AN UPPER BOUND FOR SIGNATURES OF IRREDUCIBLE, SELF-DUAL $\mathfrak{gl}(n,\mathbb{C})$-REPRESENTATIONS

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Abstract. For every irreducible, self-dual representation V of $\mathfrak{gl}(n,\mathbb{C})$ with highest weight λ, there exists a unique (up to a real scalar multiple), nondegenerate, $\mathfrak{gl}(n,\mathbb{C})$-invariant Hermitian pairing $V \to V^h$, with respect to the real structure σ such that $\sigma(X+iY) = X - iY$ for all $X, Y \in \mathfrak{gl}(n,\mathbb{R})$. Therefore, it makes sense to talk about the *signature* of a representation by looking at the signature of this pairing. In this paper, we use the Gelfand-Zetlin branching law to obtain a recursive bound for this signature in terms of the signatures of irreducible representations of $\mathfrak{gl}(n-1,\mathbb{C})$.

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1. Introduction

1.1. **Summary.** Let $\rho\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(V)$ be a finite-dimensional irreducible complex representation of $\mathfrak{gl}(n,\mathbb{C})$. We'd like to examine the properties of nondegenerate, $\mathfrak{gl}(n,\mathbb{C})$-invariant Hermitian forms on V. An important problem is to determine whether or not the form is *positive definite*, which would then make (ρ, V) a unitary representation. To do this, we can first figure out the

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signatures of the forms, which will tell us precisely when positive definiteness occurs. Our main result is this:

Theorem 1.1. *Let V_λ be an irreducible, self-dual $\mathfrak{gl}(n,\mathbb{C})$-representation with highest weight λ, and let $\langle\cdot,\cdot\rangle$ be an invariant nondegenerate Hermitian form on V_λ. The self-duality condition implies that $\lambda=(\lambda_1+it,\lambda_2+it,\dots,-\lambda_2+it,-\lambda_1+it)$ for real-valued $\{\lambda_i\}$ and t. Without loss of generality, let $t=0$; doing so only modifies V_λ by a tensor product with a 1-dimensional character $\mathbb{C}_{it}$, and so won't change the Hermitian form in a non-negligible way. Then, the following holds:*

(1) *The signature does not change under restriction to the* zero weight space*:*
$$\mathrm{Sig}\langle\cdot,\cdot\rangle=\mathrm{Sig}(\langle\cdot,\cdot\rangle|_{V_\lambda[0]\times V_\lambda[0]})$$

(2) *We have the following upper bound for its signature:*
$$\mathrm{Sig}(\langle\cdot,\cdot\rangle)\le\sum_{\eta\wr\lambda,\ \eta=w_0(-\eta)}\mathrm{Sig}(\langle\cdot,\cdot\rangle|_{V_\eta\times V_\eta}),$$
where w_0 reverses the entries of η, and $\eta\wr\lambda$ if and only if, letting $\eta:=(\eta_1,\dots,\eta_{n-1})$ and $\lambda:=(\lambda_1,\dots,\lambda_n)$, we have that $\lambda_i-\eta_j$ are integers for all i,j, and that the interleaving relation
$$\lambda_1\ge\eta_1\ge\lambda_2\ge\cdots\ge\lambda_{n-1}\ge\eta_{n-1}\ge\lambda_n$$
holds.

The paper is divided into several parts. In section 2, we first discuss the well-known classification of irreducible $\mathfrak{gl}(n,\mathbb{C})$-representations via *highest-weight theory.* Then, we introduce the *Hermitian dual* to help us formulate the notion of an invariant Hermitian pairing with respect to a real structure σ. Finally, we state and prove the necessary and sufficient conditions for $\mathfrak{gl}(n,\mathbb{C})$ to possess a nondegenerate, invariant Hermitian pairing.

In Section 3, we prove the first part of Theorem 1.1: indefiniteness of the signature outside of the zero weight space. We then introduce the *Gelfand-Zetlin branching law*, which gives an explicit decomposition of an irreducible $\mathfrak{gl}(n,\mathbb{C})$ representation when the action is restricted to $\mathfrak{gl}(n-1,\mathbb{C})$. After that, we will use the branching law along with invariance of the Hermitian form to obtain the second part of Theorem 1.1, completing the proof.

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2. Preliminaries

2.1. **Signatures.** The "signature" of a Hermitian form has several different meanings. In this paper, we will use the following definition:

Definition 2.1. If $\langle\cdot,\cdot\rangle$ is a nondegenerate Hermitian form on a finite-dimensional complex vector space V, *Sylvester's Law* states that there are unique $a,b\in\mathbb{N}$ such that V has a basis

$\{e_1,\ldots,e_a,f_1,\ldots,f_b\}$ satisfying, for all $1\le i,j,k\le \dim(V)$:

$$\langle e_i,e_j\rangle=\delta_{ij}$$
$$\langle f_k,f_l\rangle=-\delta_{kl}$$
$$\langle e_i,f_k\rangle=0.$$

The *signature* of $\langle\cdot,\cdot\rangle$ is then defined as

$$\operatorname{Sig}\langle\cdot,\cdot\rangle:=|a-b|.$$

2.2. **Highest-weight theory.** Define the following partial order on $\mathbb{C}$:

Definition 2.2. Let the partial order "$\le$" on $\mathbb{C}$ signify the following: for $w,z\in\mathbb{C}$. we have $w\le z$ if and only if $z-w\in\mathbb{R}_{\ge 0}$. Note that this extends the usual order on $\mathbb{R}$.

We will refer to this order when comparing complex numbers in this and later sections.

In its most general form, the *theorem of the highest weight* classifies all finite-dimensional irreducible representations of semisimple Lie algebras, and although $\mathfrak{gl}(n,\mathbb{C})$ isn't semisimple, we can extend it to $\mathfrak{gl}(n,\mathbb{C})$, as

$$\mathfrak{gl}(n,\mathbb{C})\cong\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}.$$

Denote $\mathfrak{t}\subset\mathfrak{gl}(n,\mathbb{C})$ to be the subalgebra of *diagonal matrices*; this is a *Cartan subalgebra* of $\mathfrak{gl}(n,\mathbb{C})$, and is isomorphic to $\mathbb{C}^n$. With this notation, we can decompose $\mathfrak{gl}(n,\mathbb{C})$ into a Lie algebra direct sum $\mathfrak{gl}(n,\mathbb{C})\cong\mathfrak{n}^-\oplus\mathfrak{t}\oplus\mathfrak{n}^+$, where

$$\mathfrak{n}^-=\operatorname{span}\{E_{ij}\}_{i>j},$$
$$\mathfrak{n}^+=\operatorname{span}\{E_{ij}\}_{i<j}.$$

Further denote $\{\mathbf{e}_i\}_{i=1}^n$ to be the basis of $\mathfrak{t}^*$ dual to the basis $\{E_{jj}\}$ of $\mathfrak{t}$, such that

$$\mathbf{e}_i(E_{jj})=\delta_{ij}.$$

In the $\mathfrak{gl}(n,\mathbb{C})$ case, the theorem of the highest weight asserts the following:

Theorem 2.3. *Finite-dimensional irreducible representations of $\mathfrak{gl}(n,\mathbb{C})$ are in bijection with n-tuples $\lambda:=[\lambda_1,\ldots,\lambda_n]\in\mathbb{C}^n$ such that $\lambda_i-\lambda_j\in\mathbb{Z}_{\ge 0}$ for all $i<j$. The associated vector space is denoted as V_λ.*

The integral weights of V_λ, denoted as Δ, is defined to be the set of all $\eta\in\mathfrak{t}^*$ such that, letting $[\eta_1,\ldots,\eta_n]$ be the entries of η but sorted in decreasing order, we have $\lambda_i-\eta_j\in\mathbb{Z}$ for all $1\le i,j\le n$, and that

$$\eta_1\le\lambda_1$$
$$\eta_1+\eta_2\le\lambda_1+\lambda_2$$
$$\vdots$$
$$\eta_1+\cdots+\eta_n=\lambda_1+\cdots+\lambda_n.$$

With Δ defined, V_λ then admits a weight space decomposition

$$V_\lambda=\bigoplus_{\eta\in\Delta}V_\lambda[\eta]$$

where

$$V_\lambda[\eta] = \{v \in V_\lambda \mid Xv = \eta(X)v \text{ for all } X \in \mathfrak{t}\}.$$

The algebra $\mathfrak{gl}(n, \mathbb{C})$ *then acts on* $v \in V_\lambda[\eta]$ *in the following manner:*

$$\begin{aligned} &E_{ij}v \in V[\eta + \mathbf{e}_i - \mathbf{e}_j] && \text{if } \eta + \mathbf{e}_i - \mathbf{e}_j \in \Delta \\ &E_{ij}v = 0 && \text{if } \eta + \mathbf{e}_i - \mathbf{e}_j \notin \Delta \\ &E_{ii}v = \eta(E_{ii})v. \end{aligned}$$

Accordingly, the elements of $\mathfrak{n}^+$ *are called raising operators, and the elements of* $\mathfrak{n}^-$ *lowering operators.*

The weight λ *is known as the highest weight, as any vector in the highest-weight space* $V_\lambda[\lambda]$ *is annihilated by all elements of* $\mathfrak{n}^+$*. Here are some properties about* $V_\lambda[\lambda]$*:*

- $V_\lambda[\lambda]$ *is 1-dimensional.*
- *If* $v \in V_\lambda[\lambda]$ *is a highest-weight vector, then* $V_\lambda = U(\mathfrak{n}^-)v$*, i.e. the orbit of the* $\mathfrak{n}^-$*-action on* v *generates a spanning set for all of* V_λ*.*
- *The set of integral weights* Δ *is invariant under the action of* $W = S_n$*, which acts by permuting the entries. That is,* $W(\Delta) = \Delta$*.*

We'll also use the following fact about weight spaces:

Proposition 2.4. *If* V *and* W *are irreducible* $\mathfrak{gl}(n, \mathbb{C})$*-representations, then in the induced representation on* $V \otimes W$ *via*

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw,$$

a functional $\lambda \in \mathfrak{t}^*$ *is an integral weight of* $V \otimes W$ *if and only if it is a sum of an integral weight of* V *and an integral weight of* W*.*

2.3. **Hermitian duals.** Denote V^h to be the *Hermitian dual*, i.e., the space of functions $\xi \colon V \to \mathbb{C}$ satisfying

$$\begin{aligned} \xi(v) + \xi(w) &= \xi(v + w) \\ \xi(cv) &= \bar{c}\xi(v), \end{aligned}$$

for all $v, w \in V$ and $c \in \mathbb{C}$.

A *sesquilinear pairing* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is naturally identified with the linear operator

$$T \in \mathrm{Hom}(V, V^h), \qquad T(v) = \langle v, \cdot \rangle\,,$$

and its *Hermitian transpose* is defined as

$$T^h \in \mathrm{Hom}(V, V^h), \qquad T^h(v) = \overline{\langle \cdot, v \rangle}.$$

A *Hermitian pairing*, with the additional constraint $\langle v, w \rangle = \overline{\langle w, v \rangle}$, is thus naturally identifed with $T \in \mathrm{Hom}(V, V^h)$ such that $T = T^h$.

2.4. **Dual representations.** Recall that the representation V of complex Lie algebra $\mathfrak{g}$ induces a representation on V^*, the space of *complex-linear functionals* on V, in the following way: for $v \in V$, elements $X \in \mathfrak{g}$ act on functionals $f \in V^*$ via

$$(Xf)(v) = -f(Xv).$$

However, representations on V^h are more complicated: functionals $f \in V^h$ now need to be *conjugate linear*, and the above won't cut it: for $z \in \mathbb{C}$, we must have

$$(zXf)(v) = -f(\overline{z}Xv) \neq -f(zXv).$$

In order to make this work, we need to have a sense of "conjugacy" for elements $X \in \mathfrak{g}$.

Definition 2.5. A *real structure* σ on a complex Lie algebra $\mathfrak{g}$ is an map $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ with the following properties:

- σ is a *Lie algebra homomorphism*, i.e. $\sigma([X,Y]) = [\sigma(X), \sigma(Y)]$ for all $X, Y \in \mathfrak{g}$.
- σ is *conjugate linear*, i.e. $z\sigma(X) = \sigma(\overline{z}X)$ for $z \in \mathbb{C}$ and $X \in \mathfrak{g}$.
- σ is an *involution*, i.e. $\sigma^2 = 1$.

What the term *real structure* refers to is the subalgebra fixed by σ, denoted as $\mathfrak{g}^\sigma$.

In the case of $\mathfrak{gl}(n,\mathbb{C})$, we will be working with matrices with the property that all their entries are *real*. Therefore, the natural real structure to use is defined by

$$\sigma(X + iY) = X - iY$$

for all $X, Y \in \mathfrak{gl}(n,\mathbb{R})$. Note that $\mathfrak{gl}(n,\mathbb{C})^\sigma = \mathfrak{gl}(n,\mathbb{R})$, as expected.

With a sense of "conjugacy" in hand, we can now define the dual representation V^h with respect to σ, the *σ-Hermitian dual* $V^{h,\sigma}$: for $v \in V$, elements $X \in \mathfrak{g}$ act on functionals $f \in V^{h,\sigma}$ by

$$(Xf)(v) = -f(\sigma(X)v).$$

We check that Xf is conjugate linear:

$$(zXf)(v) = -f(\sigma(zX)v) = -f(\overline{z}\sigma(X)v) = (Xf)(\overline{z}v).$$

We check that the representation $V^{h,\sigma}$ respects the Lie bracket:

$$([X,Y]f)(v) = -f(\sigma([X,Y])v) = f(\sigma(Y)\sigma(X)v - \sigma(X)\sigma(Y)v) = [(XY - YX)f](v).$$

2.5. **Invariant pairings.** An *σ-invariant pairing* has several equivalent meanings:

Proposition 2.6. *If (ρ, V) is a representation of a Lie algebra $\mathfrak{g}$ with a nondegenerate sesquilinear pairing $\langle \cdot, \cdot \rangle$, then the following are equivalent:*

- *For all $X \in \mathfrak{g}$ and all $v, w, \in V$, we have $\langle Xv, w \rangle + \langle v, \sigma(X)w \rangle = 0$.*
- *The associated map $T\colon V \to V^{h,\sigma}$ commutes with the action of $\mathfrak{g}$. (An operator with this property is called an intertwining operator.)*

The form is called σ-invariant *in this case.*

For $\mathfrak{gl}(n,\mathbb{C})$, using σ as described in the previous subsection, we will call a σ-invariant Hermitian form simply *invariant*, and refer to $V^{h,\sigma}$ as simply V^h.

The following states necessary and sufficient conditions for an irreducible representation V to have a σ-invariant, nondegenerate Hermitian form.

Proposition 2.7. *If an irreducible representation V has a nonzero σ-invariant Hermitian form, then $V \cong V^{h,\sigma}$, and the form is nondegenerate. Conversely, if $V \cong V^{h,\sigma}$, then a nondegenerate invariant Hermitian form exists and is unique up to a real scalar multiple.*

Proof. Since the form is nonzero, it induces a nonzero intertwining operator $V \to V^{h,\sigma}$. By Schur's Lemma, we must have $V \cong V^{h,\sigma}$, and the form is automatically nondegenerate. Conversely, if $V \cong V^{h,\sigma}$, then an intertwining operator $V \to V^{h,\sigma}$ exists and is unique up to a scalar multiple. Choose one nonzero intertwining operator $T\colon V \to V^{h,\sigma}$, and denote the form induced by this operator as $\langle \cdot, \cdot \rangle$. The Hermitian transpose $T^h\colon V \to V^{h,\sigma}$ is also an intertwining operator, and by Schur's Lemma we therefore must have, for some $z \in \mathbb{C}$,

$$T = zT^h.$$

Using the formulae for T, T^h above, then, for all $v, w \in V$,

$$\langle v, w \rangle = z\overline{\langle w, v \rangle}.$$

Note that applying the equation twice gives

$$\langle v, w \rangle = z\overline{\langle w, v \rangle} = |z|^2 \langle v, w \rangle$$

implying that $|z| = 1$, i.e. $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Define a new sesquilinear form $(\cdot, \cdot)$ as

$$(\cdot, \cdot) := e^{-i\theta/2} \langle \cdot, \cdot \rangle .$$

This form is Hermitian:

$$(v, w) = e^{-i\theta/2} \langle v, w \rangle = e^{i\theta/2}\overline{\langle w, v \rangle} = \overline{(w, v)}.$$

We see that with $(\cdot, \cdot)$, only *real* scalar multiples of $(\cdot, \cdot)$ are Hermitian, while *complex* scalar multiples fail to be Hermitian:

$$z(v, w) = z\overline{(w, v)} \neq \overline{z(w, v)}$$

if z isn't real. We finish. ◻

Proposition 2.8. *If V_λ is an irreducible representation of $\mathfrak{gl}(n, \mathbb{C})$ with highest weight λ, then we must have an isomorphism of representations*

$$(V_\lambda)^h \cong V_{w_0(-\overline{\lambda})},$$

where $w_0 \in S_n$ acts on $\lambda \in \mathbb{C}^n$ by reversing its entries, and $\overline{\lambda}$ is λ but with every entry conjugated. For example, $w_0([3, -1, -2, -4]) = [-4, -2, -1, 3]$ and

$$\overline{[3 + i, -1 + i, -2 + i, -4 + i]} = [3 - i, -1 - i, -2 - i, -4 - i].$$

Proof. The representation $(V_\lambda)^h$ is also an irreducible representation of $\mathfrak{gl}(n, \mathbb{C})$, implying that it must have a highest weight. Choose bases for each weight space $V_\lambda[\eta]$ and combine them to form a basis $\{v_i\}$ for V_λ. Further denote the set $\{v_i^*\}$ as the basis of $(V_\lambda)^h$ dual to V_λ, i.e. $v_i^*(v_j) = \delta_{ij}$ for all i, j. Let v_i be a basis element that lies in $V_\lambda[\eta]$, and let $v = \sum_j c_j v_j$ be any vector of V_λ; then, for all $X \in \mathfrak{t}$,

$$\begin{aligned}(Xv_i^*)(v) &= v_i^*(-\sigma(X)v) \\ &= v_i^*(-\sigma(X)\sum_j c_j v_j) \\ &= v_i^*(-\sigma(X)c_i v_i)\end{aligned}$$

as $-\sigma(X)$ acts on each v_j by a scalar multiple, and all summands except the $c_i v_i$ term are thus killed by v_i^*. Let $X := X_0 + iY_0$ be the decomposition of X into real and complex parts, and let $\eta := \eta_0 + i\zeta_0$ be the corresponding decomposition of $\eta \in \mathfrak{t}^*$. Continuing with the calculation:

$$\begin{aligned}v_i^*(-\sigma(X)c_i v_i) &= v_i^*(-\eta(\sigma(X))c_i v_i) \\ &= -\overline{\eta(\sigma(X))}v_i^*(c_i v_i) \\ &= -\overline{\eta(X_0 - iY_0)}v_i^*(v) \\ &= \overline{-\eta_0(X_0) - i\zeta_0(X_0) + i\eta_0(Y_0) - \zeta_0(Y_0)} \\ &= -\eta_0(X_0) + i\zeta_0(X_0) - i\eta_0(Y_0) - \zeta_0(Y_0) \\ &= (-\eta_0 + i\zeta_0)(X_0 + iY_0) \\ &= -\overline{\eta}(X).\end{aligned}$$

With the equation $Xv_i^* = -\overline{\eta}(X)v_i^*(v)$, we see that v_i^* lies in the $-\overline{\eta}$ weight space of $(V_\lambda)^h$. As a result, if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the highest weight of V_λ, then $w_0(-\overline{\lambda}) = (-\overline{\lambda_n}, \ldots, -\overline{\lambda_1})$ is the highest weight of $(V_\lambda)^h$. $\square$

Combining the previous two propositions, we now have what we want:

Corollary 2.9. *An irreducible representation V_λ of $\mathfrak{gl}(n,\mathbb{C})$ has a nondegenerate invariant Hermitian form (unique up to a real scalar) if and only if $\lambda = w_0(-\overline{\lambda})$, where w_0 is defined in the previous proposition.*

If λ satisfies the above, then note that λ must be of the form $(\lambda_1 + it, \lambda_2 + it, \ldots, -\lambda_2 + it, -\lambda_1 + it)$ for some real-valued $\{\lambda_j\}$ and t. In fact, by Proposition 2.4, we have

$$V_\lambda \cong V_{(\lambda_1,\lambda_2,\ldots,-\lambda_2,-\lambda_1)} \otimes V_{(it,\ldots,it)} \cong V_{(\lambda_1,\lambda_2,\ldots,-\lambda_2,-\lambda_1)} \otimes_{\mathbb{C}} \mathbb{C}_{it}.$$

The representation $\mathbb{C}_{it}$ is a 1-dimensional character with every vector a highest-weight vector. In particular, for $z \in \mathbb{C}_{it}$, $E_{jj}z = itz$ and $E_{jk}z = 0$ (for $j \neq k$). Therefore:

$$\begin{aligned}E_{ij}(v \otimes z) &= E_{ij}v \otimes z + v \otimes E_{ij}z = E_{ij}v \otimes z \\ E_{ii}(v \otimes z) &= E_{ii}v \otimes z + v \otimes itz,\end{aligned}$$

and thus if $\langle \cdot, \cdot \rangle$ is an invariant nondegenerate Hermitian form on $V_{(\lambda_1,\lambda_2,\ldots,-\lambda_2,-\lambda_1)}$, then the nondegenerate Hermitian form $(\cdot, \cdot)$ on V_λ defined by

$$(v \otimes 1, w \otimes 1) := \langle v, w \rangle$$

must also be invariant:

$$
\begin{aligned}
(E_{ij}(v\otimes 1),(w\otimes 1))+((v\otimes 1),E_{ij}(w\otimes 1)) &= \langle E_{ij}v,w\rangle+\langle v,E_{ij}w\rangle=0\\
(E_{ii}(v\otimes 1),(w\otimes 1))+((v\otimes 1),E_{ii}(w\otimes 1)) &= (E_{ii}(v\otimes 1),(w\otimes 1))+((v\otimes 1),E_{ii}(w\otimes 1)))\\
&= \langle E_{ii}v,w\rangle+\langle itv,w\rangle+\langle v,E_{ii}w\rangle+\langle v,itw\rangle\\
&= \langle E_{ii}v,w\rangle+\langle v,E_{ii}w\rangle=0.
\end{aligned}
$$

Remark 2.10. Hence, we can assume that $t=0$ (all entries of λ are real) without loss of generality. The Hermitian form is essentially the same, and in particular the signature will remain the same no matter what t is.

Here are some facts about the highest weight λ, if $\lambda=w_0(-\overline{\lambda})$ and all entries are real:

- If n is even, then the λ_i's are all integers or half-integers, while if n is odd, then the λ_i's are all integers, and in particular the middle entry $\lambda_{(n+1)/2}$ is simply 0.
- All the weights $\eta\in\Delta$ of V_λ are real.
- The zero weight space $V_\lambda[0]$ has positive dimension if all the λ_i's are integers, and if not, then $V_\lambda[0]=0$.
- If η is a weight, then so is $-\eta$.

3. Obtaining the inequality

3.1. **Restriction to the zero weight space.** We will now use invariance to show the first part of Theorem 1.1: indefiniteness of the form's signature outside of V_λ's zero weight space. It is restated here for convenience:

Proposition 3.1. *If V_λ is an irreducible, self-dual $\mathfrak{gl}(n,\mathbb{C})$-representation with highest weight λ, and $\langle\cdot,\cdot\rangle$ is a nondegenerate invariant Hermitian form on V_λ, then*

$$\operatorname{Sig}\langle\cdot,\cdot\rangle=\operatorname{Sig}(\langle\cdot,\cdot\rangle|_{V_\lambda[0]\times V_\lambda[0]}).$$

Proof. The proof constructs the Hermitian matrix for $\langle\cdot,\cdot\rangle$ under a suitable basis. Let $\alpha,\beta\in\mathfrak{t}^*$ be weights of V_λ. We'll consider what happens in $\langle\cdot,\cdot\rangle|_{V_\lambda[\alpha]\times V_\lambda[\beta]}$ as α, β range over all the weights.

By invariance, for all $v\in V_\lambda[\alpha]$, $w\in V_\lambda[\beta]$ and all $X\in\mathfrak{t}$, we have

$$
\begin{aligned}
0 &= \langle Xv,w\rangle+\langle v,\sigma(X)w\rangle\\
&= \langle\alpha(X)v,w\rangle+\langle v,\beta(\sigma(X))w\rangle\\
&= \left(\alpha(X)+\overline{\beta(\sigma(X))}\right)\langle v,w\rangle\,.
\end{aligned}
$$

Let $X=:X_0+iY_0$ be the decomposition with respect to $\mathfrak{gl}(n,\mathbb{R})$; then

$$\overline{\beta(\sigma(X))}=\overline{\beta(X_0)-i\beta(Y_0)},$$

and since β has all real entries (due to V_λ being self-dual), we have $\overline{\beta(X_0)-i\beta(Y_0)}=\beta(X_0)+i\beta(Y_0)=\beta(X)$ and therefore the equation

$$0=(\alpha(X)+\beta(X))\langle v,w\rangle\,.$$

There are three cases:

- If $\alpha \neq -\beta$, then $\langle v, w\rangle$ must be 0, and therefore $V_\lambda[\alpha] \perp V_\lambda[\beta]$. In particular, $\langle \cdot, \cdot\rangle|_{V_\lambda[\alpha]\times V_\lambda[\alpha]} = 0$ for nonzero α.
- If $\alpha = -\beta$ but $\alpha \neq 0$, then for a basis $\{v_i\}_i$ of $V_\lambda[\alpha]$, we claim that we can find a dual basis $\{w_j\}_j$ of $V_\lambda[-\alpha]$ such that
$$\langle v_i, w_j\rangle = \delta_{ij}.$$
  Base case: Starting with any $v_1 \in V_\lambda[\alpha]$, we are able to find a $w_1 \in V_\lambda[-\alpha]$ such that $\langle v_1, w_1\rangle \neq 0$, or else $\langle v_1, \cdot\rangle = 0$, contradicting nondegeneracy. Normalize w_1 such that $\langle v_1, w_1\rangle = 1$.
  Inductive step: Suppose now we have linearly independent $v_1, \dots, v_k \in V_\lambda[\alpha]$ and dual vectors $w_1, \dots, w_k \in V_\lambda[-\alpha]$. From $V_\lambda[\alpha]$, choose any nonzero $v \notin \mathrm{sp}\{v_1, \dots, v_k\}$ and define
$$v_{k+1} := v - \sum_{j=1}^{k} \langle v, w_j\rangle\, v_j.$$
  We note that v_{k+1} is orthogonal to all $\{w_j\}_{j=1}^k$, as
$$\begin{aligned}\langle v_{k+1}, w_j\rangle &= \left\langle v - \sum_{i=1}^{k} \langle v, w_i\rangle\, v_i, w_j \right\rangle \\ &= \langle v, w_j\rangle - \sum_{i=1}^{k} \langle v, w_i\rangle\, \delta_{ij} = 0.\end{aligned}$$
  This implies that in $V_\lambda[-\alpha]$, there must exist $w_{k+1} \notin \mathrm{sp}\{w_1, \dots, w_k\}$ such that $\langle v_{k+1}, w_{k+1}\rangle \neq 0$, or else we'll have degeneracy. Normalize w_{k+1} so that $\langle v_{k+1}, w_{k+1}\rangle = 1$, and repeat the inductive step until $\{v_i\}$ and $\{w_j\}$ span $V_\lambda[\alpha]$ and $V_\lambda[-\alpha]$, respectively.
- If $\alpha = \beta = 0$, choose any basis of $V_\lambda[0]$ and denote the Hermitian matrix of $\langle \cdot, \cdot\rangle|_{V_\lambda[0]\times V_\lambda[0]}$ as M_0.

Now, for each pair of nonzero weights $-\alpha, \alpha \in \mathfrak{h}^*$ (recall that this occurs because $\lambda = w_0(-\lambda)$), the Hermitian matrix of $\langle \cdot, \cdot\rangle$ restricted to $(V_\lambda[\alpha] \oplus V_\lambda[-\alpha]) \times (V_\lambda[\alpha] \oplus V_\lambda[-\alpha])$ is therefore
$$\begin{array}{cc} & \begin{array}{cc}\alpha & -\alpha\end{array} \\ \begin{array}{c}\alpha \\ -\alpha\end{array} & \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{array},$$
and it is well-known that this has signature equal to zero (this can be shown by observing that the number of its positive eigenvalues equals the number of its negative eigenvalues). Denote this matrix as $M_{\pm\alpha}$.

Putting this all together, the entire Hermitian matrix for $\langle \cdot, \cdot\rangle$ looks like
$$\begin{bmatrix} M_{\pm\alpha_1} & & & \\ & M_{\pm\alpha_2} & & \\ & & \ddots & \\ & & & M_0 \end{bmatrix}.$$

For each $\pm\alpha$, we saw that $M_{\pm\alpha_1}$ had a zero signature, and therefore we can throw it away. What's left is M_0, and therefore, we have what we wanted: $\mathrm{Sig}\,\langle\cdot,\cdot\rangle = \mathrm{Sig}\,\langle\cdot,\cdot\rangle|_{V_\lambda[0]\times V_\lambda[0]}$. $\square$

3.2. The Gelfand-Zetlin branching law.

For convenience, we will use the following nonstandard notation:

Definition 3.2. Let $\eta := [\eta_1, \dots, \eta_{n-1}] \in \mathbb{C}^{n-1}$ and $\lambda := [\lambda_1, \dots, \lambda_n] \in \mathbb{C}^n$. Define a new relation "$\wr$", with $\eta \wr \lambda$ if and only if the following items hold:

- The values $\lambda_i - \lambda_j$, $\eta_i - \eta_j$ and $\lambda_i - \eta_j$ are integers for all i and j.
- The *interleaving relation* holds:
$$\lambda_1 \geq \eta_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \eta_{n-1} \geq \eta_n,$$
where as before we have the partial order $\geq$ on $\mathbb{C}$ such that $w \geq z$ if and only if $w - z \in \mathbb{R}_{\geq 0}$.

We'll now state the *Gelfand-Zetlin branching law*. It gives an explicit decomposition of an irreducible representation of $\mathfrak{gl}(n,\mathbb{C})$ when the action is restricted to $\mathfrak{gl}(n-1,\mathbb{C})$.

Theorem 3.3. *Let V_λ be a representation of $\mathfrak{gl}(n,\mathbb{C})$ with highest weight λ. If we restrict the action on V_λ to the subalgebra*
$$\left\{\begin{bmatrix} * & * & \cdots & * & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right\} \cong \mathfrak{gl}(n-1,\mathbb{C}),$$
then V_λ decomposes into irreducible representations of $\mathfrak{gl}(n-1,\mathbb{C})$ in the following manner:
$$V_\lambda|_{\mathfrak{gl}(n-1,\mathbb{C})} \cong \bigoplus_{\eta\wr\lambda} V_\eta.$$

3.3. The main result.

We now prove part 2 of Theorem 1.1. It is restated here for convenience:

Theorem 3.4. *Let V_λ be an irreducible, self-dual $\mathfrak{gl}(n,\mathbb{C})$-representation with highest weight λ, and let $\langle\cdot,\cdot\rangle$ be an invariant nondegenerate Hermitian form on V_λ. Then, we have the following upper bound for its signature:*
$$\mathrm{Sig}(\langle\cdot,\cdot\rangle) \leq \sum_{\eta\wr\lambda,\ \eta=w_0(-\eta)} \mathrm{Sig}(\langle\cdot,\cdot\rangle|_{V_\eta\times V_\eta}).$$

Proof. Like before, we'll now construct the Hermitian matrix of
$$\langle\cdot,\cdot\rangle|_{V_\lambda\times V_\lambda}$$
for each block $\langle\cdot,\cdot\rangle|_{V_\eta\times V_\mu}$, as η and μ range over all highest $\mathfrak{gl}(n-1,\mathbb{C})$-weights that interleave with λ. Again, there are several cases:

- If $\mu \neq w_0(-\mu)$, then, by Schur's lemma, for all $\eta \neq w_0(-\mu)$, there exists no nonzero intertwining operator $V_\mu \to (V_\eta)^h$. As a result, $\langle\cdot,\cdot\rangle|_{V_\eta\times V_\mu} = 0$ for all $\eta \neq w_0(-\mu)$.

  For V_μ and $V_{w_0(-\mu)}$, since $V_{w_0(-\mu)} \cong (V_\mu)^h$, we can find a basis $\{v_i\}$ of V_μ and a dual basis $\{w_j\}$ of $V_{w_0(-\mu)}$ such that $\langle v_i, w_j\rangle = \delta_{ij}$. The proof of this is essentially the same as Proposition 3.1.

As a result of these conclusions, the Hermitian matrix of $\langle\cdot,\cdot\rangle$ when restricted to $V_\mu \oplus V_{w_0(-\mu)}$ will be of the form

$$\begin{array}{cc} & \begin{array}{cc} \mu & w_0(-\mu) \end{array} \\ \begin{array}{l} \mu \\ w_0(-\mu) \end{array} & \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{array},$$

which, just as in Proposition 3.1, has signature 0.

- On the other hand, if $\mu = -w_0(\mu)$, then there exists a unique, $\mathfrak{gl}(n-1,\mathbb{C})$-intertwining isomorphism $T\colon V_\mu \to V_{-w_0(\mu)} = (V_\mu)^h$, and therefore the signature of $\langle\cdot,\cdot\rangle$ when restricted to V_μ is the same as the signature of T's induced Hermitian form.

Putting this all together, the Hermitian matrix will be indefinite outside of its restriction to the self-dual highest-weight representations. Inside the self-dual highest-weight representations (all η such that $\eta = -w_0(\eta)$ and $\eta \wr \lambda$), the form

$$\langle\cdot,\cdot\rangle|_{V_\eta \times V_\eta}$$

is only unique up to a real scalar multiple, and thus, we only know the *absolute value* of the difference between the dimensions of its maximal positive and negative definite subspaces – the signed difference will flip sign if the scalar multiple is negative. Hence, the best we can say from this argument is the bound

$$\mathrm{Sig}(\langle\cdot,\cdot\rangle) \leq \sum_{\eta\wr\lambda,\ \eta=w_0(-\eta)} \mathrm{Sig}(\langle\cdot,\cdot\rangle|_{V_\eta\times V_\eta}),$$

and we have the desired result. □

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