The maximal divided power extension of the spherical Cherednik algebras for $\mathbb{Z}/p\mathbb{Z}$

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Abstract

We provide a construction for the maximal divided power extension of the spherical Cherednik algebras for $\mathbb{Z}/p\mathbb{Z}$ as a family of $\mathbb{Z}/p\mathbb{Z}$-invariant differential operators with a certain invariance condition.

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1 Introduction

Cherednik algebras, also known as double affine Hecke algebras, are a large family of algebras, which were introduced by Cherednik in his work on Macdonald's conjecture, see [?]. Since then Cherednik algebras were discovered to be interesting as the object of study by themselves and in many applications, mainly connected with mathematical physics. A good overview of the theory of Cherednik algebras is [?].

Until recently these algebras were studied mainly in the zero characteristic, but a few years ago a theory of Cherednik algebras in positive characteristic started to develop. In [?] some general structural theory of Cherednik algebras in positive characteristic was investigated, later in [?] and [?] the Hilbert polynomials of some irreducible finite dimensional representations were calculated. All of the later research was done in connection with the MIT PRIMES program.

The current paper is a continuation of this research. Our main goal was to develop a theory of Cherednik algebras for $\mathbb{Z}/p\mathbb{Z}$ with divided powers in positive characteristic. The main reason for the study of this construction is the fact that simple reduction of the Cherednik algebra in positive characteristic, makes the algebra 'too small', because a lot of operators become central and act by zero on important representations. So to make representation theory 'richer' one can try to work with the algebra extended by divided powers. To define the maximal divided power extension even in this case turned out to be an interesting problem. What we were able to provide is an alternative construction of divided power ring over $\mathbb{Z}$ as a family of differential operators satisfying some invariance conditions.

1.1 Structure of the paper

The structure of the paper is as follows. In the subsection which directly follows this one, we give some general definitions which we are going to use in the rest of the paper. Section 2 explains what do we actually mean by the divided power extension. Section 3 shows us how this definition works for the algebra of differential operators in one variable, which will be of importance for our further research. Section 4 defines what Cherednik algebra is and gives it's alternative definition as a family of operators with invariance condition. Section 5 ties this in with the calculation of divided power rings and shows that one can use the very same definition over integers for a spherical algebra to actually get a divided power ring.

1.2 Acknowledgements

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1.3 Preliminary definitions

Definition 1.1. For the sake of simplicity, we define $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$.

2 Divided power rings

Let Z be free as a $\mathbb{Z}$-module. Given a ring A_Z over a commutative ring Z, where A_Z free as a Z-module, and V_Z, which also free as a Z-module, we have a faithful representation $A_Z \xrightarrow{\phi_Z} \mathrm{End}_Z(V_Z)$. Then we let $Q := Z \otimes_{\mathbb{Z}} \mathbb{Q}$. We can then define A_Q with representation $A_Q \xrightarrow{\phi_Q} \mathrm{End}_Q(V_Q)$. The relations are shown in the diagram below.

$$\begin{tikzcd}
A_Z \arrow[d, hook] \arrow[dr, hook] & \\
A_{DP} \arrow[r, hook, "\phi_Z"] \arrow[d, hook] & \mathrm{End}_Z(V_Z) \arrow[d, hook] \\
A_Q \arrow[r, hook, "\phi_Q"] & \mathrm{End}_Q(V_Q)
\end{tikzcd}$$

Notice that there is a map from $\mathrm{End}_Z(V_Z)$ to $\mathrm{End}_Q(V_Q)$ because $\mathrm{Hom}_Q(V_Q, V_Q) = \mathrm{Hom}_Z(V_Z, V_Q) = \mathrm{Hom}_Z(V_Z, V_Z) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Now we can define A_{DP} in the two following ways:

Definition 2.1. $A_{DP} = \{a \in A_Q | \phi(a) \in \mathrm{End}_Z(V_Z)\}$

Definition 2.2. $A_{DP} = \{a \otimes \frac{1}{n} | a \in A_Z, n \in \mathbb{N}, \frac{\phi(a)}{n} \in \mathrm{End}_Z(V_Z)\}$

2.1 $A_{DP} \subset B_{DP}$

Lemma 2.3. *We show that there exists an injective map from A_{DP} to B_{DP}, according to the diagram below. Here, B_Z, B_{DP} and B_Q are defined similarly as the definition of A_Z, A_{DP} and A_Q.*

$$\begin{tikzcd}
A_Z \arrow[r, hook] \arrow[d, hook] & B_Z \arrow[d, hook] \arrow[dr] & \\
A_{DP} \arrow[r, hook, dotted] \arrow[d, hook] \arrow[rr, bend right] & B_{DP} \arrow[r, hook] \arrow[d, hook] & \mathrm{End}_Z(V_Z) \arrow[d, hook] \\
A_Q \arrow[r, hook] & B_Q \arrow[r, hook] & \mathrm{End}_Q(V_Q)
\end{tikzcd}$$

Proof. Take $p \in A_{DP}$. Then $p \in A_Q$, which means $p \in B_Q$. Notice that B_{DP} is defined as the largest subring of B_Q s.t. there exists a map to $\mathrm{End}_Z(V_Z)$. Since there is a map from A_{DP} to $\mathrm{End}_Z(V_Z)$, $p \in B_{DP}$, thus shows $A_{DP} \subset B_{DP}$. □

3 Differential Operators on $r[x]$ for any ring r

Let r be any ring. We first introduce the definition of the differential operators on $r[x]$. We will define then degree by degree.

3.1 Grothendieck's Definition of Differential Operators

Definition 3.1. [*Grothendieck*] The differential operators of degree k on $r[x]$ can be defined inductively. We denote the set of degree i differential operators as $\mathrm{Diff}(r[x])_i$, for any $i \in \mathbb{N}$. Then we define $\mathrm{Diff}(r[x])_0 = \{f | f \in r[x]\}$. For any $i \geq 1$, we have $\mathrm{Diff}(r[x])_i = \{\varphi \in \mathrm{End}(r[x]) | [\varphi, f] \in \mathrm{Diff}(r[x])_{i-1} \forall f \in r[x]\}$.

3.2 Generators of $\mathrm{Diff}(r[x])$

Definition 3.2. The set of differential operators is defined by $\bigcup_{i=0}^{\infty} \mathrm{Diff}(r[x])_i$, which is denoted as $\mathrm{Diff}(r[x])$.

Theorem 3.3. *We claim that* $\mathrm{Diff}(r[x])$ *is* $r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_i \cdots$, *where we define* $D_i x^n = \left(\frac{\partial^i}{i!}\right) x^n = \binom{n}{i} x^{n-i}$, $D_0 = I$.

Proof. Notice that $D_i \cdot x^m = \binom{m}{i} x^{m-i}$, where $\binom{m}{i} \in \mathbb{Z}$. Thus, D_i is well defined on any ring for any $i \in \mathbb{N}$.

Then, to prove this theorem, we consider the r-module that is defined as the set of all differential operators of degree i (i.e. $\mathrm{Diff}(r[x])_i$). For the sake of simplicity, we name it Diff_i.

We will prove that $\mathrm{Diff}_i = r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_i$. We can approach this by inducting on i.

First consider the base case: when $i = 0$, $r[x] = r[x]$.

To prove that this holds for $i + 1$, we first show that $D_{i+1} \in \mathrm{Diff}_{i+1}$. This implies that $r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_{i+1} \subset \mathrm{Diff}_{i+1}$.

Lemma 3.4. $D_{i+1} \in \mathrm{Diff}_{i+1}$.

Proof. We only need to check that

$$[D_{i+1}, f] \in \mathrm{Diff}_i$$

for all i.

Clearly, it's equivalent to check that

$$[D_{i+1}, x^n] \in \mathrm{Diff}_i$$

for all monomial x^n, where $n \geq 0$.

We induct on n.

Base case: when $n = 0$, $[D_{i+1}, 1] = 0 \in \mathrm{Diff}_i$. When $n = 1$, $[D_{i+1}, x] \in \mathrm{Diff}_i$.

Then we assume that $[D_{i+1}, x^n] \in \mathrm{Diff}_i$ holds for all $n \in \{0, 1, \cdots, m-1\}$. Now consider $[D_{i+1}, x^m]$.

$$
\begin{aligned}
&[D_{i+1}, x^m] = \\
&= D_{i+1}x^m - x^m D_{i+1} = \\
&= D_{i+1}x^{m-1}x - xx^{m-1}D_{i+1} = \\
&= D_{i+1}x^{m-1}x - x^{m-1}D_{i+1}x + x^{m-1}D_{i+1}x - x^{m-1}xD_{i+1} = \\
&= [D_{i+1}, x^{m-1}]x + x^m[D_{i+1}, x]
\end{aligned}
$$

Since $[D_{i+1}, x^{m-1}] \in \mathrm{Diff}_i$, and $[D_{i+1}, x] \in \mathrm{Diff}_i$,

$$[D_{i+1}, x^{m-1}]x + x^m[D_{i+1}, x]$$

is also in Diff_i, which finishes the proof.

□

Then we show that $\mathrm{Diff}_n \subset r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_n$ for all n.

Given that $r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_i$, we consider Diff_{i+1}.

We first prove the claim below.

Claim 3.5. *For any $\varphi \in \mathrm{Diff}_{i+1}$, there exists unique $\varphi_0 \in r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_{i+1} \cdots$ such that $\varphi_0(1) = \varphi(1), \varphi_0(x) = \varphi(x), \cdots \varphi_0(x^{i+1}) = \varphi(x^{i+1})$.*

Proof. Let's look φ_0 in the forms of $a_0(x)D_0 + a_1(x)D_1 + a_2(x)D_2 + \cdots + a_{i+1}(x)D_{i+1}$.

Then, we notice that for any i, D_i has the following property:

- $D_i \cdot x^i = 1$
- $D_i \cdot x^j = 0$ for any $j < i$.

Therefore, it's easy to see that $\varphi_0(x^j) = a_j(x) + \sum_{k=0}^{j-1} a_k(x) \cdot \binom{j}{k} x^{j-k}$, and we want this to equal to $\varphi(x^j)$, and thus $a_0, a_1, \cdots, a_{i+1}$ can be determined uniquely inductively. i.e. $a_0(x) = \varphi(1)$, $a_1(x) = \varphi(x) - a_0(x)x$, $\cdots$, $a_j(x) = \varphi(x^j) - \sum_{k=0}^{j-1} a_k(x) \cdot \binom{j}{k} x^{j-k}$.

□

Given the existence of φ_0, we can then let $\varphi_1 = \varphi - \varphi_0$. Since we know that $\varphi \circ f - f \circ \varphi \in \mathrm{Diff}_i$ for any polynomial f, we have $(\varphi_1 + \varphi_0) \circ f - f \circ (\varphi_1 + \varphi_0) \in \mathrm{Diff}_i$. Thus, $\varphi_1 \circ f - f \circ \varphi_1 \in \mathrm{Diff}_i$.

Claim 3.6. *Given*

- $\varphi_1 \circ f - f \circ \varphi_1 \in \mathrm{Diff}_i$
- $\varphi_1(1) = \varphi_1(x) = \cdots = \varphi_1(x^{i+1}) = 0$

It follows that $\varphi_1 = 0$.

Proof. We induct on the power of x. The base case is that $\varphi_1(1) = \varphi_1(x) = \cdots = \varphi_1(x^{i+1}) = 0$. Assume that $\varphi_1(1) = \varphi_1(x) = \cdots = \varphi_1(x^{i+m}) = 0$ for some $m \in \mathbb{N}^+$, we now consider $\varphi_1(x^{i+m+1})$.

We take $h = x^{m+1}$, $\Phi = \varphi_1 \cdot h - h \cdot \varphi_1$, $V = r[x]|_{\deg \leq i+1}$.

Since we know that $\varphi|_V = 0$, $\Phi|_V = \varphi_1 \circ h|_V$.

Thus, we have $\Phi(x^k) = \varphi_1 \circ x^{m+1}(x^k) = \varphi_1(x^{m+1+k})$, for $k \in \{0, 1, \cdots, i+1\}$. Since $\varphi_1(x^{m+1}) = \varphi_1(x^{m+2}) = \cdots = \varphi_1(x^{i+m}) = 0$, $\Phi(x^0) = \Phi(x^1) = \cdots = \Phi(x^i) = 0$. However, since $\Phi = \varphi_1 \cdot h - h \cdot \varphi_1$, $\varphi_1 \in \mathrm{Diff}_{i+1}$, $\Phi \in \mathrm{Diff}_i$. Thus, $\Phi = 0$, which means $\Phi(x^{i+1}) = 0$. This implies that $\varphi_1(x^{m+i+1}) = 0$. By the induction assumption ($\mathrm{Diff}_i \subset r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_i$), this completes the proof. □

Since $\varphi_1 = 0$, $\varphi = \varphi_0$, which means $\varphi \in r[x] \oplus r[x]D_0 \oplus r[x]D_1 \cdots \oplus r[x]D_{i+1}$, and this completes the induction. □

3.3 $\mathrm{Diff}(\mathbb{Z}[x]) = (\mathbb{Z}[\partial, x])^{\mathrm{DP}}$

Theorem 3.7. $\mathrm{Diff}(\mathbb{Z}[x]) = (\mathbb{Z}[\partial, x])^{\mathrm{DP}}$

Proof. We first show that $(\mathbb{Z}[\partial, x])^{DP} \subset \mathrm{Diff}(\mathbb{Z}[x])$.

Take a differential operator $p \in (\mathbb{Z}[\partial, x])^{DP}$. We can group p into a sum of homogeneous differential operators (i.e. each term in the differential operator has a degree), say $p = \sum p_k$, where p_k is degree k. We can then write $p_k = \sum_{i=\min(k,0)}^{\infty} \alpha_i x^{k+i}\partial^i$. We now want to show that for any homogeneous operators p_k, $n|\alpha_i i!$ for all i.

We consider two different cases, $k \geq 0$ and $k < 0$.

We prove the two cases respectively by inducting on i. We first prove the case when $k \geq 0$.

Base case: Notice that $p_k(x^0) = \alpha_0 x^k$, thus $n|p_k(x^0) \Rightarrow n|\alpha_0 \cdot 0!$

Inductive step: Assume that $n|\alpha_i \cdot i!$ for $i \in \{0, 1, \cdots, m-1\}$. We now want to show that $n|\alpha_m \cdot m!$.

Consider $p(x^m)$.

$$\begin{aligned}
&p_k(x^m) = \\
&= \sum_{i=\min(k,0)}^{\infty} \alpha_i x^{k+i}\partial^i(x^m) = \\
&= \sum_{i=0}^{m} \alpha_i x^{k+i}\partial^i(x^m) = \\
&= \sum_{i=0}^{m-1} \alpha_i x^{k+i}\partial^i x^m + \alpha_m x^{k+m}\partial^m x^m = \\
&= \sum_{i=0}^{m-1} \alpha_i x^{k+i}\partial^i x^m + \alpha_m \cdot m! \cdot x^{k+m}
\end{aligned}$$

Since $n|\alpha_i \cdot i!$ for $i \in \{0, 1, \cdots, m-1\}$, $n|\sum_{i=0}^{m-1} \alpha_i x^{k+i}\partial^i x^m$, so $n|\alpha_m \cdot m! \cdot x^{k+m}$, which finishes the induction.

We then prove the case when $k < 0$.

Base case: Consider $p_k(x^k)$.

$$\begin{aligned}
&p_k(x^k) = \\
&= \sum_{i=k}^{\infty} \alpha_i x^{k+i}\partial^i(x^k) = \\
&= \alpha_k x^0 k! = \\
&= \alpha_k k!
\end{aligned}$$

Thus $n|\alpha_{-k}(-k)!$.

Inductive step: we assume that $n|\alpha_i i!$ for $i \in \{k, k+1, \cdots, m-1\}$, and now consider $p_k(x^m)$.

$$\begin{aligned} & p_k(x^m) = \\ & = \sum_{i=\min(k,0)}^{\infty} \alpha_i x^{k+i} \partial^i (x^m) = \\ & = \sum_{i=k}^{m} \alpha_i x^{k+i} \partial^i (x^m) = \\ & = \sum_{i=k}^{m-1} \alpha_i x^{k+i} \partial^i x^m + \alpha_m x^{k+m} \partial^m x^m = \\ & = \sum_{i=k}^{m-1} \alpha_i x^{k+i} \partial^i x^m + \alpha_m \cdot m! \cdot x^{k+m} \end{aligned}$$

Similarly, since $n|\alpha_i \cdot i!$ for $i \in \{k, 1, \cdots, m-1\}$, $n| \sum_{i=k}^{m-1} \alpha_i x^{k+i} \partial^i x^m$, so $n|\alpha_m \cdot m! \cdot x^{k+m}$, which finishes the induction.

Therefore, we know that $\frac{p}{n} = \sum \frac{\alpha_i i!}{n} D_i$, which belongs to $\mathrm{Diff}(\mathbb{Z}[x])$. Thus, we proved that $(\mathbb{Z}[\partial, x])^{DP} \subset \mathrm{Diff}(\mathbb{Z}[x])$.

Also, we know that $\mathrm{Diff}(\mathbb{Z}[x]) \subset \mathbb{Q}[\partial, x]$ there exists a map from $\mathrm{Diff}(\mathbb{Z}[x])$ to $\mathrm{End}(\mathbb{Z}[x])$. According to the definition of $(\mathbb{Z}[\partial, x])^{DP}$, it's the largest subring of $\mathbb{Q}[\partial, x]$ s.t. there exists a map to $\mathrm{End}(\mathbb{Z}[x])$.

Thus, we showed that $\mathrm{Diff}(\mathbb{Z}[x]) = (\mathbb{Z}[\partial, x])^{\mathrm{DP}}$.

□

4 Cherednik Algebra

4.1 Background information

Let b be an integer. Let $G = \mathbb{Z}/b\mathbb{Z}$. G is a group that generated by s, with a single relation $s^b = I$. Let $q = e^{\frac{2\pi i}{b}}$. $k = \mathbb{Q}[q]$, and let $Q = k[c]$, where c is a formal variable.

Definition 4.1. We define $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ as following: $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}} = Q\langle x, y, s\rangle \backslash \{[y, x] = 1 - 2cs, sys^{-1} = q^{-1}y, sxs^{-1} = qx\}$, where $[,]$ is the commutator, $\langle , \rangle$ is the freely generated algebra on x, y, s.

Definition 4.2. Define $\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}} = Q[x, x^{-1}, \partial] \rtimes Q[G] = Q[x, x^{-1}, \partial] * Q[G] \backslash \{sxs^{-1} = s(x), s\partial s^{-1} = s(\partial)\}$, where $\rtimes$ is the semidirect product, $*$ is the free product and $\backslash$ is the quotient.

Now, let's consider a natural representation of those algebras. Indeed, $\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}}$ acts on $Q[x, x^{-1}]$, where $s(x^i) = q^i x^i$, for $i \in \mathbb{Z}$.

We can also define an action of $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ on $Q[x, x^{-1}]$.

We define $D = \partial_x - \frac{2c(1-s)}{(1-q)x}$. Now we define the structure of a representation of $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ on $Q[x, x^{-1}]$ as following:

$$\begin{aligned} \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}} &\to \mathrm{End}_k(Q[x, x^{-1}]) \\ y &\mapsto D \\ x &\mapsto x \\ s &\mapsto s \end{aligned}$$

Notice that this can be restricted on $Q[x]$ because if $i > 0$, then $\deg(D(x^i)) = i-1$; if $i = 0$, then $D(1) = 0$.

We now want to consider a family of twisted action of $\mathrm{Diff}_\mathrm{p}^\mathrm{Q}$ on $Q[x, x^{-1}]$. Instead, it's easier to define them using action of $\mathrm{Diff}_\mathrm{p}^\mathrm{Q}$ on $x^l|x|^n Q[x, x^{-1}]$, where we can understand the symbol as following:

$$\partial(x^{l+i}|x|^n) = (l+i+n)x^{l+i+1}|x|^n \tag{4.1}$$

$$s(x^{l+i}|x|^n) = q^{l+i}x^{l+i}|x|^n \tag{4.2}$$

Claim 4.3. *$x^l|x|^n Q[x, x^{-1}]$ has a submodule $x^l|x|^n Q[x]$ if and only if $n = \frac{2c}{1-q}(1-q^l) - l$.*

Proof. $x^l|x|^n Q[x, x^{-1}]$ has a $\mathrm{H}_\mathrm{c,p}^\mathrm{Q}$ if and only if $D(x^l|x|^n) = 0$.

$$\Leftrightarrow (x^l|x|^n)' - \frac{2c}{(1-q)x}(1-s)(x^l|x|^n) = 0 \tag{4.3}$$

$$\Leftrightarrow (lx^{l-1}|x|^n + nx^l|x|^{n-1} \cdot \frac{|x|}{x}) - \frac{2c}{(1-q)x}(1-s)(x^l|x|^n) = 0 \tag{4.4}$$

Notice that

$$s(x^l|x|^n) = q^l x^l |q|^n |x|^n = q^l x^l |x|^n$$

thus, (3.4)

$$\Leftrightarrow (l+n) - \frac{2c}{1-q}(1-q^l) = 0$$

$$\Leftrightarrow n = \frac{2c}{1-q}(1-q^l) - l$$

.

□

Since $q^p = 1$, it is enough to consider $l \in \{0, 1, \cdots, p-1\}$.

4.2 $\mathrm{H}_\mathrm{c,p}^\mathrm{Q} = \mathcal{H}_\mathrm{c,p}^\mathrm{Q}$

Definition 4.4. For $l \in \{0, 1, \cdots, p-1\}$, we define $k_l = 2c[l]_q - 2$, $V_l = x^l|x|^{k_l}Q[x, x^{-1}]$, $U_l = x^l|x|^{k_l}Q[x]$.

Then we define $\mathcal{H}_\mathrm{c,p}^\mathrm{Q}$ as following:

Definition 4.5. $\mathcal{H}_\mathrm{c,p}^\mathrm{Q} = \{p \in \mathrm{Diff}_\mathrm{p}^\mathrm{Q} | pU_l \subset U_l, l = \{0, 1, \cdots, p-1\}\}$

Then, follow from Claim 3.3, $\mathrm{H}_\mathrm{c,p}^\mathrm{Q} \subset \mathcal{H}_\mathrm{c,p}^\mathrm{Q}$.

Theorem 4.6. $\mathrm{H}_\mathrm{c,p}^\mathrm{Q} = \mathcal{H}_\mathrm{c,p}^\mathrm{Q}$.

Proof. We would like to show $\mathcal{H}_\mathrm{c,p}^\mathrm{Q} \subset \mathrm{H}_\mathrm{c,p}^\mathrm{Q}$.

Definition 4.7. We define the degree of differential operators $\in \mathrm{Diff}_\mathrm{p}^\mathrm{Q}$ as following:

$$\deg(x) = 1, \deg(\partial) = -1, \deg(s) = 0$$

.

Then $\deg(x^p\partial^q s^h) = p - q = z$.

Notice that this definition of degree can be restricted to $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ and $\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}}$, because the degree of D is -1.

Then we define the ordering of differential operators as following (as a lexicographical order):

$$x^i < sx^i < \cdots < s^{p-1}x^i < \partial x^{i-1} < s\partial x^{i-1} < \cdots$$

, for any $j, i \in \mathbb{N}^*$.

Next we take $x \in (\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$. We want to find $\mathfrak{x} \in \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ s.t. $x - \mathfrak{x} = 0$.

Lemma 4.8. *Let W be a vector space with infinite basis $f_1 < f_2 < \cdots < f_i < \cdots$, and let V be a vector space with infinity basis $e_1 < e_2 < \cdots < e_i < \cdots$. If f_i can be expressed as $c_i e_i + \sum_{i+1}^{\infty} c_j e_j$, then $W = V$.*

Proof. Since $f_i = c_i e_i + \sum_{i+1}^{\infty} c_j e_j$, $f_i = T_{i,k} e_k$ for some upper-triangular matrix $T_{i,j}$. Since the upper-triangular matrix is invertible, $e_k = \frac{f_k}{c_k} + \sum_{k+1}^{\infty} c'_j f_j$, for some constant c'_j. Thus, for any i, e_i can be expressed as a linear combination of basis of W, and for any i, f_i can be expressed as a linear combination of V. Thus $W = V$. □

Lemma 4.9. *for $i \in \mathbb{N}^*$, $(\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i = (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}})_i$.*

Proof. By definition, $(\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i \subset (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}})_i$.

We want to show that $(\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}})_i \subset (\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$. Take $p \in (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}})_i$. Notice that $px^n|x|^l = Ax^{l+n}|x|^l$ for some constant A. Because U_j is closed under multiplication by x, $pU_j \subset U_j$. Therefore, $(\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}})_i = (\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$ when i is positive. □

Claim 4.10. *The highest term in $D^n s^p x^h$ is $\partial^n s^p x^h$, according to the order we defined above.*

Proof. We prove the claim by inducting on the power of D.

Base case: when $n = 0$, $D^n s^p x^h = s^p x^h$, which has highest term $s^p x^h$.

Inductive step: Assume that for $n \in \{0, 1, \cdots, n' - 1\}$, $D^n s^p x^h$ has highest term $\partial^n s^p x^h$. We now consider $D^{n'} s^p x^h$.

$$\begin{aligned}
&D^{n'} s^p x^h = \\
&= D(D^{n'-1} s^p x^h) = \\
&= (\partial - \frac{2c(1-s)}{(1-q)x})(D^{n'-1} s^p x^h) = \\
&= (\partial - \frac{2c(1-s)}{(1-q)x})(\partial^{n'-1} s^p x^h + \cdots) = \\
&= \partial\partial^{n'-1} s^p x^h - \frac{2c(1-s)}{(1-q)x}\partial^{n'-1} s^p x^h + \cdots \\
&= \partial^{n'} s^p x^h - \frac{2c}{1-q} x^{-1}\partial^{n'-1} s^p x^h + \frac{2c}{1-q}\partial^{n'-1} s^{p+1} x^h + \cdots
\end{aligned}$$

Notice that after commuting x^{-1} and $\partial^{n'-1}$, the additional terms have lower powers on x, and thus cannot become the highest term. In addition, $\frac{2c}{1-q}\partial^{n'-1} s^p x^{h-1}$ and $\frac{2c}{1-q}\partial^{n'-1} s^{p+1} x^h$ both have lower degrees than $\partial^{n'} s^p x^h$, so $\partial^{n'} s^p x^h$ is the highest term, which completes the induction.

Therefore we can see that $D^n s^p x^h$ has highest term $\partial^n s^p x^h$ for any n, p, h. □

Therefore, $D^q s^p x^h$ can be expressed as $\partial^q s^p x^h +$ lower terms. Let $W = \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ and let $V = \mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}}$. We can then apply lemma 3.8, which proves $\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{p}} = \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$.

Then we consider the case when degree i is negative. Take $p \in (\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$, where i is a negative integer.

Claim 4.11. *$p = D' + \sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k$, where $D' \in (\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$.*

Proof. Since p is a differential operator, $p = As^m x^{j+i}\partial^j +$ lower terms, for some constant A, k, j.

Notice that when $(j+i) < 0$, the claim obviously holds. Thus we only need to consider when $(j+i) \geq 0$.

Notice that

$$\begin{aligned} &As^m x^{j+i}\partial^j - As^m x^{j+i}D^j = \\ &= As^m x^{j+i}\partial^j - As^m x^{j+i}(\partial - \frac{2c(1-s)}{(1-q)x})^j = \\ &= Bs^{m+1}x^{j+i-1}\partial^{j-1} + \cdots \end{aligned} \tag{4.5}$$

where $Bs^{m+1}x^{j+i-1}\partial^{j-1}$ (for some constant B) is the highest term in equation 4.5, and thus the degree of x is guaranteed to decrease at least by 1 by the above operation.

Since the power of x is finite, we can repeat the above operation until the power of x becomes negative. Also, notice that $As^m x^{j+i}D^j \in \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}_i}$ for any m and j, so we know that $p = D' + \sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k$, where $D' \in (\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$. □

Now we denote $Q = p - D'$.

Lemma 4.12. $Q = 0$.

Proof. To prove this lemma, we first show the following equivalence.

Lemma 4.13. *To prove that $Q = \sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k$ is zero is equivalent to*

$$\begin{vmatrix} 1 & m_1 & m_1(m_1+1) & \cdots & \prod_{g=0}^{-i-2}(m_1+g) \\ 1 & m_2-1 & (m_2-1)m_2 & \cdots & \prod_{g=0}^{-i-2}(m_2+g-1) \\ 1 & m_3-2 & (m_3-2)(m_3-1) & \cdots & \prod_{g=0}^{-i-2}(m_3+g-2) \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & m_{-i}+i-1 & (m_{-i}+i-1)(m_{-i}+i) & \cdots & \prod_{g=0}^{-i-2}(m_{-i}+g+i-1) \end{vmatrix} \neq 0$$

Here, the matrix has $(-i)$ columns and $(-i)$ rows with the term on the rth row and sth column be $\prod_{g=0}^{s-2}(m_r + g - (r-1))$, where $m_h = 2c[h+l]_q$ for any h.

Proof. We begin the proof with a claim.

Claim 4.14. *$(A(s) \cdot x^b\partial^a)(x^c|x|^d) = \prod_{h=0}^{a-1}(c+d-h)A(q^{c+b-a})x^{c+b-a}|x|^d$, where $A(s)$ is a polynomial of s, and a, b, c, d are integers.*

Proof. We first show that $\partial^a(x^c|x|^d) = \prod_{h=0}^{a-1}(c+d-h)x^{c-a}|x|^d$.

We prove by inducting on a.

Base case: when $a = 1$,

$$\begin{aligned} &\partial(x^c|x|^d) \\ &= cx^{c-1}|x|^d + dx^c|x|^{d-1}\frac{|x|}{x} \\ &= (c+d)x^{c-1}|x|^d \end{aligned}$$

Inductive step: Assume $\partial^a(x^c|x|^d) = \prod_{h=0}^{a-1}(c+d-h)x^{c-a}|x|^d$ for $a \in \{1,2,\cdots,a'-1\}$. Consider a'.

$$\begin{aligned}
&\partial^{a'}(x^c|x|^d)\\
&= \partial\partial^{a'-1}(x^c|x|^d)\\
&= \partial\prod_{h=0}^{a'-2}(c+d-h)x^{c-a'+1}|x|^d\\
&= \prod_{h=0}^{a'-2}(c+d-h)\partial(x^{c-a'+1}|x|^d)\\
&= \prod_{h=0}^{a'-1}(c+d-h)(x^{c-a'}|x|^d)
\end{aligned}$$

Which finishes the induction.

□

Given that, we can continue the proof of lemma 3.11. Let $n_l = x^l|x|^{2c[l]_q-l}$, for $l = \{0,1,\cdots,p-1\}$ Consider the operator Q operating on $n_l x^n$ for some integer n. Then

$$\begin{aligned}
&(\sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k)(n_l x^n)\\
&= (\sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k)(x^l|x|^{2c[l]_q-l}x^n)\\
&= (\sum_{k=0}^{-i-1} A_k(s)x^{k+i}\partial^k)(x^{n+l}|x|^{2c[l]_q-l})\\
&= \sum_{k=0}^{-i-1}\prod_{h=0}^{k+i-1}(n+l+(2c[l]_q-l)-h)A_k(q^{n+l+k-(k+i)})x^{n+l+k-(k+i)}|x|^n\\
&= \sum_{k=0}^{-i-1}\prod_{h=0}^{k+i-1}(n+2c[l]_q-h)A_k(q^{n+l-i})x^{n+l-i}|x|^n\\
&= 0
\end{aligned}$$

if and only if

$$\sum_{k=0}^{-i-1}\prod_{h=0}^{k+i-1}(n+2c[l]_q-h)A_k(q^{n+l-i}) = 0 \tag{4.6}$$

for $n \in \{0,1,\cdots,-i-1\}$. Equation 3.4 holds if and only if

$$\sum_{k=0}^{-i-1}\prod_{h=0}^{k+i-1}(n+2c[l-n+i]_q-h)A_k(q^l) = 0 \tag{4.7}$$

for $n \in \{0,1,\cdots,-i-1\}$.

Notice that the set of linear equations in 3.5 has a non-trivial solution if and only if the corresponding matrix M doesn't ahve determinant 0, where M is

$$\begin{pmatrix} 1 & m_1 & m_1(m_1+1) & \cdots & \prod_{g=0}^{-i-2}(m_1+g) \\ 1 & m_2-1 & (m_2-1)m_2 & \cdots & \prod_{g=0}^{-i-2}(m_2+g-1) \\ 1 & m_3-2 & (m_3-2)(m_3-1) & \cdots & \prod_{g=0}^{-i-2}(m_3+g-2) \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & m_{-i}+i-1 & (m_{-i}+i-1)(m_{-i}+i) & \cdots & \prod_{g=0}^{-i-2}(m_{-i}+g+i-1) \end{pmatrix}$$

, and this finishes the proof of Lemma 3.11. $\square$

Then we back to the proof of Lemma 3.10.

Applying lemma 3.11, we only need to show that the matrix has determinant non-zero.

Let matrix $M_c = M|_c$. i.e.

$$M_c = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & -2 & (-2)(-1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & i-1 & (i-1)(i) & \cdots & \prod_{g=0}^{-i+1}(i-1+g) \end{pmatrix}$$

Then it's easy to see that M_c is an upper-triangular matrix, and thus, the determinant is the product of all terms on the diagonal. Since each term on the diagonal is non-zero, the determinant of M_c is also non-zero, thus the determinant of M is non-zero, which means $Q = 0$. This finishes the proof of Lemma 3.10. $\square$

We can then back to the proof of Theorem 3.6. Since we know that $Q = 0$, we have $p = D'$. Since $D' \in (\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$, $p \in (\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$, and thus we proved that for all negative degrees i, $(\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i = (\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}})_i$. Therefore, $\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}} = \mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}}$, which finishes the proof of the theorem. $\square$

5 $\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}} = (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$

Define $e = \frac{1+s+\cdots+s^{p-1}}{p}$. Then let $\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} = e\,\mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}\,e$. By theorem 4.6, $\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} = e\,\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}\,e$

We now define $\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$ in a way similar to $\mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}}$ (under invariance). Take $p \in \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$. Then $epe \in \mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}}$, which means $epe \in Q[x, x^{-1}, \partial]^{\mathbb{Z}/p\mathbb{Z}} = (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$. Consider epe acts on U_l. Specifically, we consider epe acts on $x^m(\frac{x}{|x|})^l|x|^{2c[l]_q}$ for some m. Notice that $epex^m(\frac{x}{|x|})^l|x|^{2c[l]_q} = 0$ when $p \nmid m+l$, and $epex^m(\frac{x}{|x|})^l|x|^{2c[l]_q} = x^m(\frac{x}{|x|})^l|x|^{2c[l]_q}$ when $p|m+l$. Thus, we can define U_l^{inv} as following: $U_l^{inv} = |x|^{n_l+p-p\delta_{0,l}}Q[x^p]$.

Then we define $\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}} = \{p \in (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}\,|pU_l^{inv} \subset U_l^{inv}\}$.

We define $\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}}$ and $\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}}$ similarly.

$$\begin{tikzcd}
\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}} \arrow[r, hook] \arrow[d, hook] & (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/p\mathbb{Z}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}} \arrow[r] \arrow[d, hook] & \mathrm{End}_{\mathbb{Z}[c][q]}(\mathbb{Z}[c][q][x, x^{-1}]) \\
(\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP} \arrow[r, hook] \arrow[ur, hook] \arrow[d, hook] & (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP} \arrow[r, hook] \arrow[d, hook] & ((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP} \arrow[r, hook] \arrow[d, hook] & (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{DP} \arrow[ur, hook] \arrow[d, hook] & \\
\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} \arrow[r, hook] & \mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}} \arrow[r, hook] & (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}} \arrow[r, hook] & \mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c}} \arrow[r] & \mathrm{End}_{\mathbb{Q}[c][q]}\,\mathbb{Q}[c][q][x, x^{-1}]
\end{tikzcd}$$

Claim 5.1. $e\,\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c,p}}\,e = (\mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$

Remark 5.2. The proof is contained in [?].

Proposition 5.3. $\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}} = \mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}}$

Proof. We first prove that $\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} \subset \mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$. Take $p \in \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$. Since $eU_l^{inv} = U_l^{inv}$, $epeU_l^{inv} = epU_l^{inv}$. Also, U_l^{inv} is invariant under p, so we know that $epeU_l^{inv} = U_l^{inv}$. Thus by definition of $\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$, $\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} \subset \mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$.

We then show that $\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}} \subset \mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}}$. Take $T \in \mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$. Then $T = epe$, for some $p \in \mathrm{Diff}^{\mathrm{Q}}_{\mathrm{c,p}}$. Consider epe acts on U_l. Notice that $epeU_l = U_l^{inv} \subset U_l$, thus $epe \in \mathcal{H}^{\mathrm{Q}}_{\mathrm{c,p}} = \mathrm{H}^{\mathrm{Q}}_{\mathrm{c,p}}$, so $T \in \mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}}$. □

Claim 5.4. $(\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}} = ((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP}$

Proof. By section 2, we know that $\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}} = (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{DP}$.

By definition, $(\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}} \subset ((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP}$. Thus we only need to show that $((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP} \subset (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$.

Take $p \in ((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP}$. Since it's in $(\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{DP}$, it's also in $\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}}$. Since it's invariant under $\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}}$, $p \in (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$, which means that $((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP} \subset (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$, so we know $((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP} = (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}}$. □

Claim 5.5. $\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}} = (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$

Proof. By definition, we know that $\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}} \subset (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$. Now we want to show $(\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP} \subset \mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}}$. Take $p \in (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$. By definition of $(\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$, $p \in (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$ is equivalent to $(p \in ((\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}})^{DP}) \wedge (pU_l^{inv} \subset U_l^{inv})$. By claim 5.3, it's equivalent to $p \in (\mathrm{Diff}^{\mathrm{Z}}_{\mathrm{c}})^{\mathbb{Z}/\mathrm{p}\mathbb{Z}} \wedge (pU_l^{inv} \subset U_l^{inv})$, which means $p \in \mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}}$.

This proves that $\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}} = (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$. □

Claim 5.6. $(\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP} = \mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}}$.

Proof. Since $\mathrm{B}^{\mathrm{Q}}_{\mathrm{c,p}} = \mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}}$, $(\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP} = (\mathcal{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$. Also, we proved that $\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}} = (\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}})^{DP}$, so it follows that $\mathrm{B}^{\mathrm{Z}}_{\mathrm{c,p}} = (\mathcal{B}^{\mathrm{Q}}_{\mathrm{c,p}})^{DP}$. □