$$

(31)
4 Applications

4.1 Non-vanishing results

Theorem 1 allows one to systematically apply the Jacquet-Shalika method for Eisenstein series twisted by Maass forms over a parabolic $P$. Our goal is to show $E_P(\phi, g, s)$ is a cusp form, which contradicts Proposition 3. First, by Proposition 4 it suffices to consider $Q$ being maximal parabolic. Next, by Corollary 1 it suffices to consider $Q$ such that $P$ refines $Q$. Thus, we have signficantly narrowed down our choice of $Q$. Finally, for each such $Q$, we only need to consider $w \in W(P, Q)$, which is much smaller than the entire Weyl group. It is also pretty easy to list the elements in $W(P, Q)$ as they correspond to matrices with fixed row and column sums.

In summary, we get only a few parabolics $Q$ with $I_Q \neq 0$, and even for these $Q$, we have $I_Q$ is the sum of only a few terms. Then we can multiply $E_P$ be a common denominator to make it holomorphic at most values of $s$. By plugging in a special value of $s$ at $E_P^*$ we get a non-vanishing result, since it would be a contradiction if all of the constant terms were zero.

4.2 Examples

The classical example of $SL_2$ is instructive. In this case, both $P$ and $Q$ are associated to $(1, 1)$. Since $|W(P, Q)| = 2$, we predict two terms in the constant term expansion, which is exactly what happens.

Now consider $P$ associated to $(2, 1)$. We only need to test $Q = (2, 1)$ and $Q = (1, 2)$. For each case, we have $|W(P, Q)| = 1$, so each nonzero constant term has exactly one element. This agrees with Proposition 10.11.2 in [3].

In fact, if we evaluate these two constant terms and clear denominators, we obtain a non-vanishing result about certain $L$-functions, such as that in Section 10.12 in [3].

We may also extend the Jacquet-Shalika method to non-maximal parabolic subgroups. As an example, consider $P = (n - 2, 1, 1)$. Then if $I_Q \neq 0$ and $Q$ is maximal parabolic, then we must have $Q = (n - 1, 1)$ or $Q = (n - 2, 2)$. In the first case, we get a sum of two terms (since $|W(P, Q)| = 2$); in the second case, we get a single term. If we clear the denominators of the three terms, we get a completed Eisenstein series that cannot vanish on $(n - 1, 1)$ and $(n - 2, 2)$ simultaneously. This may lead to a new non-vanishing result.
5 Appendix

In this Appendix, we will show how Proposition 1 proves the PNT. The analysis here is taken from [7].

Definition 22. Let $\vartheta(x) = \sum_{p \leq x} \log p$, $\Phi(s) = \sum p^{-s} \log p$, and $H(t) = \vartheta(e^t)e^{-t} - 1$.

Lemma 5. $\Phi(s) - \frac{1}{s-1}$ extends to a meromorphic function on $\Re(s) > \frac{1}{2}$ which is holomorphic on $\Re(s) \geq 1$.

Proof. It is well known that $\zeta(s) - \frac{1}{s-1}$ is entire, so $\zeta(s)$ is meromorphic, with a simple pole at $s = 1$. By our assumption, $\zeta(s)$ has no zeroes on $\Re(s) \geq 1$, so the logarithmic derivative $\frac{\zeta'(s)}{\zeta(s)}$ has only a simple pole at $s = 1$ with residue $-1$. Thus

$$-\frac{\zeta'(s)}{\zeta(s)} = (-\log \zeta(s))' = \sum_p \log p \frac{1}{1 - p^s} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}.$$  (32)

Since the sum is locally uniformly and absolutely convergent for $\Re(s) > \frac{1}{2}$, it is a holomorphic function for such $s$. Thus, we must have $\Phi(s) - \frac{1}{s-1}$ is a meromorphic function on $\Re(s) > \frac{1}{2}$ and holomorphic on $\Re(s) \geq 1$, as desired. \hfill \square

This immediately implies $\Phi(s + 1) - \frac{1}{s}$ and $(\mathcal{L}H)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ are meromorphic functions that are holomorphic on $\Re(s) \geq 0$. Now we need the following theorem due to Newman [6].

Proposition 7. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a bounded piecewise continuous function, and suppose its Laplace transform extends to a holomorphic function $g(s)$ on $\Re(s) \geq 0$. Then $\int_0^\infty f(t)dt$ converges and equals $g(0)$.

We finally need an analytic criterion:

Proposition 8. Let $f : \mathbb{R}_{\geq 1} \to \mathbb{R}$ be a non-decreasing function. If $\int_1^\infty \frac{f(x) - x}{x^2} dx$ converges, then $f(x) \sim x$, i.e. $\lim_{x \to \infty} \frac{f(x)}{x}$ exists and equals 1.

Proof of PNT. We have that $\int_0^\infty H(t)dt = \int_0^\infty \vartheta(e^t)e^{-t} - 1dt$ converges by Proposition 7. By change of variables, we get $\int_1^\infty \frac{\vartheta(x)}{x^2} dx$ converges. This implies by Proposition 8 that $\vartheta(x) \sim x$, which is equivalent to PNT. \hfill \square
References


