Tate's thesis and arithmetic zeta functions

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Abstract

This paper is a mostly expository account of zeta functions in number theory. In Section 2 we prove the functional equation for the Dedekind zeta function using the methods of Tate's thesis. In Section 3 we introduce a natural generalization to zeta functions of schemes and show that for well-behaved schemes of finite type over the spectrum of a subring of $\mathbb{Q}$ the zeta functions can be extended by $\frac{1}{2}$ beyond their abscissa of convergence, slightly generalizing a classical result. Finally we briefly mention class field theory and adelic methods in higher dimensions.

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1 Introduction

In 1740, Leonhard Euler introduced the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

on $\operatorname{Re} s > 1$, with the equivalence of the definitions equivalent to the fundamental theorem of arithmetic. It acquired its name in 1859 when Riemann showed that it satisfied the functional equation

$$\Lambda(1-s) = \Lambda(s)$$

for $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. His ideas were critical to the proof of the prime number theorem

$$\pi(x) \sim \frac{x}{\log x}$$

in 1896 by Hadamard and de la Vallée Poussin, where $\pi(x)$ is the number of primes less than or equal to x. Four years after Riemann's paper, Richard Dedekind generalized the Riemann zeta function to the Dedekind zeta function for general number fields K

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ I \neq (0)}} \frac{1}{N(I)^s} = \prod_{\mathfrak{p}} \frac{1}{1-N(\mathfrak{p})^{-s}}$$

where the sum is taken over nonzero ideals of $\mathcal{O}_K$, the product is over nonzero prime ideals of $\mathcal{O}_K$, and $N(I) = |\mathcal{O}_K/I|$. The Riemann zeta function is the special case for $K = \mathbb{Q}$. The Dedekind zeta function in turn is at the heart of Landau's prime ideal theorem and many other results in algebraic number theory. Riemann's argument did not immediately seem to apply to the general case, and it was more than fifty years before Erich Hecke managed to modify the proof to find the functional equation for the Dedekind zeta function

$$\Lambda(s) = \Lambda(1-s)$$

for $\Lambda(s) = |\Delta_K|^{s/2}\left(\pi^{-s/2}\Gamma(s/2)\right)^{r_1}\left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s)$ where r_1 and r_2 are the number of real and complex embeddings of K respectively, as well as for the larger class of Hecke L-functions

$$L(s,\chi) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ (I,m)=1}} \frac{\chi(I)}{N(I)^s} = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s}}$$

where χ is a Hecke character with modulus m with a mildly more complicated functional equation. Emil Artin only a few years later generalized the functional equation to Artin L-functions, where we allow χ in the definition of the Hecke L-function to be the trace of any representation of the associated Galois group, and his student Margaret Matchett wrote her thesis in 1946 on interpreting zeta functions as products over ideles, introduced in 1936 by Claude Chevalley.

In 1950 John Tate, another student of Artin, used Matchett's interpretation to reprove the functional equation for Hecke L-functions. Whereas in Hecke's proof there seemed to be extra apparently artificial factors required to make the functional equation hold, in Tate's proof the completed zeta function emerged naturally, with the extra factors corresponding to archimedean places and the L-function to the nonarchimedean factors.

There are two main sections to this paper. The first is an exposition of Tate's thesis, proving the functional equation for the Dedekind zeta function. We specialize to number fields and to unramified characters, i.e. limiting ourselves to the Dedekind zeta function rather than more general Hecke L-functions, for simplicity of exposition, but the same methods can be generalized to handle both Hecke L-functions and function fields simultaneously.

The second section introduces the notion of schemes and discusses some of their properties before introducing the zeta function for schemes

$$\zeta_X(s) = \prod_{x \in \overline{X}} \frac{1}{1 - |k(x)|^{-s}}.$$

In this view, the Riemann zeta function is $\zeta_{\operatorname{Spec}\mathbb{Z}}$ and the Dedekind zeta function is $\zeta_{\operatorname{Spec}\mathcal{O}_K}$. In both cases we are viewing $\mathbb{Z}$ and $\mathcal{O}_K$ respectively as one-dimensional rings with spectra corresponding to one-dimensional topological spaces, so the natural generalization is to higher-dimensional spaces. We prove that for irreducible schemes X of finite type over the spectrum S of a subring of $\mathbb{Q}$ satisfying certain technical conditions the associated zeta functions are similar in the sense that

$$\frac{\zeta_X(s + \dim X - 1)}{\zeta_S(s)^\ell}$$

can be extended to a meromorphic function on $\operatorname{Re} s > \sigma - \frac{1}{2}$ holomorphic on $\operatorname{Re} s > \frac{1}{2}$, where $\operatorname{Re} s = \sigma$ is the abscissa of convergence of $\zeta_S(s)$. This implies that if $\zeta_S(s)$ can be meromorphically extended to $\operatorname{Re} s > \sigma - \frac{1}{2}$ then so can $\zeta_X(s + \dim X - 1)$, that the pole of $\zeta_X(s)$ at $s = \sigma + \dim X - 1$ looks like the pole of $\zeta_S(s)$ at $s = \sigma$ up to residue, and that the zeros of $\zeta_X(s)$ and $\zeta_S(s + \dim X - 1)$ coincide and are of the same order on $\operatorname{Re} s > \frac{1}{2}$. At least the first two of these facts and likely the third are already known for the case $S = \operatorname{Spec}\mathbb{Z}$, and the only original research in this paper is extending the result to all subrings of $\mathbb{Q}$. It may be possible to extend it further to subrings of number fields generally or potentially to all Dedekind domains or even higher-dimensional schemes with suitable hypotheses.

Finally we briefly discuss higher-dimensional class field theory and attempts to apply the methods of Section 2 to the problem posed in Section 3.

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2 Tate's thesis

This section is mainly based on Tate's thesis itself [20] and Professor Poonen's expository notes [17].

Consider a number field K, that is a finite field extension of $\mathbb{Q}$. The completions of K correspond precisely to the places of K, that is the equivalence classes of absolute values on K. These places are either archimedean (infinite) which correspond to the embeddings of K in the real or complex numbers, or nonarchimedean (finite, which correspond to nonzero prime ideals of the ring of integers $\mathcal{O}_K$ of K. Tate's idea was to construct an object which would behave like the product of the completions, or local fields, together with a certain function that would factor into contributions from the local fields. This gives a finitely many factors from the infinite places times a product over primes, which for the correct construction will be the Dedekind zeta function.

Now that we have the zeta function naturally arising out of this construction, we want to use this interpretation to find its analytic continuation to the complex plane. We will do this by first finding a local functional equation for each factor and then using a generalized form of Poisson summation to get a global functional equation for an object known as the zeta integral. Then by combining these we get a functional equation for the zeta function itself.

2.1 Places

Definition 2.1.1. A function $|\cdot| : K \to \mathbb{R}_{\geq 0}$, where K is a field and $\mathbb{R}_{\geq 0}$ is the group of nonnegative reals, is called an *absolute value* if it satisfies the following properties:

i) $|x| = 0$ if and only if $x = 0$;

ii) for all $x, y \in K$ we have $|x \cdot y| = |x| \cdot |y|$;

iii) for all $x, y \in K$ we have $|x + y| \leq |x| + |y|$.

The trivial absolute value is the map $|0| = 0$, $|x| = 1$ for $x \neq 0$.

We say that two absolute values $|\cdot|_1$, $|\cdot|_2$ are equivalent if there exists some positive real number c such that $|\cdot|_1 = |\cdot|_2^c$.

Definition 2.1.2. The *places* of K are the equivalence classes of nontrivial absolute values on K.

We say that an absolute value $|\cdot|$ is nonarchimedean or finite if it satisfies $|x+y| \leq \max(|x|, |y|)$ for every $x, y \in K$, and archimedean or infinite otherwise. We see that this is in fact a property of the place: if $|\cdot|_1$ and $|\cdot|_2$ are equivalent and $|\cdot|_1$ is finite, then there exists some positive c such that $|x + y|_2 = |x + y|_1^c \leq \max(|x|_1, |y|_1)^c = \max(|x|_1^c, |y|_1^c) = \max(|x|_2, |y|_2)$, so $|\cdot|_2$ is also finite. By symmetry $|\cdot|_1$ is finite if and only if $|\cdot|_2$ is, so each place can be classified as finite or infinite.

We will generally be concerned only with places rather than absolute values, primarily because of the following lemma.

Lemma 2.1.3. *The completions of a field K with respect to two absolute values are isomorphic if and only if the absolute values are equivalent.*

Proof following [15]. If $|\cdot|_1$ and $|\cdot|_2$ are equivalent, then defining a metric on K by $d(x, y) = |x-y|$ for $|\cdot|$ either absolute value and topologizing K with respect to this metric we see that the two absolute values define the same open sets, so the topologies are equivalent and so the completions are isomorphic.

Now suppose that the completions with respect to $|\cdot|_1$ and $|\cdot|_2$ are isomorphic. Then a sequence $\{a_n\}$ in K converges to 0 in the completion with respect to $|\cdot|_1$ if and only if it converges to 0 in the completion with respect to $|\cdot|_2$. Consider the sequence $a_n = x^n$ for some $x \in K$. This converges to 0 with respect to an absolute value $|\cdot|$ if and only if $|x| < 1$, so $|x|_1 < 1$ if and only if $|x|_2 < 1$. Choose some $x, y \in K$ with $|y|_1 > 1$ and let $\{a_n/b_n\}$ be a decreasing sequence of rational numbers converging to $\alpha = \frac{\log|x|_1}{\log|y|_1}$. Then $|x|_1 = |y|_1^\alpha < |y|_1^{a_n/b_n}$ for every natural n, so $\frac{|x|_1^{b_n}}{|y|_1^{a_n}} = \left|\frac{x^{b_n}}{y^{a_n}}\right|_1 < 1$, so by the above it follows that $\left|\frac{x^{b_n}}{y^{a_n}}\right|_2 < 1$, so $|x|_2 < |y|_2^{a_n/b_n}$. Taking $n \to \infty$ gives $|x|_2 \le |y|_2^{\alpha}$. If we take a_n/b_n to instead be an increasing sequence of rational numbers converging to α then the same argument gives $|x|_2 \ge |y|_2^{\alpha}$, so $|x|_2 = |y|_2^{\alpha}$. Therefore $\alpha = \frac{\log|x|_1}{\log|y|_1} = \frac{\log|x|_2}{\log|y|_2}$, so $\frac{\log|x|_1}{\log|x|_2} = \frac{\log|y|_1}{\log|y|_2}$. Fixing some y with $|y|_1 > 1$, since x is arbitrary we get $|x|_1 = |x|_2^c$ for some real c. Since $|y|_1 > 1$, we know that $|y_2| > 1$, so $\frac{\log|y|_1}{\log|y|_2} = \frac{\log|x|_1}{\log|x|_2} = c$ is positive. $\square$

Specializing to the case of a number field allows us to describe the places explicitly.

Definition 2.1.4. The *ring of integers* $\mathcal{O}_K$ of a field K containing $\mathbb{Z}$ is the set of elements of K which are zeros of monic polynomials in $\mathbb{Z}[x]$.

Example 2.1.5. Let $K = \mathbb{Q}[i]$. Then the ring of integers of K is the set of complex numbers of the form $a + bi$ for a and b integers.

We can easily define certain places. The embeddings $i : K \hookrightarrow \mathbb{R}$ of K into $\mathbb{R}$, if any exist, each induce a place via the standard absolute value $|\cdot|$ on $\mathbb{R}$: for $x \in K$ we define $|x|_i = |i(x)|$. Similarly the embeddings $i : K \hookrightarrow \mathbb{C}$ of K into $\mathbb{C}$ which are not purely real induces a place for each pair $i, \bar{i}$ according to $|x|_{\{i,\bar{i}\}} = |i(x)|^2$, where $|\cdot|$ is the standard absolute value on the complex numbers. We will refer to these as real and complex places respectively. All of these are archimedean places. We can also easily define nonarchimedean places: for $x \in K$, we can write

$$x\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{a_{\mathfrak{p}}(x)},$$

where the product is over prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ and the $a_k(x)$ are integers, all but finitely many of which are 0. Then we can define the nonarchimedean absolute value $|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-a_{\mathfrak{p}}(x)}$ and a corresponding distinct nonarchimedean place for each prime ideal $\mathfrak{p}$. In fact we claim that these are all the places of K.

Theorem 2.1.6 (Ostrowski's theorem for number fields)**.** *The only archimedean places of a number field K are those induced by its real and complex embeddings, and the only nonarchimedean places are the $\mathfrak{p}$-adic places with representatives $|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-a_{\mathfrak{p}}(x)}$.*

We first need a lemma describing the archimedean places of $\mathbb{Q}$.

Lemma 2.1.7. *The only archimedean place of $\mathbb{Q}$ is the equivalence class of the usual archimedean absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}$.*

Proof. Suppose that $|\cdot|$ is an archimedean absolute value on $\mathbb{Q}$. For any natural x we have $|x| = |1 + 1 + \cdots + 1| \le x|1| = x$, since $|1 \cdot y| = |y| = |1| \cdot |y|$ so $|1| = 1$. Then for any two natural numbers m, n both greater than 1 we can write $m = a_0 + a_1 n + \cdots + a_r n^r$ for $a_i \in \{0, 1, \ldots, n-1\}$ and $n^r \le m$, so $r \le \frac{\log m}{\log n}$ and $|a_i| \le a_i \le n$, so

$$|m| \le \sum_{j=0}^{r} |a_j||n|^j \le \sum_{j=0}^{r} n \cdot |n|^j \le (1+r)n \cdot |n|^r \le \left(1 + \frac{\log m}{\log n}\right) n \cdot |n|^{\log m/\log n},$$

so $|m| \le |n|^{\log m/\log n}$. Since m and n were arbitrary, we can do the same argument with their places reversed, so we have $|m| = |n|^{\log m/\log n}$, or $\frac{\log|m|}{\log m} = \frac{\log|n|}{\log n}$ for all natural m, n greater than

1. Fixing say n gives $\frac{\log|m|}{\log m}$ equal to a positive real constant, say c, so $|m| = m^c$ for some positive real c, so recalling that $|1| = 1$ on the naturals $|\cdot|$ is equivalent to the standard absolute value. Extending it to the rationals by multiplicativity gives the result on all of $\mathbb{Q}$. $\square$

Proof of Theorem 2.1.6, following [15]. Suppose that $|\cdot|$ is an archimedean absolute value on K, and $\overline{K}$ is the completion of K with respect to $|\cdot|$. Then since K contains $\mathbb{Q}$, $\overline{K}$ contains the completion of $\mathbb{Q}$ with respect to $|\cdot|$. The restriction of $|\cdot|$ to $\mathbb{Q}$ is an archimedean absolute value, and so by Lemma 2.1.7 must be equivalent to the standard absolute value on the reals restricted to $\mathbb{Q}$, so $\overline{K}$ must contain the real numbers, the smallest field containing $\mathbb{Q}$ complete with respect to this absolute value. Therefore we can define $f_x(z) = |x^2 - (z+\bar{z})x + z\bar{z}|$ where $x \in \overline{K}$ and z is complex, noting that $z + \bar{z}$ and $z\bar{z}$ are both real and therefore in $\overline{K}$. Since f_x is continuous in z and tends to $+\infty$ as $|z| \to \infty$ it takes a minimum value m. Let z_0 be the point such that $f_x(z_0) = m$ and if $f_x(z) = m$ then $|z| \leq |z_0|$.

Suppose that $m > 0$ and consider the polynomial $g(y) = y^2 - (z_0 + \bar{z}_0)y + z\bar{z} + a$ for some $0 < a < m$. Let z_1 and $\bar{z}_1$ be the roots of g. Then $z_1\bar{z}_1 = z_0\bar{z}_0 + a$, so that $|z_1| > |z_0|$, so by the definition of z_0 we have $f(z_1) > m$.

Fix some natural number n and let $G(z) = (g(z) - a)^n - (-a)^n$. Then $G(z_1) = (g(z_1) - a)^n - (-a)^n = (-a)^n - (-a)^n = 0$. Suppose that the roots of $G(z)$ are $z_1, \ldots, z_{2n}$. Then for $x \in \overline{K}$ as before

$$|G(x)|^2 = \prod_{j=1}^{2n} |x - z_j|^2 = \prod_{j=1}^{2n} (x^2 - (z_j + \bar{z}_j)x + z_j\bar{z}_j) = \prod_{j=1}^{2n} f_x(z_j) \geq f_x(z_1)m^{2n-1}.$$

On the other hand

$$|G(x)| \leq |x^2 - (z_0 + \bar{z}_0) + z_0\bar{z}_0|^n + |-a|^n = f_x(z_0)^n + a^n = m^n + a^n,$$

so $f_x(z_1)m^{2n-1} \leq (m^n + a^n)^2$, so

$$f_x(z_1) \leq m\left(1 + \left(\frac{a}{m}\right)^n\right)^2.$$

Since $a < m$, taking $n \to \infty$ gives $f_x(z_1) \leq m$. But from above we have $f_x(z_1) > m$, so our above assumption was incorrect and $m \leq 0$, and in fact $m = 0$ since f_x is manifestly positive. Therefore there exists some z_0 such that $x^2 - (z_0 + \bar{z}_0)x + z_0\bar{z}_0 = 0$, so every $x \in \overline{K}$ is the root of a real quadratic polynomial, so $\overline{K}$ can be embedded in either $\mathbb{R}$ or $\mathbb{C}$, and so so can K. If the roots of all such polynomial are real then the embedding is real and unique; if some are complex then there are two possible complex embeddings which are conjugate to each other.

We have shown that each archimedean absolute value on K has a corresponding real embedding i or pair of conjugate complex embeddings $i, \bar{i}$. If two absolute values correspond to the same real embedding i or complex embeddings $i, \bar{i}$ then the completions of $i(K)$ in $\mathbb{R}$ or the completions of $i(K)$ and $\bar{i}(K)$ in $\mathbb{C}$ are the same, so as the embeddings are isomorphisms the completions $\overline{K}$ are the same, so by Lemma 2.1.3 the absolute values are equivalent. Therefore the map from (real and complex) archimedean absolute values to (real and complex) embeddings is injective. Since every embedding gives rise to an absolute value as described in the discussion before the theorem, the map is also surjective, so the archimedean absolute values are as claimed.

Now suppose that $|\cdot|$ is a nonarchimedean valuation. For a natural number n we have $|n| = |1 + 1 + \cdots + 1| \leq |1| = 1$. For $x \in \mathcal{O}_K$ there exist integers $a_0, \ldots, a_{n-1}$ such that $a_0 + a_1x + \cdots +$

$a_{n-1}x^{n-1} + x^n = 0$. Therefore $|x|^n = |-a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}| \leq \max_j(a_j|x|^j) \leq \max_j |x|^j$ where j ranges over the integers from 0 to $n-1$. Therefore there exists a j in this range such that $|x|^n \leq |x|^j$, so $|x| \leq 1$.

If every element of $\mathcal{O}_K$ has absolute value 1, then since $K = \operatorname{Frac}\mathcal{O}_K$ the absolute value is trivial, so let I be the set of elements of $\mathcal{O}_K$ with norm less than 1. By the multiplicativity of the norm, the fact that it is nonarchimedean, and the fact shown above that $|x| \leq 1$ for $x \in \mathcal{O}_K$, this is an ideal of $\mathcal{O}_K$. Suppose that $xy \in I$ for $x, y \in \mathcal{O}_K$, that is $|xy| < 1$. Since $|x| \leq 1$ and $|y| \leq 1$ it follows that at least one of x and y has absolute value less than 1, so I is a prime ideal, and we have seen that it is not the zero ideal.

Write

$$x\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{a_\mathfrak{p}(x)}$$

for $x \in \mathcal{O}_K$ where the product is taken over prime ideals of $\mathcal{O}_K$, with $a_\mathfrak{p}(x) \geq 0$ and equal to 0 for all but finitely many prime ideals. Then $|x| < 1$ if and only if $x \in I$, that is if $a_I(x) \geq 1$. Writing $x\mathcal{O}_K = y\mathcal{O}_K \cdot I^n$ for some $y\mathcal{O}_K$ relatively prime to I where $n = a_I(x)$, we have $|x| = |y| \cdot |I|^n$ for some norm $|I|$ assigned to the prime ideal I. Since $y \notin I$ we have $|x| = |I|^n$ and $|x| < 1$, so $|I| < 1$, so we can choose some positive constant c such that $|I|^c = N(I)^{-1}$, so that $|x| = N(I)^{a_I(x)}$. Therefore every nonarchimedean absolute value is of the form claimed, and since all such are nonarchimedean absolute values the result follows. $\square$

2.2 Review of required analysis

We will assume a vague familiarity with the language of algebraic number theory, basic algebra, complex analysis, and measure theory. The analysis is generally fairly simple, but for reference we list a few useful definitions.

Definition 2.2.1. A *Haar measure* on a locally compact abelian group is a measure μ invariant under the action of the group: $\mu(g + S) = \mu(S)$ for every measurable set S.

Given a locally compact abelian group, a Haar measure always exists and is unique up to a scaling constant.

Definition 2.2.2. A Schwartz function on the real or complex numbers is an infinitely differentiable function $f(x)$ such that every derivative of f tends to 0 as $|x| \to \infty$ faster than x^{-n} for any natural n.

Definition 2.2.3. A Schwartz-Bruhat function on the real or complex numbers in the first case or on a finite extension of the p-adic numbers in the second case is a Schwartz function in the first case or a locally constant function of compact support in the second.

The main importance of Schwartz-Bruhat functions for our purposes is that for an appropriately defined Fourier transform on any of the fields on which we have defined them they obey the Fourier inversion theorem.

2.3 Local theory

Fixing a place v of K, we can now define the fundamental objects of the local theory. Let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}_K$ corresponding to v, $K_v = K_\mathfrak{p}$ be the completion of K at v or equivalently at $\mathfrak{p}$, and $\mathcal{O}_v = \mathcal{O}_v$ be the ring of integers of K_v. We choose an absolute value $|\cdot|_v$ in v on K normalized by $|\mathfrak{p}|_v = N(\mathfrak{p})^{-1}$ if v is nonarchimedean, the standard absolute value on the reals if v is real, and the square of the standard absolute value if v is complex (although this is not strictly an absolute value, it will work as one and satisfactorily represents the place; the square comes from the two equivalent embeddings i and $\bar{i}$).

We normalize the measure dx on the reals to be the standard Lebesgue measure, on the complex numbers to be twice the Lebesgue measure corresponding to the chosen absolute value $|\cdot|^2$, and on the $\mathfrak{p}$-adic numbers to be such that $\int_{\mathcal{O}_v} dx = N(\mathfrak{d})^{-\frac{1}{2}}$, where $\mathfrak{d}$ is the different ideal of $\mathcal{O}_v$, so that every measure is self-dual (see the proof of Lemma 2.3.7).

We are now ready to define the main object of the local theory. Let f be a Schwartz-Bruhat function on K_v and $d^\times x$ be the multiplicative Haar measure on $K_v^\times$.

Definition 2.3.1. The *local zeta integral* is

$$Z_v(f,s) = \int_{K_v^\times} f(x)|x|_v^s \, d^\times x.$$

Example 2.3.2. For some intuition on this, consider the case in which v is real. Then

$$Z_v(f,s) = \int_{\mathbb{R}^\times} f(x)|x|^s \frac{dx}{x} = 2\int_0^\infty f(x)x^{s-1}\,dx$$

is just the Mellin transform.

We compute $Z_v(f,s)$ for certain cases of f for v real, complex, or nonarchimedean in the proof of Proposition 2.3.7.

Proposition 2.3.3. *For any Schwartz-Bruhat function f, the local zeta integral $Z_v(f,s)$ converges absolutely for* $\operatorname{Re} s > 0$.

Proof following [17]. Let $\sigma = \operatorname{Re} s$. For $|x|_v$ large, $f(x)$ decays rapidly, since it is Schwartz-Bruhat, so since the integrand is bounded in absolute value by $f(x)x^\sigma$ the integral over $|x|_v > 1$ converges. On the other hand $f(x)$ is bounded for $|x|_v \leq 1$, so it is enough to show that $\int_{0<|x|_v\leq 1} x^\sigma\, d^\times x$ converges. Choose $a \in K_v^\times$ such that $|a|_v < 1$, and let $A_n = \{x : |a|_v^{n+1} < |x|_v \leq |a|_v^n\}$ for every nonnegative integer n. The closure of A_n in $K_v^\times$ is compact and $|x|_v^\sigma$ is finite, so each $I_n = \int_{A_n} |x|_v^\sigma\, d^\times x$ is finite; and $I_{n+1} = |a|_v^\sigma I_n$ since $d^\times x$ is a Haar measure with respect to multiplication, so the integral over $0 < |x|_v \leq 1$ is the sum of a geometric series, which converges absolutely if $\sigma > 0$. □

Corollary 2.3.4. *As a function of s, $Z_v(f,s)$ is holomorphic on* $\operatorname{Re} s > 0$.

Proof. By Lemma 2.3.3 $Z_v(f,s)$ is absolutely convergent for $\operatorname{Re} s > 0$, so we can differentiate under the integral, and since the integrand is differentiable so is $Z_v(f,s)$, so $Z_v(f,s)$ is holomorphic. □

Lemma 2.3.5. *For any Schwartz-Bruhat functions f and g on K_v and s with real part in the open interval* $(0,1)$*, we have*

$$Z_v(f,s)Z_v(\widehat{g},1-s) = Z_v(\hat{f},1-s)Z_v(g,s).$$

Proof. Assume without loss of generality that $d^\times x = \frac{dx}{|x|_v}$. Expanding, we have

$$\begin{aligned}
Z_v(f,s)Z_v(\widehat{g},1-s) &= \int_{K_v^\times} f(x)|x|_v^s \, d^\times x \int_{K_v^\times} \widehat{g}(y)|y|_v^{1-s} \, d^\times y \\
&= \int_{K_v^\times} f(x)|x|_v^s \, d^\times x \int_{K_v^\times} \left(\int_{K_v} g(z)\psi(yz) \, dz \right) |y|_v^{1-s} \, d^\times y \\
&= \int_{K_v^\times} \int_{K_v^\times} \int_{K_v} f(x)g(z)|xy^{-1}|_v^{1-s}\psi(yz)|y|_v \, d^\times x \, d^\times y \, dz \\
&= \int_{(K_v^\times)^3} f(x)g(z)|xy^{-1}|_v^s \psi(yz)|yz|_v \, d^\times x \, d^\times y \, d^\times z \\
&= \int_{(K_v^\times)^3} f(x)g(z)|xzt^{-1}|_v^s \psi(t)|t|_v \, d^\times x \, d^\times t \, d^\times z
\end{aligned}$$

which is manifestly symmetric in f and g, where we justified exchanging the order of integration by Proposition 2.3.3 and where we set $t = yz$. $\square$

This shows that on $0 < \operatorname{Re} s < 1$ the meromorphic function

$$\frac{Z_v(f,s)}{Z_v(\hat{f},1-s)}$$

is independent of the choice of f. Therefore it suffices to compute it for a single f. In fact we will see that it is more natural for the purposes of the functional equation to consider the adjustment of the zeta integral by certain terms known as L-factors.

Definition 2.3.6. If v is nonarchimedean, the *local L-factor* $L_v(s)$ is given by $\frac{1}{1-|\varpi|_v^s}$, where ϖ is a uniformizer, or a generator of the maximal ideal $\mathfrak{p}$. If v is archimedean, we define $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ and set $L_v(s) = \Gamma_{\mathbb{R}}(s)$ if v is real and $L_v(s) = \Gamma_{\mathbb{C}}(s)$ if v is complex, where $\Gamma(s) = \int_0^\infty e^{-x}x^s \frac{dx}{x}$ is the gamma function.

In particular, we will be interested in the function

$$\rho_v(s) := \frac{Z_v(f,s)/L_v(s)}{Z_v(\hat{f},1-s)/L_v(1-s)}.$$

Proposition 2.3.7. *For $0 < \operatorname{Re} s < 1$, $\rho(s)$ is 1 if v is archimedean and is $N(\mathfrak{d})^{s-\frac{1}{2}}$ if v is nonarchimedean.*

Proof. First, suppose that v is real. Let $f(x) = e^{-\pi x^2}$. Then $\hat{f} = f$, and

$$Z_v(f,s) = \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s \frac{dx}{x} = 2\int_0^\infty e^{-\pi x^2} x^s \frac{dx}{x}.$$

Letting $t = \pi x^2$, we have $dt = 2\pi x \, dx$, so this is

$$\frac{1}{\pi}\int_0^\infty e^{-t}(t/\pi)^{s/2} \frac{dt}{t/\pi} = \pi^{-s/2}\Gamma(s/2) = \Gamma_{\mathbb{R}}(s) = L_v(s),$$

so $Z_v(f,s)/L_v(s) = 1$, so $\rho(s) = \frac{1}{1} = 1$.

Next, suppose that v is complex. Let $f(x) = e^{-2\pi|x|^2}$. Then we have again $\hat{f} = f$ (note that here the integral in the definition of the Fourier transform is taken over $\mathbb{C}$ rather than over $\mathbb{R}$), and

$$Z_v(f,s) = \int_{\mathbb{C}^\times} e^{-2\pi|z|^2}|z|^{2s}\,d^\times z = 4\pi\int_0^\infty e^{-2\pi r^2} r^{2s-1}\,dr,$$

so letting $t = 2\pi r^2$ we have $dt = 4\pi r\,dr$, so this is

$$\int_0^\infty e^{-t}\left(\frac{t}{2\pi}\right)^{s-\frac{1}{2}}\frac{dt}{\sqrt{\frac{t}{2\pi}}} = (2\pi)^{1-s}\Gamma(s) = \pi\Gamma_{\mathbb{C}}(s) = \pi L_v(s),$$

so $Z_v(f,s)/L_v(s) = \pi$, so $\rho(s) = \frac{\pi}{\pi} = 1$.

Finally, suppose that v is nonarchimedean, with $f(x) = 1_{\mathcal{O}_v}(x)$ equal to 1 if x is a $\mathfrak{p}$-adic integer and 0 otherwise. Its Fourier transform is

$$\hat{f}(y) = \int_{\mathcal{O}_v}\psi(xy)\,dx,$$

so letting $t = xy$ we have $dt = y\,dx$, so

$$\hat{f}(y) = \frac{1}{|y|_{\mathfrak{p}}}\int_{y\mathcal{O}_v}\psi(t)\,dt,$$

so $\hat{f}(y)$ is $\int_{\mathcal{O}_v} dx$ if y is in the inverse different ideal $\mathfrak{d}^{-1}$ of $\mathcal{O}_v$ and 0 otherwise. Since we have chosen our measure dx such that $\int_{\mathcal{O}_v} dx = N(\mathfrak{d})^{-\frac{1}{2}}$, we get $\hat{f}(y) = N(\mathfrak{d})^{-\frac{1}{2}}1_{\mathfrak{d}^{-1}}(y)$.

Note that the computation of these three Fourier transforms, together with the similar computations in the reverse direction, show that $\hat{\hat{f}} = f$ and so prove our earlier claim that these are the normalizations which make each measure self-dual with respect to the standard additive character.

Integrating,

$$Z_v(f,s) = \int_{\mathcal{O}_v\setminus\{0\}}|x|_{\mathfrak{p}}^s\,d^\times x = N(\mathfrak{d})^{-\frac{1}{2}}\frac{1}{1-N(\mathfrak{p})^{-s}} = N(\mathfrak{d})^{-\frac{1}{2}}L_v(s).$$

Write A_m for the set of elements of order m and $\mathfrak{d} = \mathfrak{p}^d$, so that $\mathfrak{d}^{-1}$ is the disjoint union $\bigcup_{m\geq -d} A_m$. Then

$$\begin{aligned}
Z(\hat{f},s) &= N(\mathfrak{d})^{-\frac{1}{2}}\sum_{m=-d}^{\infty}\int_{A_m}|x|_{\mathfrak{p}}^s\,d^\times x = N(\mathfrak{d})^{-\frac{1}{2}}\sum_{m=-d}^{\infty}N(\mathfrak{p})^{-ms}\int_{\mathcal{O}_v}dx \\
&= \frac{N(\mathfrak{d})^{-1}N(\mathfrak{p})^{ds}}{1-N(\mathfrak{p})^{-s}} = \frac{N(\mathfrak{d})^{s-1}}{1-N(\mathfrak{p})^{-s}} = N(\mathfrak{d})^{s-1}L_v(s),
\end{aligned}$$

so

$$\rho(s) = \frac{N(\mathfrak{d})^{-\frac{1}{2}}}{N(\mathfrak{d})^{(1-s)-1}} = N(\mathfrak{d})^{s-\frac{1}{2}}.$$

$\square$

Remark 2.3.8. We can now extend the zeta integral to the entire complex plane via the equation of holomorphic functions $\frac{Z_v(f,s)}{L_v(s)} = \rho_v(s)\frac{Z_v(\hat{f},1-s)}{L_v(1-s)}$, since by Corollary 2.3.4 and the fact that $L_v(s)$ has no zeros the left-hand side defines a holomorphic function on $\operatorname{Re} s > 0$ and the right hand defines a holomorphic function on $\operatorname{Re} s < 1$, since $\rho_v(s)$ is itself holomorphic for all v by Proposition 2.3.7.

2.4 Adeles

Now that we at least to some extent understand the local zeta integrals, we want to see how to assemble them into some sort of global object. The key to this will be a construction called the adeles $\mathbb{A}$.

Definition 2.4.1. Let $\{\alpha\}$ be a set of indices, and for every α associate with it a locally compact abelian group G_α such that for all but finitely many α there exists a subgroup $H_\alpha \subset G$ which is open and compact. Then the *restricted product*

$$\mathbb{A} = \prod_\alpha{}' (G_\alpha, H_\alpha)$$

is the subset of the direct product $\prod_\alpha G_\alpha$ in which every vector $x = (x_\alpha) \in \mathbb{A}$ is such that for all but finitely many α we have $x_\alpha \in H_\alpha$. It is equipped with the topology defined by the basis of open sets those given by

$$\prod_\alpha U_\alpha$$

where each U_α is open in G_α and for all but finitely many of the α we have $U_\alpha = H_\alpha$.

For any restricted product, the additive characters are equivalent to the restricted product of additive characters: that is, there exists a canonical isomorphism $\psi \to (\psi\big|_{G_\alpha})$ taking ψ to the collection of its restrictions such that each $\psi\big|_{G_\alpha}$ further restricted to H_α is 1 for all but finitely many α.

Definition 2.4.2. For any number field K, the *adeles* are defined by the restricted product

$$\mathbb{A} = \prod_v (K_v, \mathcal{O}_v)$$

with the finitely many archimedean places those for which $\mathcal{O}_v$ is undefined.

We can think of the adeles as elements of the direct product, subject to additional restrictions, so that each element corresponds to an infinite vector with elements in the corresponding K_v. The restricted topology ensures that the adeles are locally compact, so that we can define a Haar measure and Fourier theory on them.

We choose the standard additive character on $\mathbb{A}$ to be the direct product of the standard characters ψ_v over v, so that at any x there are finitely many factors of $\psi(x)$ not equal to 1. The adeles are locally compact, so we can define the Fourier transform on $\mathbb{A}$. Let $\psi_a(x) = \psi(ax)$, and let $\widehat{\mathbb{A}}$ denote the group of additive characters of $\mathbb{A}$.

Proposition 2.4.3. *There is a canonical isomorphism of locally compact groups $\Psi : \mathbb{A} \to \widehat{\mathbb{A}}$ taking a to ψ_a.*

Proof. For each place v there is a corresponding canonical isomorphism $\Psi_v : K_v \to \widehat{K}_v$, so the map of restricted products

$$\prod_v{}' (K_v, \mathcal{O}_v) \xrightarrow{\Psi} \prod_v{}' (\widehat{K}_v, \widehat{K_v/\mathcal{O}_v})$$

is an isomorphism. $\square$

Definition 2.4.4. For $\{f_v\}$ a collection of Schwartz-Bruhat functions with respect to the local place v such that for all but finitely many v we have $f_v = 1_{\mathcal{O}_v}$, let $f : \mathbb{A} \to \mathbb{C}$ be the function defined by $f(x) = \prod_v f_v(x_v)$. Then the Schwartz-Bruhat functions on $\mathbb{A}$ are the space of finite $\mathbb{C}$-linear combinations of such functions.

We set the measure to be the Tamagawa measure $dx = \prod_v dx_v$, the product of the local measures, where each dx_v is normalized to be self-dual with respect to the standard additive characters ψ_v. It has the property that for any basic open set of the form $\prod_v U_v$ we have $\int_{\prod_v U_v} dx = \prod_v \int_{U_v} dx_v$.

Let $K \hookrightarrow \prod_v K_v$ be the diagonal embedding $a \to (\ldots, a, a, a, \ldots)$. By a slight abuse of notation we will also refer to the image of this embedding as K as a subset of $\prod_v K_v$. We can also define a norm on the adeles: $|x| = \prod_v |x_v|_v$. Embed K into $\mathbb{A}$ via the diagonal embedding $a \to (a)$.

Proposition 2.4.5. *For any $a \in K^\times$, $|a| = 1$.*

Proof. We have

$$\int_{a\mathbb{A}/K} dx = \frac{1}{|a|} \int_{\mathbb{A}/K} dx,$$

but the map $\mathbb{A}/K \to \mathbb{A}/K$ defined by $x \to ax$ is an isomorphism for $a \in K^\times$, so the two integrals must be equal. Since they are nonzero and finite $|a| = 1$. $\square$

Lemma 2.4.6. *The standard additive character ψ restricted to K is* 1.

Proof. Let $\{a\}_v$ be the fractional part of a with respect to v, which for archimedean v is the standard fractional part and for nonarchimedean v is the rational number with denominator a power of $N(\mathfrak{p}_v)$ such that $a - \{a\}_v$ is in $\mathcal{O}_v$. Then

$$\psi(a) = \exp\left(-2\pi i \sum_v \{a\}_v\right),$$

so it is enough to show that

$$\sum_v \{a\}_v$$

is an integer, and in fact we can reduce to the cases where v is a rational prime or a single real archimedean place, as modulo 1 we can bundle together all prime ideals lying over a single rational prime and all archimedean primes. To do so, let q be a rational prime. Then this is

$$\sum_{p \neq q} \{a\}_p + \{a\}_q - \{x\}_\infty$$

where the first sum is over prime ideals of $\mathcal{O}_K$ not lying over q, the second is over those lying over q, and the last is over archimedean places. The first sum is really a finite sum, since a is a $\mathfrak{p}$-adic integer for all but finitely many $\mathfrak{p}$, and is a q-adic integer, since the denominator of the sum written as a fraction will not be divisible by q, and by definition $\{a\}_q - \{x\}_\infty = -(x - \{a\}_q)$ is a q-adic integer, so $\sum_v \{a\}_v$ is a q-adic integer. Since q was arbitrary this is true for every rational prime q, and the only numbers which are q-adic integers for every q are the rational integers, so $\psi(a) = e^{-2\pi i n(a)}$ for some integer $n(a)$, which is just 1. $\square$

We can now define the Fourier transform

$$\hat{f}(y) = \int_{\mathbb{A}} f(x)\psi(xy)\, dx$$

where multiplication of adeles is defined elementwise: $(x_v)(y_v) = (x_v y_v)$. Let $f : \mathbb{A}/K \to \mathbb{C}$ be a K-periodic function, that is a function such that for all $x \in \mathbb{A}$ and $a \in K$ we have $f(x+a) = f(x)$. Then we can also define the Fourier transform

$$\hat{f}(a) = \frac{1}{\operatorname{Vol} D} \int_D f(x)\psi(ax)\, dx$$

where D is a fundamental domain for $\mathbb{A}/K$, that is a set of adeles such that the disjoint union $\bigcup_{a \in K}(D+a)$ gives the full set of adeles $\mathbb{A}$, and $\operatorname{Vol} D = \int_D dx$. For f continuous and L^1 with $\hat{f}$ in $L^1(K)$ the Fourier inversion theorem

$$f(x) = \sum_{a \in K} \hat{f}(a)\overline{\psi(ax)}$$

holds as usual.

Proposition 2.4.7. *If $f : \mathbb{A} \to \mathbb{C}$ is a Schwartz-Bruhat function which can be written as the product of local Schwartz-Bruhat functions f_v, then its Fourier transform is given by the product of the local Fourier transforms $\hat{f}(y) = \prod_v \hat{f}_v(y_v)$.*

Proof. We can write the Fourier transform as

$$\hat{f}(y) = \int_{K_{v_1}} \int_{K_{v_2}} \cdots f(x)\psi(xy)\, dx = \prod_v \int_{K_v} f_v(x_v)\psi_v(x_v y_v)\, dx_v = \prod_v \hat{f}_v(y_v)$$

since $f_v = 1_{\mathcal{O}_v}$ for all but finitely many v, so we are ensured that the integrals are taken only over $\mathcal{O}_v$ for all but finitely many v. $\square$

Finally we define the group of units of $\mathbb{A}$.

Definition 2.4.8. The *group of ideles* is $\mathbb{A}^\times = \prod{}'(K_v^\times, \mathcal{O}_v^\times)$.

Proposition 2.4.9. *The ideles are precisely the elements of $\mathbb{A}$ with nonzero norm.*

Proof. Since the absolute value is given by a finite product, $|x| = 0$ if and only if $|x_v|_v = 0$ for some place v. Therefore the restriction to $K_v^\times$ is necessary and sufficient, and $\mathcal{O}_K^\times$ is the corresponding open compact group under multiplication. $\square$

We define the multiplicative Haar measure on $\mathbb{A}^\times$ to be $\prod_v d^\times x_v$ where $d^\times x_v = \frac{dx_v}{|x|_v}$ if v is infinite and $d^\times x = \frac{1}{1-N(\mathfrak{p}_v)} \frac{dx_v}{|x|_v}$ otherwise, with the addition factor for finite places so that $\int_{\mathcal{O}_v^\times} d^\times x_v$ is 1 for all but finitely many places v.

2.5 Global theory

We can now define the global analogue of the local zeta integral. Let $f : \mathbb{A} \to \mathbb{C}$ be a Schwartz-Bruhat function.

Definition 2.5.1. The *global zeta integral* is

$$Z(f, s) = \int_{\mathbb{A}^\times} f(x)|x|^s \, d^\times x.$$

Lemma 2.5.2. *If $f : \mathbb{A} \to \mathbb{C}$ is the product over v of local Schwartz-Bruhat functions $f_v : K_v \to \mathbb{C}$ such that for all but finitely many v $f_v = 1_{\mathcal{O}_v}$, then we can factor the global zeta integral as*

$$Z(f, s) = \prod_v Z_v(f_v, s).$$

Proof. Ordering the places of K as in the proof of Proposition 2.4.7, we can write the global zeta integral as

$$Z(f, s) = \int_{A_K^\times} f(x)|x|^s \, d^\times x = \int_{K_{v_1}^\times} \int_{K_{v_2}^\times} \cdots f(x)|x|^s \, d^\times x = \prod_v \int_{K_v^\times} f_v(x)|x|_v^s \, d^\times x_v = \prod_v Z_v(f_v, s)$$

since $f_v = 1_{\mathcal{O}_v}$ for all but finitely many v, so we are ensured that the integrals are taken only over $\mathcal{O}_v$ for all but finitely many v. $\square$

Before examining the properties of the global zeta integral, let's explicitly compute an example.

Example 2.5.3. Let f_v be the function we chose for each place v in the proof of Proposition 2.3.7, that is $f_v(x) = e^{-\pi x^2}$ if v is real, $f_v(x) = e^{-2\pi |x|^2}$ if v is complex, and $f_v(x) = 1_{\mathcal{O}_v}(x)$ if v is nonarchimedean, and let $f(x) = \prod_v f_v(x)$. Then by Lemmas 2.5.2 and 2.3.7, letting r_1 and r_2 be the numbers of real and complex places respectively we get

$$\begin{aligned} Z(f, s) = \prod_v Z_v(f_v, s) &= \left(\prod_{v \text{ real}} L_v(s) \right) \left(\prod_{v \text{ complex}} \pi L_v(s) \right) \left(\prod_{v \text{ finite}} N(\mathfrak{d}_v)^{-\frac{1}{2}} L_v(s) \right) \\ &= |\Delta_K|^{-\frac{1}{2}} \Gamma_{\mathbb{R}}(s)^{r_1} (\pi \Gamma_{\mathbb{C}}(s))^{r_2} \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}, \end{aligned}$$

which we note looks interestingly similar to the Dedekind zeta function.

Proposition 2.5.4. *For any Schwartz-Bruhat function f the global zeta integral $Z(f, s)$ converges absolutely for* $\operatorname{Re} s > 1$.

Proof. Without loss of generality we can assume that $f = \prod_v f_v$ is the product of local functions, since any Schwartz-Bruhat function is a linear combination of such functions, so if the result holds for these then by the linearity of integration it holds for all f. We replace the integrand by its absolute value, so we can assume f nonnegative and real and s real. Decomposing by Lemma 2.5.2, we have

$$Z(f, s) = \prod_v Z_v(f_v, s).$$

For all but finitely many v, all of which are nonarchimedean, $f_v = 1_{\mathcal{O}_v}$, in which case $Z_v(f_v, s) = N(\mathfrak{d}_v)^{-\frac{1}{2}} L_v(s)$ by the proof of Proposition 2.3.7, and by Remark 2.3.8 $\frac{Z_v(f_v,s)}{L_v(s)}$ is holomorphic on the entire complex plane, so since each $L_v(s)$ is holomorphic on $\operatorname{Re} s > 0$ it follows that $Z(f, s)$ converges absolutely if and only if $\prod_{v \text{ finite}} N(\mathfrak{d}_v)^{-\frac{1}{2}} L_v(s)$ does. Splitting this product into $\left(\prod_{v \text{ finite}} N(\mathfrak{d}_v)^{-\frac{1}{2}}\right) \left(\prod_{v \text{ finite}} L_v(s)\right)$, the first product is just $|\Delta_K|^{-\frac{1}{2}}$ where Δ_K is the discriminant of the number field K, since the absolute value of the discriminant is the product of the norms of the local different ideals. The second product is just the Dedekind zeta function of K. Since there are at most $[K : \mathbb{Q}]$ places of K lying over each rational prime p, each corresponding to a prime ideal with norm at least p, for s real and greater than 1 the Dedekind zeta function of K is positive and at most

$$\prod_p (1 - p^{-s})^{[K:\mathbb{Q}]},$$

which is just the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

raised to the $[K : \mathbb{Q}]$-th power. Since the Riemann zeta function is bounded by $\int_1^{\infty} (x + 1)^{-s}\, dx = \frac{2^{1-s}}{s-1}$ it converges absolutely for $\operatorname{Re} s > 1$, so so does the Dedekind zeta function, so so does $Z(f, s)$. $\square$

We would like to find a functional equation for the global zeta integral, so that we can combine it with the functional equations for the local factors by Lemma 2.5.2 to get a functional equation for

$$L(s) := \prod_v L_v(s).$$

However, in order to do so we will need a more powerful method than the direct manipulations we could use in the proof of Lemma 2.3.5. Following in the footsteps of Riemann, we will use a form of generalized Poisson summation.

Lemma 2.5.5. *For any Schwartz-Bruhat function $f : \mathbb{A} \to \mathbb{C}$,*

$$\sum_{a \in K} f(x + a)$$

converges absolutely and uniformly on all compact subsets of $\mathbb{A}$ to a K-periodic function.

Proof. Assume without loss of generality that we can write f as a product of local Schwartz-Bruhat functions f_v. All compact subsets of $\mathbb{A}$ are contained in some set $U = \prod_v U_v$ for the U_v open subsets of K_v with $U_v = \mathcal{O}_v$ for all but finitely many v, so it suffices to show the claim on sets of the form of U. For v nonarchimedean f_v is locally constant of compact support, so we have $f_v(U_v + a)$ identically 0 unless a either has $|a|_v \leq 1$ or is one of finitely many exceptions, so that these a form a fractional ideal I_v of $\mathcal{O}_v$. Therefore the intersection of these over all v is a fractional ideal I of $\mathcal{O}_K$. Multiplying by the archimedean f_v which go to 0 rapidly and summing over this ideal gives absolute and uniform convergence on S. Further the sum is K-periodic, as adding an element of K to the input permutes the sum but by absolute convergence gives the same result. $\square$

Theorem 2.5.6. *For any Schwartz-Bruhat function $f : \mathbb{A} \to \mathbb{C}$,*

$$\sum_{a\in K} f(a) = \sum_{a\in K} \hat{f}(a).$$

Proof. Let $F(x) = \sum_{a\in K} f(x+a)$. Note that F is a function on $\mathbb{A}/K$, and so its Fourier transform $\widehat{F}$ is a function on its Pontryagin dual $\widehat{\mathbb{A}/K} \cong K$. Explicitly,

$$\widehat{F}(b) = \frac{1}{\operatorname{Vol} D} \int_D \sum_{a\in K} f(x+a)\psi(bx)\,dx.$$

By Lemma 2.5.5 we can exchange the order of the summation and integration to get

$$\begin{aligned}\widehat{F}(b) &= \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \int_D f(x+a)\psi(bx)\,dx \\ &= \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \int_{D+a} f(x)\psi(b(x-a))\,dx \\ &= \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \psi(-ab) \int_{D+a} f(x)\psi(bx)\,dx.\end{aligned}$$

By Lemma 2.4.6 ψ is trivial on K, so $\psi(-ab) = 1$, so this is

$$\frac{1}{\operatorname{Vol} D} \sum_{a\in K} \int_{D+a} f(x)\psi(bx)\,dx = \int_{\mathbb{A}} f(x)\psi(bx)\,dx = \hat{f}(b).$$

Therefore by Fourier inversion we have

$$F(x) = \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \widehat{F}(a)\overline{\psi(ax)}.$$

Substituting in the definition of $F(x)$ on the left-hand side and the above result on the right, we get

$$\sum_{a\in K} f(x+a) = \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \hat{f}(a)\overline{\psi(ax)},$$

and setting $x = 0$ gives

$$\sum_{a\in K} f(a) = \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \hat{f}(a).$$

Applying this result to $\hat{f}$ gives

$$\sum_{a\in K} \hat{f}(a) = \frac{1}{\operatorname{Vol} D} \sum_{a\in K} \hat{\hat{f}}(a) = \frac{1}{\operatorname{Vol} D} \sum_{a\in K} f(a)$$

by the self-duality of the measure, so $(\operatorname{Vol} D)^2 = 1$, so $\operatorname{Vol} D = 1$, which in combination with the above gives the result. $\square$

Corollary 2.5.7. *For any Schwartz-Bruhat function $f : \mathbb{A} \to \mathbb{C}$,*

$$\sum_{a \in K} f(ax) = \frac{1}{|x|} \sum_{a \in K} \hat{f}(a/x).$$

Proof. Let $g_z(x) = f(zx)$ for an idele z. Then g_x is a Schwartz-Bruhat function with Fourier transform

$$\hat{g}_z(y) = \int_{\mathbb{A}} g_z(x)\psi(xy)\,dx = \int_{\mathbb{A}} f(xz)\psi(xy)\,dx,$$

so letting $u = xz$ we have $du = |z|\,dx$, so this is

$$\frac{1}{|z|} \int_{\mathbb{A}} f(u)\psi(uyz^{-1})\,du = \hat{f}(y/z).$$

Applying Theorem 2.5.6 immediately gives the result. □

For $t > 0$ write $\mathbb{A}_t^\times$ for the set of ideles with norm t, and let

$$Z(f, s; t) = \int_{\mathbb{A}_t^\times} f(x)|x|^s\,d^\times x,$$

so that

$$Z(f, s) = \int_0^\infty Z(f, s; t)\,\frac{dt}{t}.$$

Let

$$g(f, s, t) = Z(f, s; t) + f(0) \int_{\mathbb{A}_t^\times/K^\times} |x|^s\,d^\times x,$$

where $d^\times x$ by an abuse of notation is the induced measure on $\mathbb{A}_t^\times/K^\times$ compatible with both the multiplicative Haar measure $d^\times$ on the ideles and the Haar measure on the group of positive reals $\frac{dt}{t}$.

Lemma 2.5.8. *For any Schwartz-Bruhat function $f : \mathbb{A} \to \mathbb{C}$ we have $g(f, s, t) = g(\hat{f}, 1 - s, t^{-1})$.*

Proof. We have

$$\begin{aligned}
Z(f, s; t) &= \int_{\mathbb{A}_t^\times} f(x)|x|^s\,d^\times x \\
&= \int_{\mathbb{A}_t^\times/K^\times} \sum_{a \in K^\times} f(ax)|ax|^s\,d^\times x \\
&= \int_{\mathbb{A}_t^\times/K^\times} |x|^s \sum_{a \in K^\times} f(ax)\,d^\times x
\end{aligned}$$

since $|a| = 1$ for $a \in K^\times$ by Lemma 2.4.5. In order to apply Theorem 2.5.6 we add a term from $a = 0$ and continue:

$$Z(f, s; t) + f(0) \int_{\mathbb{A}_t^\times/K^\times} |x|^s\,d^\times x = \int_{A_t^\times/K^\times} |x|^s \sum_{a \in K} f(ax)\,d^\times x.$$

By Corollary 2.5.7 this is

$$\int_{\mathbb{A}_t^\times/K^\times} |x|^{s-1} \sum_{a\in K} \hat{f}(a/x)\, d^\times x.$$

Letting $y = \frac{1}{x}$, since $d^\times x$ is invariant under this substitution we have

$$g(f,s,t) = \int_{\mathbb{A}_{1/t}^\times} |y|^{1-s} \sum_{a\in K} \hat{f}(ay)\, d^\times y.$$

But this is the same form as we found before for g with $f \to \hat{f}$, $s \to 1-s$, and $t \to t^{-1}$, so $g(f,s,t) = g(\hat{f}, 1-s, t^{-1})$. $\square$

Let $V = \int_{\mathbb{A}_1^\times/K^\times} d^\times x$.

Lemma 2.5.9. *For any $t > 0$,*

$$\int_{\mathbb{A}_t^\times/K^\times} |x|^s\, d^\times x = Vt^s.$$

Proof. We have

$$\int_{\mathbb{A}_t^\times/K^\times} |x|^s\, d^\times x = t^s \int_{\mathbb{A}_t^\times/K^\times} d^\times x.$$

If a is an idele of norm t, then

$$\int_{\mathbb{A}_t^\times/K^\times} d^\times x = \int_{a\mathbb{A}_1^\times/K^\times} d^\times x,$$

and since $d^\times x$ is invariant under scaling by an idele

$$\int_{a\mathbb{A}_1^\times/K^\times} d^\times x = \int_{\mathbb{A}_1^\times/K^\times} d^\times x = V.$$

Combining these gives the result. $\square$

Theorem 2.5.10. *For any Schwartz-Bruhat function f the global zeta integral extends to a meromorphic function on the complex plane and satisfies the functional equation*

$$Z(f,s) = Z(\hat{f}, 1-s).$$

Further $Z(f,s)$ is holomorphic everywhere except at $s=0$ and $s=1$, at each of which it has simple poles with residues $-f(0)V$ and $\hat{f}(0)V$ respectively.

Proof. Let

$$I_1(f,s) = \int_0^1 Z(f,s;t)\, \frac{dt}{t}, \qquad I_2(f,s) = \int_1^\infty Z(f,s;t)\frac{dt}{t},$$

so that $Z(f,s) = I_1(f,s) + I_2(f,s)$. By Lemma 2.5.9, $g(f,s,t) = Z(f,s;t) + f(0)Vt^s$, so by Lemma 2.5.8 we have $Z(f,s;t) = Z(\hat{f}, 1-s; t^{-1} + \hat{f}(0)Vt^{s-1} - f(0)Vt^s$. Therefore

$$\begin{aligned} I_1(f,s) &= \int_0^1 Z(\hat{f}, 1-s; t^{-1}\, \frac{dt}{t} + \hat{f}(0)V \int_0^1 t^{s-1}\, \frac{dt}{t} - f(0)V \int_0^1 t^s\, \frac{dt}{t} \\ &= \int_1^\infty Z(\hat{f}, 1-s; t)\, \frac{dt}{t} - \frac{\hat{f}(0)V}{1-s} - \frac{f(0)V}{s}, \end{aligned}$$

so

$$Z(f,s) = I_1(f,s) + I_2(f,s) = I_2(f,s) + I_2(\hat{f}, 1-s) + \frac{\hat{f}(0)V}{1-s} + \frac{f(0)V}{s}.$$

This is manifestly symmetric under $f \to \hat{f}$, $s \to 1-s$. Further since $Z(f,s)$ converges for $\operatorname{Re} s > 1$ by Proposition 2.5.4, so since it converges better as $\operatorname{Re} s$ becomes smaller it must converge for all complex s, so $Z(f,s)$ is holomorphic except for the simple poles coming from the latter two terms with locations and residues as claimed. □

Combining this with Lemma 2.3.5 gives the functional equation for the Dedekind zeta function. Let $\Lambda_K(s) = |\Delta_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$, where r_1 and r_2 are the numbers of real and complex places respectively and

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ I \neq (0)}} N(I)^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

is the Dedekind zeta function of K.

Corollary 2.5.11. *For every complex number s not equal to 0 or 1, $\Lambda(s) = \Lambda(1-s)$.*

Proof. Choose $f : \mathbb{A} \to \mathbb{C}$ to be a Schwartz-Bruhat function given by the product of local Schwartz-Bruhat functions f_v. By Lemma 2.5.2 we have

$$Z(f,s) = \prod_v Z_v(f_v, s).$$

Therefore

$$\prod_v \frac{Z_v(f_v,s)}{L_v(s)} = \frac{Z(f,s)}{L(s)}$$

where

$$L(s) = \prod_v L_v(s).$$

On the other hand by Remark 2.3.8 this is

$$\prod_v \rho_v(s) \frac{Z_v(\hat{f}_f, 1-s)}{L_v(1-s)},$$

which separating and again applying Lemma 2.5.2 and Lemma 2.4.7 is

$$\rho(s) \frac{Z(\hat{f}, 1-s)}{L(1-s)}$$

where

$$\rho(s) = \prod_v \rho_v(s).$$

Therefore

$$\frac{Z(f,s)}{L(s)} = \rho(s) \frac{Z(\hat{f}, 1-s)}{L(1-s)}.$$

But by Theorem 2.5.10 $Z(f,s) = Z(\hat{f}, 1-s)$, so this is the statement $L(1-s) = \rho(s)L(s)$. Now by Proposition 2.3.7 $\rho_v(s)$ is $N(\mathfrak{d}_v)^{s-\frac{1}{2}}$ if v is finite and 1 otherwise, so

$$\rho(s) = \left(\prod_{\mathfrak{p}} N(\mathfrak{d}_{\mathfrak{p}})\right)^{s-\frac{1}{2}} = |\Delta_K|^{s-\frac{1}{2}}.$$

Therefore $L(1-s) = |\Delta_K|^{s-\frac{1}{2}} L(s)$, or more symmetrically $L(1-s)|\Delta_K|^{(1-s)/2} = L(s)|\Delta_K|^{s/2}$. Further $L(s)$ is the product of of r_1 factors of $L_{\mathbb{R}}(s) = \Gamma_{\mathbb{R}}(s)$, r_2 factors of $L_{\mathbb{C}}(s) = \Gamma_{\mathbb{C}}(s)$, and a factor of $\frac{1}{1-N(\mathfrak{p})^{-s}}$ for each nonarchimedean place v at a nonzero prime ideal $\mathfrak{p}$. Therefore

$$L(s) = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2} \prod_{\mathfrak{p}} \frac{1}{1-N(\mathfrak{p})^{-s}} = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s),$$

so $|\Delta_K|^{s/2}L(s)$ is both equal to $\Lambda(s)$ and is symmetric under $s \to 1-s$. □

Example 2.5.12. We can recover the functional equation for the Riemann zeta function by setting $K = \mathbb{Q}$, so that $\Delta_{\mathbb{Q}} = 1$, $r_1 = 1$, and $r_2 = 0$, so $\Lambda_{\mathbb{Q}}(s) = \Gamma_{\mathbb{R}}(s)\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is symmetric under $s \to 1-s$.

Remark 2.5.13. The theory extends naturally to Hecke L-functions by viewing the functions $|\cdot|_v^s$ as multiplicative characters of the local field K_v and generalizing to other "ramified" characters. Similarly the global theory views $|\cdot|^s$ as the unramified character and generalizes to Hecke characters, which in this view are the characters of the idele class group $\mathbb{A}^\times/K^\times$. This extension is conceptually not much more difficult but computationally intensive; the interested reader should feel free to explore it themselves.

3 Schemes and the arithmetic zeta function

We introduce another generalization of the Riemann and Dedekind zeta functions, this time with a more geometric flavor. Each ring of integers $\mathcal{O}_K$ is a one-dimensional ring, in a sense to be made precise below, and can in some sense be viewed as a curve. Therefore it is natural to generalize zeta functions to higher-dimensional spaces. In order to do so rigorously, we need to introduce the notion of schemes. We assume familiarity with commutative algebra, in particular smooth ring maps and the properties of Noetherian rings, and some basic topology.

Much of the expository material on schemes is based on material from the Stacks Project and Peter Nelson's lecture notes [14].

3.1 Schemes

Let R be a commutative ring with identity. Then we define $X = \operatorname{Spec} R$ to be the topological space consisting of the set of prime ideals of R under the Zariski topology, defined by having closed sets

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subseteq \mathfrak{p}\}.$$

In this topology the closed points are the maximal ideals. Henceforth the topology is assumed to be the Zariski topology. For each closed point x of X we can associate to it a field $k(x)$, known as its residue field, given by R/x.

A sheaf of rings $\mathcal{O}_X$ takes each open subset U of X a ring $\mathcal{O}_X(U)$, the ring of regular functions on U, with the requirement that these rings be compatible: if $U \subseteq V$ are open sets of X, then there is a natural restriction map $\mathrm{res}_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ such that $\mathrm{res}_{U,U}$ is the identity, if $V \subseteq W$ then $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$, and for any collection of open sets if there is a corresponding regular function on each which agree on the intersections of the sets then there exists a unique regular function on the union of the sets which is equal to each function when restricted to the corresponding set. An affine scheme X is a locally ringed topological space defined by $\operatorname{Spec} R$ for some ring R equipped with the sheaf of rings $\mathcal{O}_X$ fixed by $\mathcal{O}_X(X) = R$. (By a mild abuse of notation we will write $\operatorname{Spec} R$ for the affine scheme with space $\operatorname{Spec} R$ and sheaf of rings the "structure sheaf" defined by the compatibility relations and $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$.)

Example 3.1.1. Let K be an algebraically closed field. Then $\operatorname{Spec} K[x_1, \ldots, x_n]$ is affine n-space over K.

Example 3.1.2. Keeping K as above, $\operatorname{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ has closed points corresponding by the Nullstellensatz to the zero loci cut out by $f_1, \ldots, f_m$.

Example 3.1.3. The affine scheme $\operatorname{Spec} \mathbb{Z}$ consists of a closed point (p) for each prime p as well as the open or generic point (0), whose closure is the entire space. The residue field of each closed point (p) is $\mathbb{Z}/(p) \cong \mathbb{F}_p$.

More generally, a scheme X is a locally ringed space space which is locally isomorphic to an affine scheme, or explicitly a topological space equipped with a sheaf of rings such that there exists an open cover $\{U_\alpha\}$ of X with each U_α isomorphic to an affine scheme. We can still define the field $k(x)$ for any closed point x of X: let $\mathcal{O}_{X,x}$ be the local ring given by the stalk of the sheaf $\mathcal{O}_X$ at x and let $\mathfrak{m}_{X,x}$ be its maximal ideal. Then $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.

Note that schemes generalize varieties: for an algebraically closed field K, the maximal ideals of the ring $R = K[x_1, \ldots, x_n]/I$ for some ideal $I = (f_1, \ldots, f_m)$ are by the Nullstellensatz $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ for $a_i \in K$, that is to solutions to the system of equations $f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0$, so that the closed points of the affine scheme $\operatorname{Spec} R$ correspond to the variety cut out by these equations. We can similarly view the solution set of polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ over a general field K in its algebraic closure $\overline{K}$ as a scheme, with structure sheaf $K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$.

Similarly projective schemes, which we can form by gluing together affine schemes, generalize projective varieties.

We will need to be able to discuss several properties of schemes and of morphisms between them. Specific references can be found in the references section; generally all of these definitions can be found in the Stacks Project ([1], [5], [11], [16], [18], [19]), and several of the proofs are also due to articles in it.

Definition 3.1.4. A morphism of rings $f : A \to B$ is *of finite type* if B is finitely generated as an A-algebra, or equivalently if $B \cong A[x_1, \ldots, x_n]/I$ for some nonnegative integer n and some ideal I of $A[x_1, \ldots, x_n]$.

Definition 3.1.5. A morphism of schemes $f : X \to Y$ is *locally of finite type* if there exists an affine open cover $\{U_\alpha\}$ of Y such that for every U_α there exists an open cover $\{V_{\alpha\beta}\}$ of $f^{-1}(U_\alpha)$ such that the induced map $\mathcal{O}_Y(U_\alpha) \to \mathcal{O}_X(V_{\alpha\beta})$ is a ring morphism of finite type for every α, β. We further say that f is *of finite type* if the open covers $\{V_{\alpha\beta}\}$ can be chosen to be finite.

Definition 3.1.6. The *Krull dimension* of a ring, if it exists, is the largest nonnegative integer n such that there exists a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ where all inclusions are strict. If there is no such number then the ring is said to have infinite Krull dimension.

Definition 3.1.7. The *dimension* of a scheme X, if it exists, is the maximum over affine open sets $S \subseteq X$ of the Krull dimension of $\mathcal{O}_X(S)$. If there exists such an S with $\mathcal{O}_X(S)$ of infinite Krull dimension or with arbitrarily high dimension then X is said to be infinite-dimensional.

Example 3.1.8. A field K has Krull dimension 0, so that $\operatorname{Spec} K$ is topologically a point, while $K[x_1, \ldots, x_n]$ has Krull dimension n, so affine n-space over K $\operatorname{Spec} K[x_1, \ldots, x_n]$ is n-dimensional. Less intuitively, $\mathbb{Z}$ has Krull dimension 1, so that $\operatorname{Spec} \mathbb{Z}$ is one-dimensional and can be thought of as a curve, while $\mathbb{Z}[x_1, \ldots, x_n]$ is $n + 1$-dimensional.

Definition 3.1.9. We say that a morphism of schemes $f : X \to Y$ is *dominant* or that X dominates Y if the image of f is dense in Y.

Definition 3.1.10. Given two pairs of scheme morphisms $X \to Z$, $Y \to Z$, the *fiber product* of X and Y with respect to Z is the scheme $X \times_Z Y$ equipped with scheme morphisms to X and to Y such that the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

commutes which is universal among such diagrams.

Remark 3.1.11. Since the fiber product is defined by a universal property, it is unique up to a unique isomorphism. In the affine case it is the dual construction to the tensor product of rings, with $X \times_Z Y = \operatorname{Spec}(A \otimes_C B)$ if $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$, and so exists; gluing together the affine subschemes gives existence in the general case. The morphism $X \times_Z Y \to Y$ is the pullback of the morphism $X \to Z$ along the morphism $Y \to Z$.

Example 3.1.12. Let $Z = \operatorname{Spec} \mathbb{Z}$, $Y = \operatorname{Spec} \mathbb{F}_p$, and $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I$ for some ideal I of $\mathbb{Z}[x_1, \ldots, x_n]$. Then $X \times_Z Y = \operatorname{Spec}(\mathbb{Z}[x_1, \ldots, x_n]/I \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{Spec} \mathbb{F}_p[x_1, \ldots, x_n]/\bar{I}$ where $\bar{I}$ is the reduction of I modulo (p).

Proposition 3.1.13. *Let $f : X \to Y$ be a morphism of schemes, let $Z \subseteq Y$ be an open or closed subscheme of Y, and let h be the projection map $X \times_Y Z \to X$. Then $f(x) \in Z \subset Y$ if $x = h(p)$ for some $p \in X \times_Y Z$.*

Proof. Let g be the natural inclusion map $Z \hookrightarrow Y$. Letting f^* be the induced map $X \times_Y Z \to Z$, by the definition of the fiber product we have $f(h(p)) = g(f^*(p))$, so since the image of f^* is Z and g is an inclusion map we get the result. □

In fact we will identify $f^{-1}(Z)$ with the topological space underlying $X \times_Y Z$, and define $f^{-1}(Z)$ as a scheme by equipping it with the sheaf structure of $X \times_Y Z$.

Given a morphism of schemes $f : X \to Y$, we call X_Z the fiber of X lying over Z for any subscheme Z of Y.

Proposition 3.1.14. *If $f : X \to Y$ is a morphism of schemes of finite type and Y is finite-dimensional, then each fiber of X is finite-dimensional.*

Proof. Since Y is finite-dimensional, each ring $\mathcal{O}_Y(U)$ has bounded Krull dimension for each open neighborhood of every $y \subseteq Y$ so since f is of finite type $f^{-1}(U)$ can be covered by finitely many finitely generated $\mathcal{O}_Y(U)$-algebras, each of which is finite-dimensional. Therefore there exists a finite maximum of the Krull dimensions of these rings, so every fiber X_y is finite-dimensional. $\square$

Proposition 3.1.15. *If $f : X \to Y$ and $g : Z \to Y$ are of finite type, then the induced morphism $f^* : X \times_Y Z \to Z$ is of finite type.*

Proof. Cover Y by open affines $\{U_\alpha\}$. Since g is of finite type, for each α there exist a finite set of open affines $\{V_{\alpha\beta}\}$ covering $g^{-1}(U_\alpha)$, for each of which $\mathcal{O}_Y(U_\alpha) \to \mathcal{O}_Z(V_{\alpha\beta})$ is a ring homomorphism of finite type. By Lemma 3.1.13, $(f^*)^{-1}(V_{\alpha\beta}) = (X \times_Y Z) \times_Z V_{\alpha\beta} = X \times_Y V_{\alpha\beta} = (X \times_Y U_\alpha) \times_{U_\alpha} V_{\alpha\beta} = f^{-1}(U_\alpha) \times_{U_\alpha} V_{\alpha\beta}$. Since f is of finite type, we can cover $f^{-1}(U_\alpha)$ with finitely many open affine $W_{\alpha\gamma}$, for each of which $\mathcal{O}_Y(U_\alpha) \to \mathcal{O}_X(W_{\alpha\gamma})$ is a ring homomorphism of finite type. Therefore $(f^*)^{-1}(V_{\alpha\beta})$ is covered by a set of finitely many affines of the form $W_{\alpha\gamma} \times_{U_\alpha} V_{\alpha\beta} = \operatorname{Spec}(\mathcal{O}_X(W_{\alpha\gamma}) \otimes_{\mathcal{O}_Y(U_\alpha)} \mathcal{O}_Z(V_{\alpha\beta}))$, which is the spectrum of a ring which is a finitely generated $\mathcal{O}_Z(V_{\alpha\beta})$-algebra, so by definition f^* is of finite type. $\square$

Definition 3.1.16. We say that a morphism of schemes $f : X \to Y$ of finite type is of *relative dimension d* if for every $x \in X$ we have $\dim X_{f(x)} = d$.

Example 3.1.17. The morphism $\operatorname{Spec} \mathbb{Z}[x] \to \operatorname{Spec} \mathbb{Z}$ is of relative dimension 1, as each fiber is $\operatorname{Spec} \mathbb{F}_p[x]$ for some p, which is one-dimensional (see Example 3.1.8).

Lemma 3.1.18. *If $f : X \to Y$ is of finite type and of relative dimension d and $Z \to Y$ is of finite type, then the induced morphism $f^* : X \times_Y Z \to Z$ is also of finite type and of relative dimension d.*

Proof. Choose a point $p \in X \times_Y Z$. Then the fiber of $X \times_Y Z$ of $f^*(p)$ is determined by the commutative diagram

$$\begin{tikzcd}
X \times_Y \operatorname{Spec} k(f^*(p)) \arrow[r] \arrow[d] & X \times_Y Z \arrow[r] \arrow[d, "f^*"] & X \arrow[d, "f"] \\
\operatorname{Spec} k(f^*(p)) \arrow[r] & Z \arrow[r] & Y
\end{tikzcd}.$$

Since the morphism $\operatorname{Spec} k(f^*(p)) \to Y$ factors through $k(f^*(p)) \to k(f(\bar{p})) \to Y$ where $\bar{p}$ is the projection of $p \in X \times_Y Z$ onto X, we get a commutative diagram

$$\begin{tikzcd}
X \times_Y \operatorname{Spec} k(f^*(p)) \arrow[r] \arrow[d] & X \times_Y \operatorname{Spec} k(f(\bar{p})) \arrow[r] \arrow[d, "f^*"] & X \arrow[d, "f"] \\
\operatorname{Spec} k(f^*(p)) \arrow[r] & \operatorname{Spec} k(f(\bar{p})) \arrow[r] & Y
\end{tikzcd}.$$

We want to show that $X \times_Y \operatorname{Spec} k(f^*(p))$ is d-dimensional for every p if $X \times_Y \operatorname{Spec} f(\bar{p})$ is d-dimensional for every d, so this is equivalent by the second diagram to showing that the result holds for finite field extensions $k(f^*(p))$ of $k(f(\bar{p}))$, which is a finite field extension because the morphism of schemes $\operatorname{Spec} k(f^*(p)) \to \operatorname{Spec} k(f(\bar{p}))$ is dual to the morphism of rings $k(f(\bar{p})) = k(f^*(p))$, so this defines a field extension and since the original morphism was of finite type the field extension must be finite. But in this case each scheme has only one fiber, so it is sufficient to show that if $X \times_Y \operatorname{Spec} K$ is d-dimensional then $X \times_Y \operatorname{Spec} K'$ is d-dimensional if K' is a finite field extension of K. Let $U = \operatorname{Spec} A$ be an affine open subscheme of $X \times_Y \operatorname{Spec} K$ with dimension d. Then A is a finitely-generated K-algebra $A = K[x_1, \ldots, x_n]/I$ with inverse image $\operatorname{Spec}(A \times_K K') = \operatorname{Spec} K'[x_1, \ldots, x_n]/\bar{I}$ which has the same dimension as A, so the dimension of $X \times_Y \operatorname{Spec} K'$ is at least that of $X \times_Y \operatorname{Spec} K$. On the other hand mapping any affine open subscheme of $X \times_Y \operatorname{Spec} K'$ to its image in $X \times_Y \operatorname{Spec} K$ gives something of this form, and all of the affine open subschemes of $X \times_Y \operatorname{Spec} K$ have dimension at most d, so the dimension of $X \times_Y \operatorname{Spec} K'$ can also be at most d. Therefore the dimension of both is d. □

Definition 3.1.19. A scheme is *irreducible* if the underlying topological space is irreducible.

Definition 3.1.20. A scheme is *connected* if the underlying topological space is connected.

Definition 3.1.21. A morphism of schemes $f : X \to Y$ locally of finite type is *smooth* if for every $x \in X$ there exist open affine neighborhoods $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that if $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ then the induced ring homomorphism $B \to A$ is smooth.

In particular if Y is the spectrum of a field then this $X \to Y$ is smooth if it has no singular points. We can partially extend this to general Y via the following proposition.

Proposition 3.1.22. *Smoothness is stable under base change, that is if $X \to Y$ is smooth then $X \times_Y Z \to Z$ is smooth.*

Proof. Smoothness as a property of ring maps is stable under base change via tensor products, so the same is true locally for morphisms and so smoothness is preserved at every point. □

Therefore it is a necessary, though not quite sufficient, condition for $X \to Y$ to be smooth that every fiber over a point be smooth. In order to make this sufficient we would need an extra condition about variation over the fibers, but intuitively we can think of $X \to Y$ being smooth as requiring that all fibers are nonsingular.

Definition 3.1.23. A scheme X is *Noetherian* if there exists an affine open neighborhood U of every point x such that $U = \operatorname{Spec} R$ for a Noetherian ring R and the underlying topological space of X is quasi-compact.

Lemma 3.1.24. *If a scheme has a finite cover by open affine subschemes then it is quasi-compact.*

Proof. If X has such a cover, then since X is locally affine every open cover must contain the affine cover and so must have a finite open subcover. □

Proposition 3.1.25. *If $X \to Y$ is a morphism of schemes of finite type and Y is Noetherian, then X is Noetherian.*

Proof. This reduces to the affine case as usual, where it is the dual statement to the fact that a finitely generated ring over a Noetherian ring is Noetherian. □

Proposition 3.1.26. *A Noetherian scheme has finitely many irreducible components.*

Proof. The underlying topological space is also Noetherian, so the result follows from the fact that Noetherian topological spaces have finitely many irreducible components. □

Proposition 3.1.27. *A Noetherian scheme X smooth over* $\operatorname{Spec} K$ *for K a field is irreducible if and only if it is connected.*

Proof. If X is irreducible then it is connected, so the difficulty lies in establishing the other direction. Suppose X has at least two irreducible components. Since X is connected and Noetherian, they have a nonempty intersection containing some closed point x. Since f is smooth, the induced homomorphism $k \to \mathcal{O}_{X,x}$ is smooth. Since X is Noetherian $\mathcal{O}_{X,x}$ is Noetherian as well as being smooth over k and a local ring, so it must be regular, so $\mathcal{O}_{X,x}$ must be an integral domain. On the other hand U must contain the generic points of both irreducible components. Therefore $\mathcal{O}_{X,x}$ contains two ideals dividing (0), so it cannot be an integral domain. Therefore X must be irreducible. $\square$

Lemma 3.1.28. *For any scheme X and regular function f on X, the zero locus of f, that is the set of points $x \in X$ such that the image of f in $\mathcal{O}_{X,x}/m_{X,x}$ is* 0 *where $\mathcal{O}_{X,x}$ is the local stalk at x and $\mathfrak{m}_{X,x}$ is its maximal ideal, is closed.*

Proof. For every $x \in X$, let $U = \operatorname{Spec} A$ be an open affine neighborhood of x. Then the zero locus of f on U is the set of prime ideals of A containing f, which by the definition of the Zariski topology is closed. Therefore X is everywhere locally closed and so closed. $\square$

Definition 3.1.29. A morphism of schemes $X \to Y$ is *separated* if the diagonal morphism $X \to X \times_Y X$ is a closed immersion.

Proposition 3.1.30. *Separation is stable under base change, that is if $X \to Y$ is separated then $X \times_Y Z \to Z$ is separated.*

Proof. The diagonal morphism $X \times_Y Z \to (X \times_Y Z) \times_Z (X \times_Y Z) = X \times_Y (Z \times_Z X) \times_Y Z = (X \times_Y X) \times_Y Z$ is just the base change of the original diagonal morphism $X \to X \times_Y X$, so since closed immersions are preserved under base change $X \times_Y Z \to Z$ is separated if $X \to Y$ is. $\square$

Definition 3.1.31. A scheme $X \to \operatorname{Spec} K$ for some field K is *geometrically connected* if $X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ is connected for every field extension K' of K. We say that $X \to \operatorname{Spec} R$ is geometrically connected if $X \times_{\operatorname{Spec} R} \operatorname{Spec}(\operatorname{Frac} R) \to \operatorname{Spec}(\operatorname{Frac} R)$ is geometrically connected.

Definition 3.1.32. A scheme $X \to \operatorname{Spec} K$ for some field K is *geometrically irreducible* if $X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ is irreducible for every field extension K' of K. We say that $X \to \operatorname{Spec} R$ is geometrically irreducible if $X \times_{\operatorname{Spec} R} \operatorname{Spec}(\operatorname{Frac} R) \to \operatorname{Spec}(\operatorname{Frac} R)$ is geometrically irreducible.

Example 3.1.33. The scheme $X = \operatorname{Spec} \mathbb{R}[x]/(x^2+1)$ is irreducible, since x^2+1 is a maximal ideal, but is not geometrically irreducible, since x^2+1 splits into $(x+i)(x-i)$ over $\mathbb{C}$.

Definition 3.1.34. A scheme X is *integral* if it has an open cover by affine schemes $\{U_\alpha\}$ with each U_α equal to $\operatorname{Spec} A_\alpha$ for A_α an integral domain.

This is equivalent to requiring that $\mathcal{O}_X(U)$ be an integral domain for every open subscheme U.

Definition 3.1.35. A scheme X is *reduced* if it has an open cover by affine schemes $\{U_\alpha\}$ with each U_α equal to $\operatorname{Spec} A_\alpha$ with A_α having no nilpotent elements.

Proposition 3.1.36. *A scheme which is both irreducible and reduced is integral.*

Proof. Suppose that X is such a scheme which is not integral. Choose an affine irreducible open subscheme $U = \operatorname{Spec} A$. Suppose that f and g are nonzero elements of A such that $fg = 0$. Then the sets V and W of points $x \in U$ on which f and g respectively are 0, that is are in the prime ideal of A associated to x, are both closed subsets of U by Lemma 3.1.28, since they are the zero loci of regular functions, and have union equal to U, since the associated ideals to each x are prime, so since fg is in every maximal ideal by definition at least one of f and g is in each. Since U is irreducible, it follows that one of V or W, say V, must be equal to the entire set U. Therefore f restricted to any open affine subscheme of U is in the intersection of the prime ideals of A and therefore nilpotent, and since X is reduced the only nilpotent element is 0. Therefore A is an integral domain, so since U was arbitrary X must be integral. □

3.2 The zeta function and arithmetic schemes

In light of this perspective on the prime and maximal ideals of a ring, it is natural to look back at the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

Thinking of the primes as the closed points of $\operatorname{Spec} \mathbb{Z}$ and their numerical value as $|\mathbb{Z}/(p)|$, the size of the residue field at p, it is natural to generalize the zeta function as follows.

Definition 3.2.1. Let X be a scheme with set of closed points $\overline{X}$, and let $k(x)$ be the residue field at a closed point x. Then we define the *zeta function of* X to be the function defined on some complex half-plane by

$$\zeta_X(s) = \prod_{x \in \overline{X}} \frac{1}{1 - |k(x)|^{-s}}.$$

Example 3.2.2. If $X = \operatorname{Spec} \mathbb{Z}$, we get the Riemann zeta function $\zeta_X = \zeta$. If $X = \operatorname{Spec} \mathcal{O}_K$ for $\mathcal{O}_K$ the ring of integers of a number field, we get $\zeta_X = \zeta_K$, the Dedekind zeta function.

We interpret $|k(x)|^{-s}$ to be 0 in the case in which $k(x)$ is infinite, so that the product is essentially over x with finite residue field. For the moment we will consider this as a formal product and neglect convergence.

We can view the zeta function as the multiplicative generating function for 0-cycles on X, that is formal sums of closed points of X, possibly with multiplicity, analogous to the integers in the case $X = \operatorname{Spec} \mathbb{Z}$. In particular note that $\zeta_X(s) \in \mathbb{Z}[[p^{-s}]]$.

For schemes of finite type over a finite field we have the following equivalent form. For a variety X over a field K, we can study its solution set in its algebraic closure $\overline{K}$, and then in any subextension $K \subset L \subset \overline{K}$. The points of X lying in L are the L-rational points of X, which we will write as $X(L)$, and also form a variety.

Example 3.2.3. Let $X = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$. Then for example $\left(\frac{3}{5}, \frac{4}{5}\right)$ is a $\mathbb{Q}$-rational point, while $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is an $\mathbb{R}$-rational point and $(i, \sqrt{2})$ is a $\mathbb{C}$-rational point.

We can generalize this by defining $X(L)$ to be the set of morphisms of K-schemes $\operatorname{Spec} L \to X$. For a finite field $K = \mathbb{F}_q$ and a finite extension $L \cong \mathbb{F}_{q^m}$, note that although $X(L)$ depends on the extension all extensions of degree m differ by an isomorphism, so that $N_m = |X(L)|$ depends only on m (as well as implicitly on X and q).

Proposition 3.2.4. *Let $X \to \mathbb{F}_q$ be of finite type. Then*

$$\zeta_X(s) = \exp \sum_{m=1}^{\infty} \frac{N_m}{m} p^{-ms}.$$

Proof. Since $X \to \mathbb{F}_q$ is of finite type, each N_m is finite, and each closed point $x \in X$ has residue field a finite extension of $\mathbb{F}_q$. The elements of $X(\mathbb{F}_{q^m})$ are in one-to-one correspondence with the homomorphisms from $k(x)$ to $\mathbb{F}_{q^m}$ for each x such that $k(x)$ is a subfield of $\mathbb{F}_{q^m}$. Writing $k(x) = \mathbb{F}_{q^r}$, the union is over r dividing m, and there are $\frac{|\operatorname{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)|}{|\operatorname{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_{q^r})|} = \frac{m}{m/r} = r$ homomorphisms from $\mathbb{F}_{q^r} \to \mathbb{F}_{q^m}$, so letting a_r be the number of x with $k(x) = \mathbb{F}_{q^r}$ we get

$$|X(\mathbb{F}_{q^m})| = \sum_{r|m} r a_r.$$

Therefore

$$\zeta_X(s) = \prod_{x \in \overline{X}} \frac{1}{1 - |k(x)|^{-s}} = \prod_{r=1}^{\infty} (1 - q^{-rs})^{-a_r} = \exp \sum_{r=1}^{\infty} a_r \sum_{k=1}^{\infty} \frac{1}{k} q^{-rks}.$$

The inner sum is absolutely convergent for $s > 0$, so we can exchange the order of summation to get

$$\exp \sum_{k=1}^{\infty} \frac{1}{k} \sum_{r=1}^{\infty} a_r q^{-rks} = \exp \sum_{m=1}^{\infty} q^{-ms} \sum_{r|m} a_r \frac{r}{m} = \frac{N_m}{m} q^{-ms}.$$

Since both forms of the zeta function are analytic, they must agree on every domain to which the zeta function can be defined. $\square$

Proposition 3.2.5. *Given a morphism of schemes $f : X \to Y$ we can factor the zeta function of X according to*

$$\zeta_X(s) = \prod_{y \in \overline{Y}} \zeta_{X_y}(s)$$

where X_y is the fiber of X lying over y and ζ_{X_y} is the corresponding zeta function.

Proof. This is just the statement that the set $\overline{X}$ of closed points of X is the disjoint union of the closed points of each X_y, which follows from the facts that if $x_1 \in X_{y_1}$ and $x_2 \in X_{y_2}$ are closed points for $y_1 \neq y_2$ then $f(x_1) = y_1 \neq y_2 = f(x_2)$ and that every closed point of X is mapped to a closed point of Y by continuity, since f inherits a morphism of topological spaces from the scheme structure. $\square$

We will mostly be interested in *arithmetic schemes*, or schemes of finite type over $\operatorname{Spec} \mathbb{Z}$.

Example 3.2.6. Let $\mathbb{A}^n_{\mathbb{Z}}$ be affine space over $\operatorname{Spec} \mathbb{Z}$ defined by $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. Then by Proposition 3.2.5 we can find the zeta function by multiplying together the zeta functions of its fibers over closed points, which correspond to the rational primes p. By Proposition 3.1.13, the fiber of $\mathbb{A}^n_{\mathbb{Z}}$ over p is given by $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} k(p) = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$. Recalling that the fiber product is dual to the tensor product, by Remark 3.1.11 this is $\operatorname{Spec}(\mathbb{Z}[x_1, \ldots, x_n] \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{Spec} \mathbb{F}_p[x_1, \ldots, x_n]$; in general if A is a finitely-generated B-algebra then $A \otimes_B C$ corresponds to the base change from B to C, and so the same holds for fiber products of affine schemes. The zeta function of $\operatorname{Spec}\mathbb{F}_p[x_1,\ldots,x_n]$ by Proposition 3.2.4 is

$$\exp\sum_{m=1}^{\infty}\frac{p^{mn}}{m}p^{-ms}=\exp\log\frac{1}{1-p^{n-s}}=\frac{1}{1-p^{n-s}}$$

since every point $(x_1,\ldots,x_n)\in\mathbb{F}_{p^m}^n$ corresponds to a closed point of $\mathbb{A}^n_{\mathbb{F}_p}$, so $N_m=p^{mn}$. Therefore by Proposition 3.1.13

$$\zeta_{\mathbb{A}^n_{\mathbb{Z}}}(s)=\prod_p\frac{1}{1-p^{n-s}}=\zeta(s-n)$$

where $\zeta(s)$ is the Riemann zeta function.

Example 3.2.7. Let $\mathbb{P}^n_{\mathbb{Z}}$ be projective n-space over $\mathbb{Z}$. By the same method as in the previous example we reduce this to projective n-space over $\mathbb{F}_p$, which consists of the disjoint union of affine 0-space, affine 1-space, etc. up to affine n-space, so that $\zeta_{\mathbb{P}^n_{\mathbb{F}_p}}(s)=\zeta_{\mathbb{A}^0_{\mathbb{F}_p}}(s)\zeta_{\mathbb{A}^1_{\mathbb{F}_p}}(s)\cdots\zeta_{\mathbb{A}^n_{\mathbb{F}_p}}(s)$. From Example 3.2.8 we know that this is $\prod_{k=0}^n\frac{1}{1-p^{k-s}}$, so

$$\zeta_{\mathbb{P}^n_{\mathbb{Z}}}(s)=\prod_p\prod_{k=0}^n\frac{1}{1-p^{k-s}}=\prod_{k=0}^n\zeta(s-k).$$

Example 3.2.8. Let $R=\mathbb{Z}[x,y]/(xy-1)$. This is equal to the direct sum $\mathbb{Z}[x]\oplus\mathbb{Z}[y]$ and so has maximal ideals of the form $(p,f(x))$ and $(q,g(y))$ for polynomials f and g irreducible modulo p and q respectively, so that letting $X=\operatorname{Spec}R$ we have $X_p=\operatorname{Spec}\mathbb{F}_p[x,y]/(xy-1)=\operatorname{Spec}\mathbb{F}_p[x]\cup\operatorname{Spec}\mathbb{F}_p[y]$, so $\zeta_{X_p}(s)=(1-p^{-s})^{-2}$ as in Example , so that

$$\zeta_X(s)=\prod_p\frac{1}{(1-p^{-s})^2}=\zeta(s)^2.$$

3.3 Convergence and continuation

Analogous to the definition of the Riemann zeta function, we gave a definition of the zeta function of an arithmetic scheme for some half-plane and want to extend it to a larger domain by analytic continuation. In some special cases, such as Examples 3.2.8 and 3.2.7, we can define the zeta function on the whole complex plane, except for finitely many well-understood poles. In general, however, we can say much less.

Theorem 3.3.1. *The zeta function $\zeta_X(s)$ of an arithmetic scheme X dominating* $\operatorname{Spec}\mathbb{Z}$ *of dimension r converges and is nonzero for* $\operatorname{Re}s>r$*, and extends to a meromorphic function on* $\operatorname{Re}s>r-\frac{1}{2}$ *with a single pole at $s=r$ of order equal to the number of dominant irreducible components of codimension* 0*.*

We will not prove this theorem directly, but will instead prove a mild generalization in subsection 3.4.

3.4 A generalization

We replace $\mathbb{Z}$ in the definition of an arithmetic scheme with any subring R of $\mathbb{Q}$. Note that a scheme of finite type over $\operatorname{Spec} R$ is not necessarily an arithmetic scheme: suppose $R = \mathbb{Z}\left[\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \ldots\right]$ is the integers adjoin the inverse of every prime congruent to 3 modulo 4. Then R is certainly not finitely generated as a $\mathbb{Z}$-algebra, so $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ is not of finite type; but $\operatorname{Spec} R \to \operatorname{Spec} R$ trivially is.

We aim to prove the following generalization of Theorem 3.3.1.

Theorem 3.4.1. *Let R be a subring of $\mathbb{Q}$ and X be a scheme separated, dominant, and of finite type over $S = \operatorname{Spec} R$ with ℓ irreducible components of codimension 0, all of which are geometrically irreducible, and suppose that $\zeta_S(s)$ converges for $\operatorname{Re} s > \sigma$. Then $\frac{\zeta_X(s+\dim X-1)}{\zeta_S(s)^\ell}$ converges and is nonzero for $\operatorname{Re} s > \sigma$ and extends to the product of a rational function and a holomorphic function on $\operatorname{Re} s > \sigma - \frac{1}{2}$, which is holomorphic on $\operatorname{Re} s > 0$.*

Theorem 3.3.1 follows immediately for arithmetic schemes with every irreducible component geometrically irreducible by choosing $R = \mathbb{Z}$. If X has irreducible components of codimension 0 which are not geometrically irreducible, each contributes another factor to the pole, but the methods necessary are different, using the Lang-Weil bound rather than the Weil conjectures, and do not generalize well, so we will satisfy ourselves with this slightly weaker result.

Example 3.4.2. Choose R to be as above the subring of $\mathbb{Q}$ localized away from primes congruent to 1 or 2 modulo 4, that is $R = \mathbb{Z}\left[\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \ldots\right]$. Then

$$\zeta_S(s) = \prod_{p \equiv 1 \pmod 4} \frac{1}{1 - p^{-s}},$$

which has abscissa of convergence $\operatorname{Re} s = 1$ and as $s \to 1$ we have $\zeta_S(s) = A\sqrt{\zeta(s)} + O(1) = A\sqrt{s-1}^{-1} + O(1)$ for some nonzero multiplicative constant A (approximately 0.8623339.)

In order to see that Theorem 3.4.1 is general, and indeed implies even this more limited form of Theorem 3.3.1, it remains to see that every scheme of finite type over $S = \operatorname{Spec} R$ with all irreducible components of codimension 0 geometrically irreducible has finitely many irreducible components.

Lemma 3.4.3. *Every subring of $\mathbb{Q}$ is Noetherian.*

Proof. Let R be a subring of $\mathbb{Q}$. Since $1 \in R$ we have $\mathbb{Z} \subseteq R$. Let $\mathcal{P}$ be the set of rational primes which are not units in R. Then R contains the localization of $\mathbb{Z}$ away from $\mathcal{P}$, i.e. if T is the complement of $\mathcal{P}$ in the primes than $T^{-1}\mathbb{Z} \subseteq R$. If $p \in \mathcal{P}$, then either (p) is prime in R or $\frac{a}{p} \in R$ for some integer a relatively prime to p, in which case (p) is replaced by (a) for a minimal a, which can be prime only if a is prime and in $\mathcal{P}$. Therefore every prime ideal of R is of the form (p) for some $p \in \mathcal{P}$, and so is finitely generated, so R is Noetherian. $\square$

Proposition 3.4.4. *If R is a subring of $\mathbb{Q}$ and $X \to S = \operatorname{Spec} R$ is of finite type, then X has finitely many irreducible components.*

Proof. By Proposition 3.4.3, Lemma 3.1.24, and Definition 3.1.23, S is Noetherian, since it is affine and R is Noetherian. Therefore by Proposition 3.1.25 X is also Noetherian, and so by Proposition 3.1.26 it has finitely many irreducible components. $\square$

We prove a weaker statement than Theorem 3.4.1 that will be useful later.

Proposition 3.4.5. *Let R be a subring of $\mathbb{Q}$ and X be a scheme of finite type over $S = \operatorname{Spec} R$, and suppose that $\zeta_S(s)$ has abscissa of convergence σ. Then $\zeta_X(s)$ converges and is nonzero for $\operatorname{Re} s > \sigma + \dim X - 1$.*

Proof. By the proof of Proposition 3.4.4, X is Noetherian and therefore quasi-compact, so an open cover of X by affines has a finite subcover, so there exists a finite cover of X by open affines. Since X is of finite type over S, each of these open affines U is the spectrum of a finitely-generated R-algebra A of finite Krull dimension, the maximal ideals of which are a subset of the maximal ideals of $R[x_1, \dots, x_{\dim A-1}]$ since R is itself 1-dimensional, so $\zeta_X(s)$ is bounded by $\zeta_{\mathbb{A}_S^{\dim X}}(s)^n$ for some positive integer n. By the same logic as in Example 3.2.8 we have $\zeta_{\mathbb{A}_S^{\dim X-1}}(s) = \zeta_S(s - \dim X + 1)$, so since $\zeta_S(s)$ converges for $\operatorname{Re} s > \sigma$ we get that $\zeta_X(s)$ converges and is nonzero for $\operatorname{Re} s > \sigma + \dim X - 1$. $\square$

Our main tool in the proof of Theorem 3.4.1 will be the Weil conjectures.

Theorem 3.4.6 (Weil conjectures). *If $X \to \operatorname{Spec} \mathbb{F}_q$ is smooth, separated, and of finite type and X is geometrically connected and integral of dimension r, then*

$$\zeta_X(s) = \prod_{k=0}^{2r} P_k(q^{-s})^{(-1)^{k+1}}$$

where $P_k(t)$ are polynomials with $P_0(t) = 1-t$, $P_{2r}(t) = 1-q^r t$, and for $1 \le k \le 2r-1$ $P_k(0) = 1$ and the roots of P_k have absolute value $q^{-\frac{k}{2}}$. Further if there exists a scheme $\tilde{X} \to \operatorname{Spec} R$ where R is a ring containing a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p} \cong \mathbb{F}_q$ such that $\tilde{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/\mathfrak{p}) = X$, then writing $b_k^{\mathfrak{p}}$ for the degree of each P_k for the reduction modulo $\mathfrak{p}$ defined by $\tilde{X} \to \tilde{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R/\mathfrak{p}) = X = X(\mathfrak{p})$ we have $b_k^{\mathfrak{p}}$ independent of $\mathfrak{p}$ provided $R/\mathfrak{p}$ is finite. Further $\zeta_X(s)$ is a rational function holomorphic on $\operatorname{Re} s > \dim X$ even if X is not geometrically connected.

The proof of the Weil conjectures, due in various parts to Dwork, Grothendieck, and Deligne, with contributions from Serre, Artin, and Verdier, is far beyond the scope of this paper. See for example [7] for an exposition of the proof, and Dwork's original paper [3] for the proof of rationality in the general case.

We can, however, make use of the Weil conjectures to show convergence of the zeta functions as in Theorem 3.4.1. In particular, we will first show the result in the case of smooth projective varieties over S by factoring the zeta function according to Proposition 3.2.5 so that we can apply the Weil conjectures to the factors and thence conclude the general case.

First we want to show that the product takes every closed point into account. Henceforth we assume that $f : X \to S$ is of finite type with $S = \operatorname{Spec} R$ for R a subring of $\mathbb{Q}$.

Proposition 3.4.7. *If $x \in X$ is a closed point, then its residue field $k(x)$ is finite.*

Proof following [13]. Let $U = \operatorname{Spec} A$ be an affine open neighborhood of x. Then x is closed in A, so it corresponds to a maximal ideal $\mathfrak{m}$ of A. Since f is of finite type and $f(x)$ is a closed point of S corresponding to a maximal ideal (p) for some $p \in \mathcal{P}$, A is a finitely-generated $\mathbb{F}_p$-algebra, the quotients of which by maximal ideals are finite extensions of $\mathbb{F}_p$, so $k(x) = \mathbb{F}_{p^m}$ for some m. $\square$

Let's now show the result for smooth projective varieties, where by varieties we mean separated and integral schemes of finite type over S, which we do not require to be the spectrum of a field. Suppose that $X \subset \mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$ is a projective variety, that is the zero locus of the family of homogeneous polynomials $f_1, \dots, f_m$ in $n+1$ variables with coefficients in R. We will suppose further that X is smooth.

Let R be a subring of $\mathbb{Q}$.

Lemma 3.4.8. *If $f \in R[x_0, \dots, x_n]$ does not factor over $\mathbb{C}$, then its reduction modulo p for p not a unit of R does not factor in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ for all but finitely many p.*

Proof following [6]. The statement that f does not factor in $\mathbb{C}$ is equivalent to the statement that the polynomial equation $gh - f = 0$ has no zeros for polynomials $g, h \in \mathbb{C}[x_0, \dots, x_n]$ of degree at most $r = \deg f - 1$. We can view $gh - f = 0$ as a system of equations with the coefficients of g and h as determinants, fixing that $gh - f$ be a polynomial with every coefficient equal to 0. By Hilbert's Nullstellensatz this system has a solution in $\mathbb{C}$, corresponding to a factoring of f over $\mathbb{C}$, if and only if the ideal generated by solutions to $gh - f = 0$ contains 1, so there exist $g_1, \dots, g_m$, $h_1, \dots, h_m$ such that $g_1 h_1 + \cdots + g_m h_m = 1$. We can reduce this equation modulo p for all but finitely many p and reverse the logic to get f modulo p irreducible over $\overline{F}_p$ for all but finitely many p. □

Lemma 3.4.9. *If X is geometrically irreducible, then so is the fiber X_p for all but finitely many primes p.*

Proof. We will prove this in the case in which X is a hypersurface. For a full proof see Theorem 9.7.7 of [9].

Assume X is cut out by a single polynomial f in projective space. If X is geometrically irreducible, then $X \times_S \operatorname{Spec} \mathbb{Q}$ is geometrically irreducible, that is $X \times_S \operatorname{Spec} \mathbb{C}$ is irreducible, which is equivalent to the statement that f does not factor over $\mathbb{C}$. Therefore we can apply Lemma 3.4.8 to get that X geometrically irreducible implies that for all but finitely many primes $p \in \mathcal{P}$ we have f irreducible over $\overline{\mathbb{F}}_p$, and by reversing the logic we get X_p geometrically irreducible for all but finitely many $p \in \mathcal{P}$. □

By Proposition 3.1.22 if X is smooth then so is each X_p. With X as above we have $\zeta_X(s) = \prod_p \zeta_{X_p}(s)$, and for all but finitely many $p \in \mathcal{P}$ splitting X into its irreducible components we get from Lemma 3.4.8 that X_p has the same number of irreducible components and all are geometrically irreducible. We say for a prime $p \in \mathcal{P}$ that X has good reduction at p if this holds and has bad reduction at p if p is one of the finitely many primes in $\mathcal{P}$ for which it fails.

Lemma 3.4.10. *Projective varieties are separated.*

Proof following [2]. If $X \to S$ is a projective variety, then there is a natural closed immersion into $\mathbb{P}^n_S$, so it suffices to show that $\mathbb{P}^n_S$ is separated over S, since then the diagonal morphism of X has image a closed subset of the image of the diagonal morphism of $\mathbb{P}^n_S$ and therefore is a closed immersion.

The diagonal morphism of $\mathbb{P}^n_S$ has image pairs of points (x, y) in projective space $\mathbb{P}^n_S \times_S \mathbb{P}^n_S$ such that $x_i y_j = x_j y_i$ for all $0 \le i, j \le n$, since this is when the coordinates represent the same point in projective space. Therefore this is the zero locus of a family of polynomials homogeneous in each of x and y, and therefore is closed, so $\mathbb{P}^n_S$ is separated over S, which by the above gives the result. □

From Lemma 3.4.10 and Proposition 3.1.30 we get that each $X_p \to \mathbb{F}_p$ is separated. To apply the Weil conjectures we need to show only that each irreducible component of X_p is integral over $\mathbb{F}_p$. By Proposition 3.1.36 this is equivalent to showing that each irreducible component is reduced, which for a projective variety with defined by polynomials irreducible over $\mathbb{C}$ is clear. Therefore we can prove Theorem 3.4.1 for smooth projective varieties with irreducible components geometrically irreducible.

Proof of Theorem 3.4.1 in the special case. For all but finitely many primes p in $\mathcal{P}$, by the above the Weil conjectures apply to each irreducible component $Y_{p,i}$ of X_p. Therefore if $Y_{p,i}$ has dimension r_i then we have by the Weil conjectures

$$\zeta_{Y_{p,i}}(s) = \prod_{j=0}^{2r_i} P_{ij}(p^{-s})^{(-1)^{j+1}} = \prod_{j=0}^{2r_i} \prod_{k=1}^{b_{ij}} (1 - \alpha_{ijk} p^{-s})^{(-1)^{j+1}}$$

with $|\alpha_{ijk}| = p^{\frac{j}{2}}$, with b_{ij} independent of p. Since X is smooth, by Propositions 3.1.22 and 3.1.27 the $Y_{p,i}$ are just the connected components of X_p, so $\zeta_{X_p}(s) = \prod_i \zeta_{Y_{p,i}}(s)$. Let $\mathcal{P}_b$ be the finite set of primes in $\mathcal{P}$ at which X has bad reduction and $\mathcal{P}_g$ its complement in $\mathcal{P}$. Then

$$\begin{aligned}
\zeta_X(s) &= \left(\prod_{p \in \mathcal{P}_b} \zeta_{X_p(s)} \right) \left(\prod_{p \in \mathcal{P}_g} \zeta_{X_p}(s) \right) \\
&= \left(\prod_{p \in \mathcal{P}_b} \zeta_{X_p(s)} \right) \prod_{p \in \mathcal{P}_g} \prod_{i=1}^{m} \prod_{j=0}^{2r_i} \prod_{k=1}^{b_{ij}} \left(1 - \alpha_{ijk} p^{-s}\right)^{(-1)^{j+1}} \\
&= \left(\prod_{p \in \mathcal{P}_b} \zeta_{X_p(s)} \right) \prod_{i=1}^{m} \prod_{j=0}^{2r_i} \prod_{k=1}^{b_{ij}} \prod_{p \in \mathcal{P}_g} \left(1 - \alpha_{ijk} p^{-s}\right)^{(-1)^{j+1}} .
\end{aligned}$$

The innermost product converges absolutely if

$$\prod_{p \in \mathcal{P}_g} \left(1 + |\alpha_{ijk}| p^{-s}\right) = \prod_{p \in \mathcal{P}_g} \left(1 + p^{\frac{j}{2} - s}\right)$$

converges, which is bounded by

$$\prod_{p \in \mathcal{P}} \left(1 + p^{\frac{j}{2} - s} + p^{2\left(\frac{j}{2} - s\right)} + \cdots \right) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{\frac{j}{2} - s}} = \zeta_S\left(s - \frac{j}{2}\right)$$

converges, which by definition is for $\operatorname{Re} s > \sigma + \frac{j}{2}$. Therefore $\zeta_X(s)$ converges for $\operatorname{Re} s > \sigma + \max_i r_i$. By Lemma 3.1.18 and the fact that $\operatorname{Spec} R$ is one-dimensional while $\operatorname{Spec} \mathbb{F}_p$ is zero-dimensional, each r_i is one less than the dimension of the ith connected component Y_i of X, and so $\max_i r_i = r - 1$. Therefore $\zeta_X(s)$ converges for $\operatorname{Re} s > \sigma + r - 1$.

Now suppose that X has ℓ connected components of codimension 0, that is of dimension r. Applying the Weil conjectures to each separately, each of these contributes a factor of $\frac{1}{1 - p^{r-1-s}}$ to each ζ_{X_p}. Other than these terms, we get factors converging for $\operatorname{Re} s > \sigma + \frac{j}{2}$ for $j \leq 2(r-2)$ for connected components of codimension at least 1 and for $j \leq 2(r-1) - 1 = 2r - 3$ for the remaining factors from the components of codimension 0, so in all the contribution from primes at which X has good reduction to

$$\frac{\zeta_X(s)}{\zeta_S(s-r+1)^\ell} = \prod_{p\in\mathcal{P}} \left(1-p^{r-1-s}\right)^\ell \zeta_{X_p}(s)$$

converges and is nonzero for $\operatorname{Re} s > \sigma + \frac{j}{2} \geq \sigma + r - \frac{3}{2}$. There are only finitely many primes at which X has bad reduction, and since $\zeta_{X_p}(s)$ is still rational the contribution from them is a rational function which is holomorphic and nonzero for $\operatorname{Re} s > r - \frac{1}{2}$. Replacing s with $s + r - 1$ gives the result. □

To generalize to Theorem 3.4.1 in full, we need to eliminate the assumption that X is a smooth projective variety. To do so we need to discuss birational equivalence.

Definition 3.4.11. For U, V dense open subschemes of a scheme X, two scheme morphisms $U \to Y$, $V \to Y$ are *equivalent* if they are equal on $U \cap V$.

Definition 3.4.12. A *rational map* from X to Y is an equivalence class of morphisms from open subschemes of X to Y.

Definition 3.4.13. A rational map f from X to Y is *birational* if there exists a rational map g from Y to X such that $f \circ g$ is the identity on Y and $g \circ f$ is the identity on X. If there exists a birational map from X to Y we say that X is birational to Y, or that X and Y are birationally equivalent.

Lemma 3.4.14. *Suppose that X and Y are birationally equivalent schemes satisfying the hypotheses of Theorem 3.4.1. Then Theorem 3.4.1 holds for X if and only if it holds for Y.*

Proof. Let U be an open subset of X such that $\dim(X \setminus U) < \dim X$. Then $\zeta_X(s) = \zeta_U(s)\zeta_{X\setminus U}(s)$, and by Lemma 3.4.5 and the definition of U $\zeta_{X\setminus U}(s)$ converges and is nonzero for $\operatorname{Re} s > \sigma + \dim X - 2$, so the theorem holds for X if and only if it holds for U. Since every open subset of X is isomorphic to an open subset of Y by birationality, the theorem holds for X if and only if it holds for Y. □

We now need the following results.

Lemma 3.4.15 (Hironaka's Theorem)**.** *For any reduced scheme X over a field K of characteristic* 0 *there exists a scheme $\tilde{X}$ over* $\operatorname{Spec} K$ *such that the morphism of schemes $\tilde{X} \to X$ is birational and the composition $\tilde{X} \to$* $\operatorname{Spec} K$ *is smooth.*

Lemma 3.4.16 (Chow's Lemma)**.** *For any integral separated scheme of finite type $X \to S$ there exists an integral separated projective scheme $\tilde{X}$ of finite type over S birational to X.*

We will not prove these, but refer the reader to [10] and [8] respectively for the classical references.

We are now ready to complete the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1 in the general case. We can assume that X is reduced, since the closed points of X and their residue fields do not change upon replacing X with its reduced form, and we can assume for Theorem 3.4.1 that X is separated. Since the geometric connectedness property is preserved by the birational morphism and we can assume by the above that X is smooth, the case in which X is irreducible implies the general case, since we can consider each irreducible component separately and multiply the zeta functions of each to get the overall zeta function, so we can also assume that X is irreducible, and therefore by Lemma 3.1.36 that X is integral. We can then apply Lemmas 3.4.15 and 3.4.16 to $X \times_S \operatorname{Spec} \mathbb{Q}$ to get a geometrically irreducible projective smooth integral separated scheme of finite type over $\operatorname{Spec} \mathbb{Q}$ birational to $X \times_S \operatorname{Spec} \mathbb{Q}$. Therefore $\tilde{X} \times_{\operatorname{Spec} \mathbb{Q}} S$ is birational to X has the same properties, so by Lemma 3.4.14 we can assume for the purposes of proving Theorem 3.4.1 that X is smooth, projective, separated, integral, and of finite type over S. Therefore by Lemma 3.4.14 the result in full generality follows from the special case proven above. □

4 A taste of higher-dimensional class field theory

We have managed to extend the zeta function of arithmetic schemes, as well as other S-schemes, to a region slightly larger than its original domain on which it converges, but compared to the results of Section 2 this is a fairly weak result. Can we do better?

For general subrings of $\mathbb{Q}$ the associated zeta function can have a natural barrier, that is a line $\operatorname{Re} s = \sigma$ on which points at which the zeta function has a pole become dense, so it is probably unreasonable to hope for a full duality-type result extending the zeta function to the entire complex plane, so we will restrict ourselves to true arithmetic schemes.

The natural approach is to attempt to generalize the methods of Section 2, viewing Tate's thesis as a result on the zeta functions of certain one-dimensional schemes. We relied on essentially one-dimensional objects such as local fields and rings of integers. A number of mathematicians, in particular Alexey Parshin, Kazuya Kato, and Ivan Fesenko, have shown that it is possible to generalize these. In particular, local fields are fields with ring of integers a local ring with residue field a finite field. We can view the operation $F \to \mathcal{O}_F \to \mathcal{O}_F/\mathfrak{m}_F$ as a field map for F a valuation field, that is such that $\mathcal{O}_F$ is a local ring, and we say that F has dimension n as a valuation field if we can iterate this map n times. Thus for example a local field is 1-dimensional as a valuation field, since we can take this map once and then get a finite field, which is not a valuation field and so has dimension 0 as a valuation field. We define n-dimensional local fields to be fields such that n iterations of this map gives a finite field.

Example 4.1. Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $k = \mathbb{Q}_p((t))$ be the field of Laurent series over $\mathbb{Q}_p$. Then the ring of integers $\mathcal{O}_k$ of k is the ring $\mathbb{Q}_p[[t]]$ of power series over $\mathbb{Q}_p$, which has the unique maximal ideal (t), so that the residue field is $\mathcal{O}_k/(t) = \mathbb{Q}_p$. Iterating the map, the ring of integers of $\mathbb{Q}_p$ is the ring $\mathcal{O}_p = \mathbb{Z}_p$ of p-adic integers, which has the unique maximal ideal (p), and the second residue field $\mathcal{O}_p/(p) = \mathbb{F}_p$ is a finite field. Therefore k is a valuation field of dimension 2.

This then lets us define integral structures of different dimensions on each local field, which we can use to define a set of adelic objects for higher-dimensional global fields and follow the analysis of Section 2, with some modifications. For example, although Section 2 does not explicitly use class field theory, in the background we have the local Artin reciprocity map

$$F^\times \to \operatorname{Gal}(\overline{F}/F)^{\mathrm{ab}}$$

where $\operatorname{Gal}(\overline{F}/F)^{\mathrm{ab}}$ denotes the abelianization of the Galois group $\operatorname{Gal}(\overline{F}/F)$ for F a local field. If we take a global field, say $\mathbb{Q}$, and assemble the local reciprocity maps at each completion $\mathbb{Q}_v$, we get the global reciprocity map in much the same way as we got the global zeta integral from the local zeta integrals over each place. In higher dimensions, the analogue of $F^\times$ is the topological Milnor K-group $K_n^{\text{top}}(F)$, from which we can define a similar reciprocity map and which we will use in place of the multiplicative groups $K_v^\times$ to assemble the ideles. See e.g. [13] for an exposition of higher-dimensional class field theory, and [6] for an analysis of the zeta functions of higher-dimensional arithmetic schemes (in particular in dimension two) and for the analysis below.

Unfortunately the higher-dimensional analogue of Theorem 2.5.10 does not follow in the same way from this analysis. Although we can use an analogue of Poisson summation to show the appropriate analogue of Lemma 2.5.8, now with $g(f, s, t) = g(\hat{f}, 2 - s, t^{-1})$, the boundary terms which in the one-dimensional case were the straightforward integrals

$$\hat{f}(0)V \int_0^1 t^{s-1} \frac{dt}{t} - f(0)V \int_0^1 t^s \frac{dt}{t}$$

are in general apparently intractable. The analysis does allow us to reduce the analogue of Theorem 2.5.10 to a hypothesis on the mean-periodicity of the boundary term, but as of yet the boundary term for the most part remains a mystery.

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