On constant terms of Eisenstein series
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Abstract

In this paper, we analyze the general properties of the Jacquet-Shalika method. This method has applications to topics such as a different proof of the Prime Number Theorem. Specifically, we look at the constant terms of Eisenstein series twisted by Maass forms over arbitrary proper parabolic subgroups. We then discuss a general framework for applying the Jacquet-Shalika method to the non-vanishing of certain functions, and we specialize to certain examples.

1 Introduction

The Prime Number Theorem (PNT) is a theorem in analytic number theory that states the number of primes less than n is asymptotically equal to $\frac{n}{\log n}$ as $n \to \infty$ [7]. In the Appendix we will prove the PNT can be deduced from the following fact.

Proposition 1. *The Riemann zeta function $\zeta(s)$ does not vanish on the line $\Re(s) = 1$.*

There are several known proofs of Proposition 1, including one involving Merten's lemma [7, Lemma 16.4]. There is also a very interesting proof using the tools of modular forms and Eisenstein series. This proof was first outlined by Jacquet and Shalika in [4] and relies on the computation of the constant term of an Eisenstein series over SL_2. In this paper, we will explain how to generalize this method to SL_n for $n \geq 2$.

1.1 Organization of the Paper

Section 2 will discuss preliminaries, including notation and definitions to be used throughout the paper. Section 3 will carry out the actual computation of the constant terms in a general fashion. Finally, Section 4 will give some special examples and applications of our formula.

1.2 Acknowledgments

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2 Preliminaries

2.1 L-functions

The zeta function is an example of an L-function.

Definition 1. *An L-**function** is a series* $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *that admits:*

- *An Euler product, i.e. a factorization* $\prod_p f_p(p^{-s})$ *for some functions* f_p
- *A meromorphic continuation to the entire complex plane*
- *A functional equation* $\Lambda(s) = \varepsilon\Lambda(1-s)$ *for some* ε *and* $\Lambda(s) = F(s)L(s)$ *for some function* F*.*

The zeta function is just the L-function when we take $a_n = 1$ for all n.

2.2 Eisenstein series

Definition 2. *Let* $\mathfrak{H}$ *be the upper half plane, i.e. those complex numbers* $x + yi$ *with* $y > 0$*.*

Definition 3. *The **standard Eisenstein series** over* $\mathfrak{H}$ *is*

$$E_s(z) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}. \tag{1}$$

The following expansion is well-known, see e.g. [3].

Proposition 2. $E_s(z)$ *has a Fourier expansion*

$$E_s(z) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n\geq 1} \sigma_{s-\frac{1}{2}}(n) K_{s-\frac{1}{2}}(2\pi n|y|)e^{2\pi i x}. \tag{2}$$

where $\phi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$.

2.3 Proof of Proposition 1

We will now prove Proposition 1. First, a general criterion that is stated in [2].

Proposition 3. *Let* $\langle f, g\rangle = \int_{G_k\backslash G_{\mathbb{A}}} \overline{f(x)} g(x)\, dx$ *be the Petersson inner product. With this inner product, cuspidal Maass forms are orthogonal to Eisenstein series.*

Proof of Proposition 1. Suppose $\zeta(1+iy) = 0$. Since $\zeta(\overline{z}) = \overline{\zeta(z)}$ we have $\zeta(1-iy) = 0$. Then $E_s(z)\zeta(2s)$ has constant Fourier term equal to zero for $s = \frac{1+iy}{2}$, which means E_s is a cuspidal form. Since cuspidal forms are orthogonal to Eisenstein series, we must have $\langle E_s(z), E_s(z)\rangle = 0$. This forces $E_s(z) = 0$, which is a contradiction since the non-constant terms do not vanish for sufficiently large y. $\square$

This proof can generalize and forms the basis of the Jacquet-Shalika method [4]. In the rest of the paper we will outline how to apply the Jacquet-Shalika method in general.

2.4 Adeles

We will be using the adelic framework of modular forms.

Definition 4. *The* ***adele group*** $\mathbb{A}$ *over a field* $k = \mathbb{Q}$ *is the set of all sequences* $(x_\infty; x_2, x_3, x_5, x_7, \dots)$ *where* $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$ *for all primes* p, *and* $x_p \in \mathbb{Z}_p$ *for all but finitely many primes* p.

The rationals are diagonally embedded into $\mathbb{A}$ by $x = (x; x, x, x, \dots)$.

2.5 Definition of Maass form

Let K be the maximal compact subgroup of $G_{\mathbb{A}}$, and let $k = \mathbb{Q}$.

Definition 5. *An* ***adelic Maass form*** *is a function* $f : G_k\backslash G_{\mathbb{A}}/K \to \mathbb{C}$ *such that* f *is an eigenfunction of the invariant differential operators.*

We will be considering adelic Maass forms that are also eigenfunctions of the spherical Hecke algebra $\mathcal{H}^\circ$.

Definition 6. *For a prime* p, *the* ***spherical Hecke algebra*** $\mathcal{H}_p^\circ$ *is the space of locally constant* $\mathbb{C}$*-valued functions on* $GL(2,\mathbb{Q})$ *that are* K_p *bi-invariant,* *i.e.* $\Phi(kgk') = \Phi(g)$ *for* $k, k' \in K_p$, $g \in G(\mathbb{Q}_p)$. *The spherical Hecke algebra is the restricted direct product*

$$\mathcal{H}^\circ = \bigotimes_{p\leq\infty}' \mathcal{H}_p^\circ. \tag{3}$$

which is the unit for all but finitely many primes p. *The action of* H° *on a function* f *is*

$$T * f(g) = \int_{GL_{\mathbb{A}}} T(h) f(gh) dh. \tag{4}$$

2.6 Weyl group

Definition 7. *The Weyl group* W *of* $G = SL_n$ *is the set of equivalence classes of matrices in* SL_n *with entries* $+1, 0, -1$ *such that there is exactly one nonzero entry in each row and each column. Two matrices are equivalent if by replacing each element by its absolute value we get the same matrix.*

It is clear that W is in bijective correspondence with permutations of $[n] = \{1, 2, \ldots, n\}$ and in fact we will associate the permutation σ with the equivalence class in W with the property that a_{ij} is nonzero exactly when $j = \sigma(i)$. The following lemma is easy to verify.

Lemma 1. *If* $A = \{a_{ij}\}$ *and* $B = \{b_{ij}\}$ *are related by* $B = w^{-1}Aw$, *then* $b_{\sigma(i)\sigma(j)} = \pm a_{ij}$, *with* $+$ *sign if* $i = j$.

2.7 Parabolic subgroups

Definition 8. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ *be an ordered partition of* $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. *Then for* $1 \leq k \leq n$, *we define* $gr_\lambda(k)$ *as the smallest integer* $s \geq 1$ *such that* $\lambda_1 + \lambda_2 + \cdots + \lambda_s \geq k$. *In that case, we say* k *is in group* s *of* λ.

Definition 9. *Call a subgroup* H *of* G *free at position* (i, j) *if some element of* H *is nonzero at* (i, j). *Otherwise, we say* H *is zero at* (i, j).

Definition 10. *The (proper)* ***parabolic subgroup*** P_λ *of rank* r *associated to an ordered partition* λ *of* n *is defined as the group of invertible matrices that is free at* (i, j) *if and only if* $gr_\lambda(i) \leq gr_\lambda(j)$. *Also define* $gr_{P_\lambda}(k) = gr_\lambda(k)$, *and say* k *is in the* s*-th* P*-group where* $s = gr_P(k)$. *When the context is clear, we omit the subscript* λ. *A proper parabolic subgroup is a parabolic subgroup not equal to* G.

A special parabolic subgroup is the Borel subgroup B associated to $(1, 1, \ldots, 1)$, consisting of all upper triangular invertible matrices. Also the whole group G is just a parabolic subgroup associated to (n).

Definition 11. *The* ***unipotent radical*** $N(P)$ *of a parabolic subgroup* P *is those matrices in* P *whose* (i, j) *entry is* $\delta_{i,j}$ *if* $gr_P(i) = gr_P(j)$.

Definition 12. *The* ***block form*** $M(P)$ *of a parabolic subgroup* P *is defined as the group of invertible matrices that is free if and only if* $gr_P(i) = gr_P(j)$. *Let* M_k *be the* $\lambda_k \times \lambda_k$ *submatrix formed from the rows and columns with indices in the* k*-th* P*-group.*

Definition 13. *We define the compact subgroup* $K(P)$ *to be the embedding of* $K(\lambda_1) \times K(\lambda_2) \times \cdots \times K(\lambda_r)$ *in* $M(P)$.

We have the Levi decomposition $P = N(P)M(P)$ for any parabolic subgroup P, and in general we have the Langlands decomposition $G = N(P)M(P)K(P)$. Note also that $N(P)$ normalizes $M(P)$, i.e. $mN(P)m^{-1} = N(P)$ for any $m \in M(P)$.

Definition 14. *A* ***cuspidal Maass form*** *is a Maass form* f *such that*

$$I_P = \int_{N_k \backslash N_{\mathbb{A}}} f(ng)dn = 0 \tag{5}$$

for all unipotent radicals $N(P)$ *of proper parabolic subgroups of* G.

In fact, it only suffices to verify this for a special type of parabolic subgroup.

Definition 15. *A* ***maximal parabolic*** *subgroup is a parabolic subgroup of rank* 2.

Proposition 4. *If* $I_P = 0$ *for all maximal parabolic subgroups* P *then* f *is cuspidal.*

Proof. For any parabolic P, there is a maximal parabolic Q that contains P. This means $N = N(P) \supset N(Q)$. Let N' be the subgroup of N that is unity along the diagonal and zero at the free (non-diagonal) indices of $N(Q)$. (In a sense, N' is the complement of $N(Q)$ with respect to N). Then

$$\int_{N_k \backslash N_{\mathbb{A}}} \phi(ng)dn = \int_{N'_k \backslash N'_{\mathbb{A}}} \int_{N_k(Q) \backslash N_{\mathbb{A}}(Q)} \phi(n_1 n_2 g)dn_1 dn_2 = 0 \tag{6}$$

since the inner integral vanishes. □

Now we are ready to define Maass forms on parabolic subgroups.

Definition 16. *A Maass form on P_λ with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a function $f : P_k \backslash P_{\mathbb{A}}$ such that $f(p) = \prod_{i=1}^{r} \phi_i(M_i)$, where each ϕ_i is a Maass form on $SL(\lambda_i)$.*

2.8 Parabolic maps

The terminology in this section is non-standard but will simplify the exposition.

Definition 17. *A parabolic subgroup map, or **group map**, is a permutation of $[n]$. Let P, Q be arbitrary parabolic subgroups. A **proper group map** from P to Q is a group map w such that $i > j$ and $gr_P(i) = gr_P(j)$ implies $w(i) > w(j)$, and also $w(i) > w(j)$ and $gr_Q(w(i)) = gr_Q(w(j))$ implies $i > j$.*

Definition 18. *A P-preserving group map is a group map from P to P such that $gr_P(i) = gr_P(j)$ implies $gr_P(w(i)) = gr_P(w(j))$.*

2.9 Characters

Let Z be scalar multiples of the identity in G. For a sequence of complex numbers $s = (s_1, s_2, \dots, s_r)$ satisfying $p_1 s_1 + p_2 s_2 + \cdots + p_r s_r = 0$, define the character $\chi_s : N_{\mathbb{A}} M_k Z \backslash P_{\mathbb{A}}$ by $\chi_s(m) = \prod_{i=1}^{r} |\det(M_i)|^{s_i}$ for $m \in M_{\mathbb{A}}$.

2.10 Twisted Eisenstein series

Definition 19. *An Eisenstein series twisted by a Maass form ϕ on parabolic subgroup P is given by*

$$E_P(g, \phi, s) = \sum_{P_k \backslash G_k} \phi(\gamma g) \chi_s(\gamma g). \tag{7}$$

2.11 Modular function

When integrating over a region R controlled by a variable r, we would sometimes like to make the substitution $r \to g r g^{-1}$, for some g. The extra multiplicative factor this substitution creates is called the modular function, $\delta(R, g)$.

2.12 Previous results

Jacquet and Shalika were the first to obtain a non-vanishing result via looking at constant terms of Eisenstein series. Langlands [5] developed this method further and computed the constant term where P and Q are two maximal parabolics. In this paper, we will look at general parabolics P, Q and provide a formula for the constant term. Specifically, we will prove:

Theorem 1. *Let $W' = W(P, Q)$ be defined as in Section 3.2. Then for each choice of w, there exist subgroups M^+ of $M = M(Q)$ and N^- of $N = N(Q)$ such that for some functions $M_\phi(w, s)$ that depend only on ϕ, w, s and for all $g \in B$, we have*

$$\int_{N_k\backslash N_{\mathbb{A}}} E_P(g, \phi, s)dn = \sum_{w\in W'} M_\phi(w, s) \sum_{\beta\in M_k^+\backslash M_k} \delta(N^-, \beta g)\chi_s(w\beta g w^{-1})\phi(w\beta g w^{-1}). \tag{8}$$

Remark 1. *Because the LHS is right K-invariant, knowing the values for $g \in B$ allows us to determine the values for $g \in P$.*

3 Constant Term Calculations

Let P, Q be parabolic subgroups of G, and let $f(g) = \phi(g)\chi_s(g)$, where ϕ is a Maass form on P. We will evaluate the constant term of $E_P(g, \phi, s)$ over Q. Recall $E_P(g, \phi, s) = \sum_{\gamma\in P_k\backslash G_k} f(\gamma g)$. Let N be the unipotent radical of Q. Let $W_P = W \cap P$ and $W_Q = W \cap Q$.

$$\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in P_k\backslash G_k} f(\gamma n g)dn = \int_{N_k\backslash N_{\mathbb{A}}} \sum_{w\in W_P\backslash W/W_Q} \sum_{\gamma\in(w^{-1}P_k w\cap Q_k)\backslash Q_k} f(w\gamma n g)dn \tag{9}$$

$$= \sum_{w\in W_P\backslash W/W_Q} \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(w^{-1}P_k w\cap Q_k)\backslash Q_k)} f(w\gamma n g)dn \tag{10}$$

$$= \sum_{w\in W_P\backslash W/W_Q} \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(w^{-1}P_k w\cap Q_k)\backslash Q_k)} f(w n\gamma g)dn. \tag{11}$$

An explanation of why is as follows.

Recall the Bruhat decomposition $G_k = B_k W B_k$ where W is the Weyl group. Since $B_k \subset P_k$ and $B_k \subset Q_k$, we have $G_k = P_k W Q_k$, hence G_k can be decomposed into double cosets $P_k w Q_k$ for $w \in W$. In fact, we have the disjoint union

$$G = \bigsqcup_{w \in W_P \backslash W / W) Q} P_k w Q_k.$$

Thus, the summation over $P_k \backslash G$ can be written as a double summation, one over $P_k \backslash G / Q_k$ and the other over $P_k \backslash P_k w Q_k$. The set $P_k \backslash G / Q_k$ is isomorphic to $W_P \backslash W / W_Q$, while the second set $P_k \backslash P_k w Q_k$ is isomorphic to $w \cdot (w^{-1} P_k w \cap Q_k) \backslash Q_k$.

In the second step, we change the order of summation. We are allowed to do so if we assume the choice of w and the integration $N_k \backslash N_{\mathbb{A}}$ are independent.

In the third step, we replace n by $\gamma^{-1} n \gamma$. Since $\gamma \in Q_k$, the change of variables is uni-modular, i.e. no extra multiplicative factors are introduced.

3.1 Weyl group double cosets

Now we provide a new characterization of $W_P \backslash W / W_Q$. Let P, Q be associated to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ respectively.

Proposition 5. *(a) The double cosets of $W_P \backslash W / W_Q$ are in bijective correspondence with $r \times s$ non-negative integer matrices whose i-th row sums to λ_i and j-th column sums to μ_j via the map*

$$\rho : W_P w W_Q \to \{a_{ij} = \textit{number of } i\textit{-th } P\textit{-group elements mapped to } j\textit{-th } Q\textit{-group elements under } w\} \quad (12)$$

(b) Each double coset of $W_P \backslash W / W_Q$ contains an element that is a proper group map from P to Q.

Proof. (a) Well-defined: The elements in W_P are P-preserving and the elements in W_Q are Q-preserving. These operations clearly leave a_{ij} unchanged.

Injective: let w_1, w_2 be such that $\rho(w_1) = \rho(w_2)$. For a fixed i and w let $g_{ij}(w)$ be the image of the i-th P-block under w intersected with the j-th Q-block. Then since $g_{ij}(w_1)$ has the same cardinality as $g_{ij}(w_2)$, there is a bijection between $g_{ij}(w_1)$ and $g_{ij}(w_2)$. Since this analysis works for all i, j, we see there exists an element w of W_Q such that $g_{ij}(w_1) = g_{ij}(w_2 w)$. Now perform a similar process to find an element w' of W_P such that $w_1 = w' w_2 w$.

Surjective: assign $w(1), w(2), \ldots, w(n)$ in order. The choice for each $w(i)$ is unique because w is proper and $\rho(w)$ is given. Note that this construction also proves (b). □

3.2 Evaluating Constant Terms

Let ρ be the bijection defined in (12). Since ϕ is a cusp form, we are able to cancel many terms in the expansion (9). Let

$$I(w) = \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\beta\in(w^{-1}P_k w\cap Q_k)\backslash Q_k} f(wn\beta g)dn. \tag{13}$$

$$= \sum_{\beta\in(w^{-1}P_k w\cap Q_k)\backslash Q_k} \int_{N_k\backslash N_{\mathbb{A}}} f(wn\beta g)dn \tag{14}$$

Here we change the order of integration and summation, assuming that the integration and summation are independent.

Proposition 6. *We have $I(w) = 0$ whenever some row in $\rho(w)$ contains two (or more) nonzero elements.*

Proof. Let row ι contain two or more nonzero elements. First, by Proposition 5 we may assume w is a proper group map. Then

$$\int_{N_k\backslash N_{\mathbb{A}}} f(wn\beta g)dn = \int_{N_k\backslash N_{\mathbb{A}}} f(wnw^{-1}w\beta g)dn. \tag{15}$$

Let $\rho(w) = \{a_{ij}\}$, and let N' be the subgroup of G with 1's on the diagonal and zeroes everywhere else except $M(P)_i$, which is the unipotent subgroup of the parabolic associated to $\lambda = (a_{\iota 1}, a_{\iota 2}, \ldots, a_{\iota s})$. If (i,j) represents a free index of N', then $\operatorname{gr}_Q(w(i)) < \operatorname{gr}_Q(w(j))$, so (i,j) is a free index of wNw^{-1}. Hence, N' is a subgroup of wNw^{-1}. Let N'' be a subgroup of wNw^{-1} that is unity on the main diagonal and zero at all the non-diagonal free indices of N'. We know χ_s is left-invariant on N'. Hence we can factor the integral

$$\int_{N_k\backslash N_{\mathbb{A}}} f(wnw^{-1}w\beta g)dn$$

$$= \int_{N''_k\backslash N''_{\mathbb{A}}} \chi_s(n_1 w\beta g) \int_{N'_k\backslash N'_{\mathbb{A}}} \phi(n_2 n_1 w\beta g)dn_2 dn_1 = 0. \tag{16}$$

Hence, since ϕ_i is cuspidal, the inner integral must vanish, so we have $I(w) = 0$. □

Corollary 1. *We have $I(w) = 0$ unless the partition associated to P refines the partition associated to Q, i.e. P is formed from Q by successively merging two numbers in the partition. (For instance, $(1,1,2,2,3)$ refines $(3,3,3)$.)*

Proof. If $I(w) \neq 0$, then every row in $\rho(w)$ contains exactly one nonzero element, which implies every μ_j is the sum of one or more of the λ_i. □

Let $W(P,Q)$ be the set of $w \in W$ such that every row in $\rho(w)$ contains exactly one nonzero element.

3.3 Nonzero terms

We will now assume $I(w) \neq 0$. The computation we want is

$$I(w) = \sum_{\gamma \in N_k^+ \backslash N_k} \sum_{\beta \in M_k^+ \backslash M_k} \int_{N_k \backslash N_{\mathbb{A}}} f(wn\gamma\beta g)dn \tag{17}$$

$$= \sum_{\beta \in M_k^+ \backslash M_k} \int_{N_k^+ \backslash N_{\mathbb{A}}} f(wn\beta g)dn \tag{18}$$

$$= \sum_{\beta \in M_k^+ \backslash M_k} \int_{N_{\mathbb{A}}^-} f(wn\beta g)dn \tag{19}$$

$$= \sum_{\beta \in M_k^+ \backslash M_k} J_w(\beta g), \tag{20}$$

where

$$J_w(g) = \int_{N_{\mathbb{A}}^-} f(wng)dn \tag{21}$$

$$= \int_{N_{\mathbb{A}}^-} \delta(N^-, g) f(wgn)dn \tag{22}$$

$$= \int_{N_{\mathbb{A}}^-} \delta(N^-, g) f((wgw^{-1})wn)dn \tag{23}$$

$$= \delta(N^-, g)\chi_s(wgw^{-1}) \int_{N_{\mathbb{A}}^-} \phi((wgw^{-1})wn)\chi_s(wn)dn. \tag{24}$$

We will justify each step. First, we may assume w is a proper group map.

Definition 20. *Call (i,j) positive if $i < j$, $w^{-1}(i) < w^{-1}(j)$, and $gr_Q(i) \neq gr_Q(j)$. Call (i,j) negative if $i > j$, $w^{-1}(i) < w^{-1}(j)$, and $gr_Q(i) \neq gr_Q(j)$. Let N^+ be the subgroup of N that is free on the positive elements (and zero on the negative elements), and N^- be the subgroup of N that is free on the negative elements (and zero on the positive elements). Clearly, $N = N^+N^-$.*

Definition 21. *Let M^+ be the subgroup of M that is free at the positions (i, j) such that $gr_P(w^{-1}(i)) \leq gr_P(w^{-1}(j))$.*

Lemma 2. *We have $w^{-1}P_k w \cap Q_k = N_k^+ M_k^+$.*

Proof. Let (i, j) be a position that is free in both $w^{-1}P_k w$ and Q_k. Then $(w^{-1}(i), w^{-1}(j))$ is free in P_k, so $\mathrm{gr}_P(w^{-1}(i)) \leq \mathrm{gr}_P(w^{-1}(j))$. Also (i, j) is free in Q_k, so $\mathrm{gr}_Q(i) \leq \mathrm{gr}_Q(j)$. Now if $\mathrm{gr}_Q(i) = \mathrm{gr}_Q(j)$ then (i, j) is free in M^+. If $\mathrm{gr}_Q(i) < \mathrm{gr}_Q(j)$ then $i < j$, and the fact that $\mathrm{gr}_P(w^{-1}(i)) \leq \mathrm{gr}_P(w^{-1}(j))$ and w being proper implies $w^{-1}(i) < w^{-1}(j)$. This means (i, j) is free in N^+. Hence (i, j) is free in $N_k^+ M_k^+$. This proves $w^{-1}P_k w \cap Q_k \subset N_k^+ M_k^+$.

For the reverse inclusion, $N_k^+ M_k^+ \subset N_k M_k = Q_k$ is obvious. If (i, j) is free in N_k^+, then $w^{-1}(i) < w^{-1}(j)$, so (i, j) is free in $w^{-1}P_k w$. If (i, j) is free in M_k^+, then $\mathrm{gr}_P(w^{-1}(i)) \leq \mathrm{gr}_P(w^{-1}(j))$, which also implies (i, j) is free in $w^{-1}P_k w$. This completes the proof of the lemma. □

Equation (18) depends on the following Lemma.

Lemma 3. *Every element in $(w^{-1}P_k w \cap Q_k)\backslash Q_k$ is the semidirect product of $N_k^+\backslash N_k$ and $M_k^+\backslash M_k$.*

Proof. First, since $Q = NM$, every $q \in (w^{-1}P_k w \cap Q_k)\backslash Q_k$ is expressible as nm for some $n \in N_k^+$ and $m \in M_k^+$. To prove uniqueness, suppose $N_k^+ n_1 M_k^+ m_1 = N_k^+ n_2 M_k^+ m_2$ for some $n_1, n_2 \in N_k$ and $m_1, m_2 \in M_k$. Then $n_2^{-1} n' n_1 = m' m_2 m_1^{-1} m''$ for some $n' \in N_k^+$ and $m', m'' \in M_k^+$. The left-hand side is in N_k and the right-hand side is in M_k, so since $N_k \cap M_k = \{I\}$, both sides are the identity. This immediately proves $N_k^+ n_1 = N_k^+ n_2$ and $M_k^+ m_1 = M_k^+ m_2$, as desired. □

Note that (19) is a standard unfolding argument. Equation (20) uses the fact that $f(wng)$ is constant on $N_k^+\backslash N_{\mathbb{A}}^+$ (since $wn'w^{-1} \in N(P)$ for $n' \in N_K^+$) to factor the integral $\int_{N_k^+\backslash N_{\mathbb{A}}^+} 1\, dn$ out, which we may assume equals 1. Equation (23) uses the fact that $ng \in Q$, so $ng = g_1 g_2 n'$ for some $g_1 \in N_{\mathbb{A}}^+, g_2 \in M_{\mathbb{A}}(Q)$, and $n' \in N_{\mathbb{A}}^-$. The change of variables $n \to n'$ introduces an extra factor $\delta(N^-, g)$, and also $\phi(g_1 g_2 n) = \phi(g_2 n)$.

Equation (24) depends on the multiplicativity of χ_s and the fact that $wgw^{-1} \in P$, which follows from g being upper triangular and in $M(Q)$. To simplify further, we need the following Lemma.

Lemma 4. *Let $g \in P$. The function $F(g, \phi, s) = \int_{N_{\mathbb{A}}^-} \phi(gwn)\chi_s(wn)dn$ can be expressed as $\phi(g)M_\phi(w, s)$ for some function M depending on ϕ, w, and s.*

Proof. The analysis presented here generalizes Section 5 of [5]. Since ϕ is left-$N(P)$ invariant, we may assume $g \in M(P)$. Use the Langlands decomposition to write $gwn = n(gwn)m(gwn)k(gwn)$, where $n \in N(P), m \in M(P), k \in K(P)$. Since $g \in M(P)$ and since both ϕ and χ_s are left $N(P)$ and right-K invariant, we get

$$F(g,\phi,s) = \int_{N_{\mathbb{A}}^-} \phi(gm(wn))\chi_s(m(wn))dn \tag{25}$$

$$= \int_{GL_{\mathbb{A}}(\lambda_1)\times GL_{\mathbb{A}}(\lambda_2)\times\cdots\times GL_{\mathbb{A}}(\lambda_r)} \phi(gm)G(w,s,\phi,m)dm \tag{26}$$

$$= \int_{GL_{\mathbb{A}}(\lambda_1)}\int_{GL_{\mathbb{A}}(\lambda_2)}\cdots\int_{GL_{\mathbb{A}}(\lambda_r)} \tag{27}$$

$$\prod_{i=1}^{r}\phi_i(g_i m_i)\cdot G(w,s,\phi,m_1,m_2,\ldots,m_r)dm_1 dm_2\ldots dm_r, \tag{28}$$

where we used the map $n \to m(wn)$ to effect a change of variables, which introduces a new bi-K-invariant function G.

Consider the integral over $GL_{\mathbb{A}}(\lambda_1)$, holding $m_2, m_3, \ldots, m_r$ fixed. We immediately recognize

$$\int_{GL_{\mathbb{A}}(\lambda_1)} \phi_1(g_1 m_1)G(m_1,m_2,\ldots,m_r)dm_1 \tag{29}$$

as the action of a Hecke operator T_1 on ϕ_1. Since ϕ_1 is an eigenfunction of T_1, we can replace (29) with $G_1(w,s,\phi,m_2,m_3,\ldots,m_r)\phi_1(g_1)$ for some function G_1 that notably does not depend on g. We can repeat this process with $GL_{\mathbb{A}}(\lambda_2)$, using G_1 instead of G. This process obviously being iterable, we finally end up with

$$F(g,\phi,s) = G_r(w,s,\phi)\phi_1(g_1)\phi_2(g_2)\ldots\phi_r(g_r). \tag{30}$$

The conclusion follows immediately. □

Remark 2. *Take $g = 1$ to obtain $M_\phi(w,s) = \frac{1}{\phi(1)}\int_{N_{\mathbb{A}}^-}\phi(wn)\chi_s(wn)$.*

Putting all of our contributions together we get

$$I_P = \sum_{w\in W'} M_\phi(s) \sum_{\beta\in M_k^+\backslash M_k} \delta(\beta g)\chi_s(w\beta g w^{-1})\phi(w\beta g w^{-1}). \tag{31}$$

4 Applications

4.1 Non-vanishing results

Theorem 1 allows one to systematically apply the Jacquet-Shalika method for Eisenstein series twisted by Maass forms over a parabolic P. Our goal is to show $E_P(\phi, g, s)$ is a cusp form, which contradicts Proposition 3. First, by Proposition 4 it suffices to consider Q being maximal parabolic. Next, by Corollary 1 it suffices to consider Q such that P refines Q. Thus, we have significantly narrowed down our choice of Q. Finally, for each such Q, we only need to consider $w \in W(P, Q)$, which is much smaller than the entire Weyl group. It is also pretty easy to list the elements in $W(P, Q)$ as they correspond to matrices with fixed row and column sums.

In summary, we get only a few parabolics Q with $I_Q \neq 0$, and even for these Q, we have I_Q is the sum of only a few terms. Then we can multiply E_P be a common denominator to make it holomorphic at most values of s. By plugging in a special value of s at E_P^* we get a non-vanishing result, since it would be a contradiction if all of the constant terms were zero.

4.2 Examples

The classical example of SL_2 is instructive. In this case, both P and Q are associated to $(1, 1)$. Since $|W(P, Q)| = 2$, we predict two terms in the constant term expansion, which is exactly what happens.

Now consider P associated to $(2, 1)$. We only need to test $Q = (2, 1)$ and $Q = (1, 2)$. For each case, we have $|W(P, Q)| = 1$, so each nonzero constant term has exactly one element. This agrees with Proposition 10.11.2 in [3]. In fact, if we evaluate these two constant terms and clear denominators, we obtain a non-vanishing result about certain L-functions, such as that in Section 10.12 in [3].

We may also extend the Jacquet-Shalika method to non-maximal parabolic subgroups. As an example, consider $P = (n - 2, 1, 1)$. Then if $I_Q \neq 0$ and Q is maximal parabolic, then we must have $Q = (n-1, 1)$ or $Q = (n-2, 2)$. In the first case, we get a sum of two terms (since $|W(P, Q)| = 2$); in the second case, we get a single term. If we clear the denominators of the three terms, we get a completed Eisenstein series that cannot vanish on $(n-1, 1)$ and $(n-2, 2)$ simultaneously. This may lead to a new non-vanishing result.

5 Appendix

In this Appendix, we will show how Proposition 1 proves the PNT. The analysis here is taken from [7].

Definition 22. *Let* $\vartheta(x) = \sum_{p \le x} \log p$*,* $\Phi(s) = \sum_p p^{-s} \log p$*, and* $H(t) = \vartheta(e^t)e^{-t} - 1$*.*

Lemma 5. $\Phi(s) - \frac{1}{s-1}$ *extends to a meromorphic function on* $\Re(s) > \frac{1}{2}$ *which is holomorphic on* $\Re(s) \ge 1$*.*

Proof. It is well known that $\zeta(s) - \frac{1}{s-1}$ is entire, so $\zeta(s)$ is meromorphic, with a simple pole at $s = 1$. By our assumption, $\zeta(s)$ has no zeroes on $\Re(s) \ge 1$, so the logarithmic derivative $\frac{\zeta'(s)}{\zeta(s)}$ has only a simple pole at $s = 1$ with residue -1. Thus

$$\frac{-\zeta'(s)}{\zeta(s)} = (-\log \zeta(s))' = \sum_p \frac{\log p}{1 - p^s} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}. \tag{32}$$

Since the sum is locally uniformly and absolutely convergent for $\Re(s) > \frac{1}{2}$, it is a holomorphic function for such s. Thus, we must have $\Phi(s) - \frac{1}{s-1}$ is a meromorphic function on $\Re(s) > \frac{1}{2}$ and holomorphic on $\Re(s) \ge 1$, as desired. □

This immediately implies $\Phi(s+1) - \frac{1}{s}$ and $(\mathcal{L}H)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ are meromorphic functions that are holomorphic on $\Re(s) \ge 0$. Now we need the following theorem due to Newman [6]

Proposition 7. *Let* $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ *be a bounded piecewise continuous function, and suppose its Laplace transform extends to a holomorphic function* $g(s)$ *on* $\Re(s) \ge 0$*. Then* $\int_0^\infty f(t)dt$ *converges and equals* $g(0)$*.*

We finally need an analytic criterion:

Proposition 8. *Let* $f : \mathbb{R}_{\ge 1} \to \mathbb{R}$ *be a non-decreasing function. If* $\int_1^\infty \frac{f(x)-x}{x^2}dx$ *converges, then* $f(x) \sim x$*, i.e.* $\lim_{x\to\infty} \frac{f(x)}{x}$ *exists and equals* 1*.*

Proof of PNT. We have that $\int_0^\infty H(t)dt = \int_0^\infty \vartheta(e^t)e^{-t} - 1dt$ converges by Proposition 7. By change of variables, we get $\int_1^\infty \frac{\vartheta(x)-x}{x^2}dx$ converges. This implies by Proposition 8 that $\vartheta(x) \sim x$, which is equivalent to PNT. □

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