On GKM Description of the Equivariant Cohomology of Affine Flag Varieties and Affine Springer Fibers

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Abstract

For a projective variety endowed with a torus action, the equivariant cohomology is determined by the fixed points of codimension 1 subtori. Especially, when the fixed points of the torus are finite and fixed varieties under the action of codimension 1 subtori have dimension less than or equal to 2, equivariant cohomology can be described by discrete conditions on the pair of fixed points via GKM description of localizaton theorem. We provide explicit formula for the equivariant cohomology of affine flag varieties and prove the isomorphism between the i-th cohomology of affine flag variety and that of affine Springer fibers for i less than the rank of affine flag variety.

1 Introduction

Given a topological space X and a commutative ring R, the cohomology ring $H^*(X,R)$ encapsulates the topological properties of X up to homotopy equivalence. However for a topological space X with the action of topological group G, the cohomology ring fails to capture the information about the group action. On the other hand, equivariant cohomology, our main target of interest, aptly reflects the topological properties along with the group action. Throughout the paper, we assume the coefficients equal to $R = \mathbb{Q}$.

In Section 2, we provide the preliminaries of equivariant cohomology and statement of the prodigious localization theorem by Goresky, Kottwitz and MacPherson [3]. In short, the information of the equivariant cohomology of projective variety with torus action is contained inside the one dimensional orbits and fixed points of the torus action. The GKM description is our main methodology for computing the equivariant cohomology of varieties.

In Section 3, we identify the fixed points and one dimensional orbits of affine flag variety $\tilde{Fl}$ under the extended torus $\tilde{T}$ action. Through GKM description, we provide complete description of the equivariant cohomolgy as a module over polynomial ring $\mathbb{S}$ generated by the characters of $\tilde{T}$. Theorem 3.1 explicitly states the generators of a module.

In Section 4, we give GKM description of the equivariant cohomology of $\tilde{Fl}_1$, an affine Springer fiber generated by ts with a regular semisimple element s. Subsequently, we prove that the i-th cohomology of $\tilde{Fl}_1$ is isomorphic to the i-th cohomology of $\tilde{Fl}$ whenever i is less than the rank of affine flag variety:

$$h^i(\tilde{Fl}) = h^i(\tilde{Fl}_1).$$

Additionally, we suggest a counterexample of non isomorphism when i is the rank of the affine flag variety.

In Section 5, we reformulate the classical results in equivariant cohomology of flag varieties under GKM description. Furthermore, we derive the ring structure of the equivariant cohomology of affine Grassmannians Gr and affine flag varieties $\tilde{Fl}$ endowed with extended torus $\tilde{T}$ action:

$$H^*_{\tilde{T}}(Gr) \cong \mathbb{S} \otimes_{\mathbb{Q}} \mathbb{Q}[g_1, \cdots, g_N]$$

$$H^*_{\tilde{T}}(\tilde{Fl}) \cong \mathbb{S}[t] \otimes_{\mathbb{S}[t]^W} H^*_{\tilde{T}}(Gr)$$

where the degrees of g_i are the degrees of the generators of $\mathbb{S}^W$, polynomial ring fixed by Weyl group.

In Section 6, we briefly discuss the equivariant cohomology ring of affine Springer fibers of $sl(2)$. Additionally we suggest a conjecture which states that the equivariant cohomology of affine flag variety surjects to the equivariant cohomology of affine Springer fibers fixed under the lattice action.

2 Equivariant cohohmology and localization theorem

We follow the terminology and notations used in [1]. We assume X to be the topological space with the action of a topological group G. While it is most natural to consider the cohomology of X/G to encapsulate the information about the group action, the orbit space X/G is often not a well behaved topological space whenever the action is not free. Thus we first construct topological space which is homotopy equivalent to X with free G-action.

Given a topological group, there exists a contractible space E_G with a free G-action inducing a universal G-bundle $E_G \to B_G = E_G/G$. Therefore $X \times E_G$ is homotopy equivalent to X with G acting freely. The equivariant cohomology $H^*_G(X)$ is then defined as the cohomology of the orbit space of $X \times E_G$ with diagonal G-action:

$$H^*_G(X) = H^*((X \times E_G)/G).$$

By the universality of principal G-bundle, equivariant cohomology is invariant under the construction of G-bundle $E_G \to B_G$. From the fibration $(X \times E_G)/G \to E_G/G$ with fiber X, we obtain the ring homomorphism $H^*_G(pt) \to H^*_G(X)$ which induces $H^*_G(X)$ as an algebra over $H^*_G(pt)$

For a compact torus $T = (S^1)^n$, we define $\Xi(T) = Hom(T, S^1)$, a character lattice of T. From a universal T-bundle

$$(S^\infty)^n \to (\mathbb{CP}^\infty)^n,$$

we have

$$H_T^*(pt) = H^*((\mathbb{CP}^\infty)^n) = \mathbb{S}$$

where $\mathbb{S} = Sym^*(\Xi(T))$ is a polynomial ring over the field $\mathbb{Q}$ with each indeterminates of degree 2. In particular we have the following localization theorem in [1] Proposition 2 and Theorem 6.

Proposition 2.1. *Let X be a complex projective manifold with compact torus T-action. For all subtori $T' \subset T$, let $X^{T'}$ be the set of fixed points of T'-action. Then $H_T^*(X)$ is free S-module with isomorphism $H^*(X) \cong H_T^*(X)/\mathbb{S}^+$. Furthermore, we have the injective map of equivariant cohomology*

$$i_T^* : H_T^*(X) \to H_T^*(X^T)$$

with the image equal to the intersection of the images of maps

$$i_{T,T'}^* : H_T^*(X^{T'}) \to H_T^*(X^T)$$

where T' runs over all subtori of codimension 1.

Under the condition that X^T is finite and $\dim(X^{T'}) \leq 2$ for all subtori of codimension 1, $X^{T'}$ is the union of points and $\mathbb{CP}^1$ with $\mathbb{CP}^1$ connecting two fixed points. On each $\mathbb{CP}^1$, the action of T is induced by character $\chi : T \to T/T'$. Therefore when we denote every fixed complex projective curves E_j, connecting points $j_0, j_\infty \in X^T$, the image $H_{T'}$ of the inclusion $i_{T,T'}^* : H_T^*(X^{T'}) \to H_T^*(X^T) = Map(X^T, \mathbb{S})$ is given by

$$H_{T'} = \left\{ f \in Map(X^T, \mathbb{S}) \mid f(j_0) \equiv f(j_\infty) \mod \chi, \forall E_j = \mathbb{CP}^1 \right\}.$$

On the same line, Goresky, Kottwitz and MacPherson [3] established the following localization theorem under the aforementioned conditions.

Proposition 2.2. (Goresky, Kottwitz, and MacPherson [3]) *Let X a complex projective manifold with compact torus T-action. Suppose X has finite fixed points and* $\dim(X^{T'}) \leq 2$ *for all subtori $T' \subset T$ of codimension 1. Let E_j index all complex projective line $\mathbb{CP}^1$ fixed by subtori of codimension 1 inducing a character χ. Then under the inclusion $i_T^* : H_T^*(X) \to H_T^*(X^T)$, we have*

$$H_T^*(X) = \left\{ f \in Map(X^T, \mathbb{S}) \mid f(w) \equiv f(w') \mod \chi, \ \forall j, \forall w, w' \in E_j \cap X^T \right\}.$$

Furthermore, $p^* : H_T^*(pt) \to H_T^*(X)$ is an injection defined by $p^*(f) = (f, f, \cdots, f)$.

Our goal for the rest of the paper is to find the equivariant cohomology of affine flag varieties and affine Springer fibers under the torus action.

Let $F = \mathbb{C}((\epsilon))$ be the field of Laurant series over $\mathbb{C}$ and let $\mathfrak{o} = \mathbb{C}[[\epsilon]]$. For a connected reductive algebraic group G over $\mathbb{C}$ and its lie algebra $\mathfrak{g}$, affine flag variety is the quotient $\tilde{Fl} = \tilde{G}/\tilde{B}$ where $\tilde{G} = G(F)$ and $\tilde{B}$ is the set of Iwamori subalgebras of $\mathfrak{g} \otimes_{\mathbb{C}} F$.

Affine Grassmannian is defined by $Gr = G(F)/G(\mathfrak{o})$. For an element $\gamma \in \mathfrak{g}(F)$, the fixed point set of a vector field induced by γ is defined to be the affine Springer fiber:

$$\tilde{Fl}_\gamma = \left\{ gG(\mathfrak{o}) \in Gr \mid Ad(g^{-1})(\gamma) \in \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{o} \right\}.$$

For a regular semisimple element s and $\gamma = t^n s$ for a natural number n, we define the affine Springer fibers $\tilde{Fl}_n = \tilde{Fl}_\gamma$. For a maximal torus T of G, we define extended torus $\tilde{T}(\mathbb{C}) = T(\mathbb{C}) \times \mathbb{C}^*$, then the aformentioned varieties - affine flag varieties, affine Grassmannians, and affine Springer fibers - have natural actions of T and $\tilde{T}$.

From the classical results on affine flag varieties and affine Grassmannians, the varieties are direct limits of finite dimensional complex projective varieties with induced torus action. Additionally, affine Springer fiber $\tilde{Fl}_1$ admits a Hessenberg paving which gives a direct limit of complex projective varieties [5]. Since direct limit commutes with cohomology, the localization theorem, which is valid in complex projective varieties, holds in these infinite dimensional complex varieties.

To apply the localization theorem, we need to determine the fixed points and one dimensional orbits $\mathbb{CP}^1$ of each varieties. Section 13 of [4] gives an explicit description of those in terms of affine Weyl groups and root system which will be described along with the equivariant cohomologies in the remaining sections.

3 Equivariant cohomology of affine flag varieties

Our goal of this section is to compute equivariant cohomology of affine flag varieties $\tilde{Fl}$ as a free module over polynomial ring of characters. Denote T a maximal torus of the flag variety and $\tilde{T}(\mathbb{C}) = T(\mathbb{C}) \times \mathbb{C}^*$ an extended torus. We denote $\mathbb{S} = Sym^*\Xi(T)$ as a polynomial ring of characters over $\mathbb{Q}$ and $\tilde{\mathbb{S}} = \mathbb{S}[t]$ as a polynomial ring of extended characters. Additionally we denote Φ as a root system of the affine Weyl group $\tilde{W}$ corresponding to the affine flag variety.

To compute the equivariant cohomology $H^*_{\tilde{T}}(\tilde{Fl})$, we first identify the fixed points and one dimensional orbits of $\tilde{T}$. The fixed points are the elements of affine Weyl group $\tilde{W}$. For a character $\chi \in \Xi(\tilde{T})$, $\tilde{Fl}^{ker\chi} \neq \tilde{Fl}^{\tilde{T}}$ if and only if $\chi \in \Phi$. If $\chi = \alpha \in \Phi$, complex projective lines fixed by $ker\chi$ connect the fixed points w and $s_\alpha w$. Therefore, the localization theorem yields the following GKM description of equivariant cohomology:

$$H^*_{\tilde{T}}(\tilde{Fl}) = \left\{ f \in Map(\tilde{W}, \tilde{\mathbb{S}}) \mid f(w) \equiv f(s_\alpha w) \mod \alpha, \forall \alpha \in \Phi, \forall w \in \tilde{W} \right\}.$$

Degree of f should be finite since we are taking the colimit with respect to the projecting pavings.

We extend this notion to arbitrary Coxeter system (W, S) with root system Φ. We follow the notations and terminologies in [7]. Given the lattice V generated by the simple roots $\{\alpha_s \mid s \in S\}$, define $\mathbb{S}(V)$ be the symmetric algebra of V with coefficient in $\mathbb{Q}$. Define the analogous ring

$$H^*(W, S) = \{ f \in Map(W, \mathbb{S}(V)) \mid f(w) \equiv f(s_\alpha w) \mod \alpha, \forall \alpha \in \Phi, \forall w \in W \}$$

which yields $H^*(\tilde{W}, S) = H^*_{\tilde{T}}(\tilde{Fl})$. Let Π be the set of positive roots of Φ. In this section we establish the following theorem.

Theorem 3.1. *For an element $w \in W$, there exists a unique class $f_w \in H^*(W, S)$ of degree $l(w)$ satisfying the following condition.*

- $f_w(w') = 0$, *for all* $l(w') \leq l(w)$, $w' \neq w$
- $f_w(w) = \prod_{\alpha \in \Pi, s_\alpha w < w} \alpha$

Additionally, f_w *is a free module generator of* $H^*(W, S)$ *over* $\mathbb{S}(V)$.

Before proving the theorem, we state some properties on Coxeter groups in [7]. Recall that a Coxeter group is a poset with respect to the Bruhat order $<$.

Lemma 3.2. [7] Let $w \in W, \alpha \in \Pi$. Then the following holds.

$$ws_\alpha > w \text{ if and only if } w\alpha > 0.$$

$$s_\alpha w > w \text{ if and only if } w^{-1}\alpha > 0.$$

It is implied in the above lemma that the number of reflections s_α such that $s_\alpha w < w$ in the Bruhat ordering is equal to $l(w)$ since the number of positive roots sent to the negative roots via w is equal to the length of w.

Lemma 3.3. [7] For an element $w \in W$ with reduced expression $s_1 \cdots s_n$ i.e. $l(w) = n$, then the set of elements in the Bruhat interval $[1, w]$ is

$$\{s_{i_1} \cdots s_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n\},$$

all the subexpressions of the reduced expression $s_1 \cdots s_n$.

Especially, whenever $s_\alpha w < w$, then $s_\alpha = s_1 \cdots \hat{s}_i \cdots s_n$ for some $1 \leq i \leq n$. This criterion is called Strong Exchange Condition.

f_w in Theorem 3.1 will be constructed inductively by assigning a polynomial for each $w' \in W$ in the increasing length. We call

$$f(w) \equiv f(s_\alpha w) \mod \alpha$$

the *GKM condition* throughout the paper. Under the poset representation of the Coxeter group W, the following theorem implies the inductive construction of $f \in H^*(W, S)$.

Theorem 3.4. *If* $f : [1, w] \setminus \{w\} \to \mathbb{S}(V)$ *satisfies the GKM condition, then* f *can be extended to the function* $F : [1, w] \to \mathbb{S}(V)$ *so that* F *satisfies the GKM condition in the interval* $[1, w]$.

Proof. We proceed by induction on $l(w) \geq 0$. $l(w) = 0$ is a trivial case. Suppose the theorem holds for $l(w) = n - 1$. When $l(w) = n$, let $w = s_1 \cdots s_n$ be the reduced expression where $s_1 = s_\alpha$ and define $w_1 = s_1 w < w$.

Notice that $p \equiv q \mod \beta$ if and only if $wp \equiv wq \mod w\beta$. Hence,

$$s_1 f(s_1 x) \equiv s_1 f(s_1 s_\beta x) \mod \beta \Leftrightarrow f(s_1 x) \equiv f(s_1 s_\beta x) \mod s_1\beta \tag{1}$$

for all $x \in [1, w_1]$. Since $s_{s_1\beta} = s_1 s_\beta s_1$, $s_1 f(s_1\ \cdot)$ restricted to $[1, w_1] \setminus \{w_1\}$ satisfy the GKM condition. By the induction hypothesis, $s_1 f(s_1\ \cdot) : [1, w_1] \setminus \{w_1\} \to \mathbb{S}(V)$ can be extended to $G : [1, w_1] \to \mathbb{S}(V)$:

$$G(w_1) \equiv s_1 f(s_1 s_\gamma w_1) \mod \gamma, \quad \forall \gamma \in \Pi \text{ such that } s_\gamma w_1 < w_1.$$

Therefore we have

$$f(w_1) - G(w_1) \equiv f(s_\gamma w_1) - s_1 f(s_1 s_\gamma w_1) \mod \gamma, \quad \forall \gamma \in \Pi \text{ such that } s_\gamma w_1 < w_1.$$

Since $f(\cdot) - s_1 f(s_1 \ \cdot) : [1, w_1] \setminus \{w_1\} \to \mathbb{S}(V)$ satisfies GKM condition on $[1, w_1] \setminus \{w_1\}$ by the same logic in equation 1 and is divisible by α, we can apply induction hypothesis on $\frac{f(\cdot) - s_\alpha f(s_\alpha \ \cdot)}{\alpha}$ to obtain a polynomial $H \in \mathbb{S}(V)$ such that

$$\alpha H \equiv f(s_\gamma w_1) - s_1 f(s_1 s_\gamma w_1) \mod \gamma, \quad \forall \gamma \in \Pi \text{ such that } s_\gamma w_1 < w_1.$$

Therefore, $f(w_1) - G(w_1) \equiv \alpha H \mod \prod_{\gamma \in \Pi, s_\gamma w_1 < w_1} \gamma$ which is equivalent to

$$s_1 f(w_1) - s_1 G(w_1) \equiv -\alpha s_1 H \mod \prod_{\beta \in \Pi \setminus \alpha, s_\beta w < w} \beta$$

by Lemma 3.3 or Strong Exchange Condition. Therefore $s_1 G(w_1)$ can be modified by some multiple of $\prod_{\beta \in \Pi \setminus \alpha, s_\beta w < w} \beta$ so that the resulting $F(w)$ satisfies $F(w) \equiv f(w_1) \mod \alpha$. Since

$$s_1 G(w_1) \equiv f(s_\beta w) \mod \beta, \quad \forall \beta \in \Pi \setminus \{\alpha\} \text{ such that } s_\beta w < w$$

by the same logic in equation 1, this implies

$$F(w) \equiv f(s_\beta w) \mod \beta, \quad \forall \beta \in \Pi \text{ such that } s_\beta w < w.$$

Therefore, f can be extended to w via $F(w)$ under GKM condition. □

Now we prove of our main Theorem 3.1 of this section.

Proof of Theorem 3.1. Let W_n be the subposet of W with elements of length less than or equal to n. If $f \in Map(W_n, \mathbb{S}(V))$ is defined over W_n with GKM condition, f can be extended to $Map(W_{n+1}, \mathbb{S}(V))$ with GKM condition by Theorem 3.4.

The initial condition given by the statement of the theorem defines $f \in Map(W_{l(w)}, \mathbb{S}(V))$ satisfying GKM condition. Hence f is aptly extended to $f_w \in Map(W, \mathbb{S}(V))$ inductively so that the GKM condition is satisfied. Especially for each element w' with $l(w') > l(w)$, $f(w')$ should satisfy $l(w')$ modular equations. Hence $f(w')$ is uniquely determined when we restrict the degree of $f(w')$ to be $l(w)$. Therefore, $f_w \in H^*(W, S)$ of degree $l(w)$ exists uniquely with the conditions in the theorem.

To prove f_w's are the generators of $H^*(W, S)$ over $\mathbb{S}(V)$ as a free module, suppose $f \in H^*(W, S)$. If $f(w') = 0$ for all $l(w') < n$, then $f(w)$ is multiple of $f_w(w)$ by the GKM condition on w when $l(w) = n$. We can subtract the multiple of f_w by a polynomial in $\mathbb{S}(V)$ from f to induce $f(w) = 0$. Therefore for $N > \deg f$, there exists a linear combination F of f_w over $\mathbb{S}(V)$ such that $F(w) = f(w)$ for all $l(w) \leq N$. Therefore $F = f$ by the GKM condition on the rest of the domain.

The linear independence of f_w follows directly from the initial conditions of f_w, hence f_w form a free module generators of $H^*(W, S)$ over $\mathbb{S}(V)$. □

We end our discussion by reinterpreting the classical result on the Betti number of affine flag variety in the GKM description.

Corollary 3.5. *For an affine flag variety $\tilde{Fl}$ with affine Weyl group $\tilde{W}$,*

$$h^i(\tilde{Fl}) = \#(w \in \tilde{W} \mid l(w) = i).$$

Proof. By Theorem 3.1, we have identified the generators of $H^*_{\tilde{T}}(\tilde{Fl})$ as a free module over $\tilde{\mathbb{S}}$, namely f_w, $\forall w \in \tilde{W}$. From the isomorphism $H^*(\tilde{Fl}) = H^*_{\tilde{T}}(\tilde{Fl})/\tilde{\mathbb{S}}^+$, $h^i(\tilde{Fl})$ is equal to the number of f_w with degree equal to i. □

Recall that the affine flag varieties $\tilde{Fl} = \tilde{G}/\tilde{B}$ admits a stratification through Bruhat decomposition:

$$\tilde{Fl} = \amalg_{w \in \tilde{W}} \tilde{B}w\tilde{B}/\tilde{B}$$

with each cells being complex affine space. Therefore, Bruhat decomposition alternatively yields the Corollary 3.5.

4 Equivariant cohomology of affine Springer fibers

Our goal of this section is to analyze the equivariant cohomology $H^*_T(\tilde{Fl}_1)$ as a module over $\mathbb{S}$. We follow the same notations in the previous section. Let Φ_0 be a root system with respect to the maximal torus T and t be the additional root induced by extended torus $\tilde{T}(\mathbb{C}) = T(\mathbb{C}) \otimes \mathbb{C}^*$. Then $\Phi = \Phi_0 + \mathbb{Z}t$. Following the results in [3], we can identify the fixed points and one dimensional orbits of the torus action.

The fixed points of $\tilde{Fl}_1$ over T are the elements of the affine Weyl group $\tilde{W}$. For a character $\chi \in \Xi(T)$, $\tilde{Fl}_1^{ker\chi} \neq \tilde{Fl}_1$ if and only if $\chi = \alpha_0 \in \Phi_0$. If $\chi = \alpha_0 \in \Phi_0$, $\tilde{Fl}_1^{ker\chi}$ is the disjoint union of the affine Springer fibers X_1 of sl_2 induced by regular semisimple element. Recall from the theory of affine springer fibers that X_1 is an infinite chain of $\mathbb{CP}^1$. Therefore $w, w' \in X_1$ if and only if $w' = s_{\alpha_0+nt}w$ for some $n \in \mathbb{Z}$.

From the above description, the localization theorem states:

$$H^*_T(\tilde{Fl}_1) = \left\{ f \in Map(\tilde{W}, \mathbb{S}) \mid f(w) \equiv f(s_{\alpha_0+nt}w) \mod \alpha_0,\ \forall \alpha_0 \in \Phi_0, \forall n \in \mathbb{Z}, \forall w \in \tilde{W} \right\}.$$

By the isomorphism $\mathbb{S} = \mathbb{S}[t]/(t)$, the relation $f(w) \equiv f(s_{\alpha_0+nt}w) \mod \alpha_0$ is equivalent to $f(w) \equiv f(s_\alpha w) \mod \alpha$ over $\mathbb{S}[t]/(t)$ where $\alpha = \alpha_0 + nt \in \Phi$. Therefore the description of $H^*_T(\tilde{Fl}_1)$ can be translated to the root system of affine Weyl group:

$$H^*_T(\tilde{Fl}_1) = \left\{ f \in Map(\tilde{W}, \mathbb{S}[t]/(t)) \mid f(w) \equiv f(s_\alpha w) \mod \alpha,\ \forall \alpha \in \Phi, \forall w \in \tilde{W} \right\}.$$

From the inclusion $i : \tilde{Fl}_1 \to \tilde{Fl}$, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^*_{\tilde{T}}(\tilde{Fl}) & \longrightarrow & H^*_{\tilde{T}}(\tilde{Fl}_1) & \longrightarrow & H_{\tilde{T}}(\tilde{W}) \\
\downarrow & & \downarrow & & \downarrow \\
H^*_T(\tilde{Fl}) & \longrightarrow & H^*_T(\tilde{Fl}_1) & \longrightarrow & H_T(\tilde{W}) \\
\downarrow & & \downarrow & & \\
H^*(\tilde{Fl}) & \longrightarrow & H^*(\tilde{Fl}_1) & &
\end{array}$$

where the second row is the quotient of the first row by t through localization theorem. In the remaining section, we may observe the image of generators f_w described in Theorem 3.1 and prove the following theorem.

Theorem 4.1. *Given rank N of affine Weyl group $\tilde{W}$,*

$$h^i(\tilde{Fl}) = h^i(\tilde{Fl}_1)$$

for all $i < N$.

We first suggest the following lemma essential for the proof.

Lemma 4.2. *The images of $\{f_w \mid l(w) < N\}$ in $H_T^*(\tilde{Fl}_1)$ generate a free module over $\mathbb{S}$ which contains every elements of degree less than N.*

Proof. By the description of f_w given in Theorem 3.1, $i^*(f_w)$ is indeed free over $\mathbb{S}$. For $w \in \tilde{W}$ with reduced expression $w = s_1 s_2 \cdots s_n$, consider the first $m = \min(n, N)$ reflections, namely $s_1 s_2 \cdots s_m$. Let α_i be the simple root of the reflection s_i. Since the proper subset of the simple roots of affine Weyl group span a lattice with positive definite bilinear form, the lattice spanned by α_i for $i = 1, 2, \cdots, m$ remains the same rank even after quotienting out by t. (Notably, span of t is the kernel of the bilinear form given by the affine Weyl group.) Therefore, α is distinct in $\mathbb{S}[t]/(t)$ for all reflections s_α such that $s_\alpha s_1 s_2 \cdots s_m < s_1 s_2 \cdots s_m$.

Suppose $f \in Map(\tilde{W}, \mathbb{S}[t]/(t))$ is an element of $H_T^*(\tilde{Fl}_1)$ with degree less than N. As in the proof of Theorem 3.1, we proceed by subtracting f by the linear combination of $i^*(f_w)$ inductively. Starting from the identity element, we subtract multiple of $i^*(f_1)$ to induce $f(1) = 0$. Suppose we have obtained $f(w) = 0$ for all $w \in \tilde{W}$ with $l(w) < n$. Then for all element with reduced expression $w' = s_1 \cdots s_n$, $f(w')$ is the multiple of $\prod_{s_\alpha \bar{w}' < \bar{w}'} \alpha$ where $\bar{w}' = s_1 \cdots s_m$. This is because $\alpha \in \Phi$ with $s_\alpha \bar{w}' < \bar{w}'$ are all distinct in $\mathbb{S}[t]/(t)$. Since $\deg f < N$, $f(w')$ is either multiple of $i^*(f_{w'})(w')$ or 0. Therefore, $f(w')$ can be eliminated by the multiple of $i^*(f_{w'})$. Consequently, we can express f as a linear combination of the images of f_w. □

Proof of Theorem 4.1. Localization theorem induces $H_T^*(\tilde{Fl}_1)$ as a free module over $\mathbb{S}[t]/(t)$ with each generators homogeneous of degree i. The number of generators of degree i is equal to $h^i(\tilde{Fl}_1)$. By Lemma 4.2, $\{i^*(f_w) \mid l(w) < N\}$ and generators of $H_T^*(\tilde{Fl}_1)$ of degree less than N span the same space over $\mathbb{S}[t]/(t)$. Therefore, the number of generators of degree $i < N$ is the number of elements of degree i in $\{i^*(f_w) \mid l(w) < N\}$ which proves the theorem. □

Indeed there is a counterexample where Theorem 4.1 does not hold for $i = N$. If we consider the affine Springer fibers $\tilde{Fl}_1$ of $sl(3)$, $h^2(\tilde{Fl}_1)$ turns out to be infinite while $h^2(\tilde{Fl})$ is finite.

5 Ring structure of equivariant cohomology

In the previous sections, we have investigated the module structure of the equivariant cohomology over polynomial ring of characters. In this section, we provide the ring structure of the equivariant cohomology of affine flag variety. We would like to mention the classical result below which explicitly describes the ring structure of equivariant cohomology of the flag varieties.

Proposition 5.1. *(Borel Description [6]) For a compact semisimple Lie group G and a maximal torus T, we have*

$$H_T^*(G/T) = \mathbb{S} \otimes_{\mathbb{S}^W} \mathbb{S}$$

where W is a Weyl group and $\mathbb{S}$ is a polynomial ring of characters.

Recall Section 3 where we had an isomorphism $H_T^*(G/T) \cong H^*(W,S)$ for a Coxeter system (W,S) of a flag variety. The explicit mapping $\mathcal{K} : \mathbb{S} \otimes_{\mathbb{S}^W} \mathbb{S} \to H^*(W,S)$ is given [6] by

$$\mathcal{K}(p \otimes q)(w) = p(wq)$$

for $p, q \in \mathbb{S}$ and $w \in W$. Before diving directly into the affine flag variety, we first digest the ideas behind the proof of Proposition 5.1. These involve the following operations on $H^*(W,S)$.

Define an action of $w \in W$ on $H^*(W,S)$ by $w \cdot f(v) = f(vw^{-1})$. Since $w \cdot f$ also satisfy GKM condition, this induces well defined group action on $H^*(W,S)$. Consequently, we can define the following operations analogous to the differentiation in Schubert polynomials.

Definition 5.2. For a positive root $\alpha \in \Pi$, we define the following differentiation on $f \in H^*(\Gamma)$.

$$\partial_\alpha f(w) = \frac{f(w) - f(ws_\alpha)}{-w\alpha}.$$

It is an easy check that this operation is well defined with the property that $\partial_\alpha^2 = 0$.

Lemma 5.3. *Ring homomorphism $\mathcal{L} : \mathbb{S} \otimes_{\mathbb{Q}} \mathbb{S} \to H^*(W,S)$ defined by*

$$\mathcal{L}(p \otimes q)(w) = p(wq)$$

is surjective.

Proof. Let $f \in H^*(W,S)$ where f is a homogeneous. Since $H^*(W,S)$ is a graded ring, we can proceed by induction on the degree of f, namely n.

When $n = 0$, then f is in the image of $\mathcal{L}$. Suppose every element of $H^*(W,S)$ with degree less than n is contained in the image. We use the strategy of summing over the W action on f.

For $\alpha \in \Pi$, $\deg \partial_\alpha f = n - 1$. By induction hypothesis, $\partial_\alpha f$ is contained in the image of $\mathcal{L}$ which implies that $f - s_\alpha \cdot f$ is contained in the image. Consequently, for any $w \in W$, we have $w \cdot f - ws_\alpha \cdot f$ contained in the image. Since reflections generate the Weyl group, we have that $f - w \cdot f$ is contained in the image for all $w \in W$.

Since $\sum_{w \in W} w \cdot f$ is constant over the action of W, $\sum_{w \in W} w \cdot f$ is a constant function and hence an element in $\mathbb{S}$. From the fact that f is a linear combination of $\sum_{w \in W} w \cdot f$ and $f - w \cdot f$, f is also in the image of $\mathcal{L}$. □

The following definition is useful for the proof of Proposition 5.1.

Definition 5.4. For a root $\alpha \in \Phi$, we define the following differentiation D_α on $p \in \mathbb{S}$.

$$D_\alpha p = \frac{p - s_\alpha p}{-\alpha}.$$

Proof of Proposition 5.1. Since we have a surjection $\mathcal{L}$, it is sufficient for us to identify the kernel of $\mathcal{L}$ as an ideal I generated by $\{h \otimes 1 - 1 \otimes h \mid \forall h \in \mathbb{S}^W\}$.

Suppose $\sum p_i \otimes q_i$ is contained in the kernel of $\mathcal{L}$. We again proceed by the induction on n, namely the degree of $\sum p_i \otimes q_i$. Since initial condition is trivial, suppose the kernel contained in I for degree less than n. $\mathcal{L}(\sum p_i \otimes q_i) = 0$ is equivalent to $\sum p_i(wq_i) = 0$ for all $w \in W$. Therefore, for positive root α, $\mathcal{L}(\sum p_i \otimes D_\alpha q_i) = 0$ and induction hypothesis implies that $\sum p_i \otimes (q_i - s_\alpha q_i)$ is contained in I. As in the proof of Lemma 5.3, $\sum p_i \otimes q_i$ is a linear combination of $\sum p_i \otimes (\sum_{w \in W} wq_i) = 0$ and $\sum p_i \otimes (q_i - wq_i)$. Therefore, the kernel of $\mathcal{L}$ is I which proves the proposition. □

In the remaining section, we follow the same notations about Coxeter groups used in the previous sections. Let $\tilde{T}$ be an extended torus with the root system $\Phi = \Phi_0 + \mathbb{Z}t$ of $\tilde{W}$. Since t is in the kernel of the Coxeter bilinear form $<,>$, t is invariant under $\tilde{W}$ action. Recall that $\tilde{W}$ is a subgroup of affine transformations $\mathrm{Aff}(V)$ for a Euclidean space V spanned by roots Φ_0. The reflection $s_{\alpha,n} := s_{\alpha+nt}$ is equivalent to the reflection with respect to the affine hyperplane $H_{\alpha,n} = \{\lambda \in V \mid <\lambda, \alpha>= n\}$. Consult Chapter 4 of [7] for the complete details.

Under this representation of affine Weyl group, let Λ be the lattice spanned by $\Phi_0^\vee = \{\alpha^\vee = 2\alpha/ <\alpha, \alpha> \mid \alpha \in \Phi_0\}$. Then $\tilde{W} = \Lambda \rtimes W$ where Λ is the translation by the lattice element and W is the Weyl group generated by the reflections through the origin. Hence $\tilde{W}$ induces an action on the lattice Λ. By averaging the function $f \in Map(\tilde{W}, \tilde{\mathbb{S}})$ over the Weyl group W we can reduce the equivariant cohomology of affine flag variety to the equivariant cohomology of affine Grassmannians. Therefore, we first investigate the ring structure of affine Grassmannians.

5.1 Equivariant cohomology ring of affine Grassmannian Gr

Given the extended torus action $\tilde{T}$ on Gr, the set of fixed points of Gr is the lattice Λ. Likewise for a character $\chi \in \Xi(\tilde{T})$, $Gr^{ker\chi} \neq Gr^{\tilde{T}}$ if and only if $\chi \in \Phi$. In particular, $Gr^{ker\chi}$ is union of $\mathbb{CP}^1$ connecting λ and $s_{\alpha,n}\lambda$ if $\chi = \alpha + nt \in \Phi$. Therefore, the localization theorem yields

$$H^*_{\tilde{T}}(Gr) = \left\{ g \in Map(\Lambda, \tilde{\mathbb{S}}) \mid g(\lambda) \equiv g(s_{\alpha,n}\lambda) \mod \alpha + nt,\ \forall \alpha \in \Phi_0, \forall n \in \mathbb{Z} \right\}.$$

Recall from the theory of Weyl group that the subalgebra of $\mathbb{S}$ fixed by Weyl group W is a polynomial ring $\mathbb{S}^W$ [7]. Let $\{d_1, \cdots, d_n\}$ be the degrees of the generators of polynomial ring $\mathbb{S}^W$. Ginzberg [2] proved that the cohomology ring of affine grassmannian $H^*(Gr)$, is the polynomial ring generated by the elements of degree $d_i - 1$. From the isomorphism of $\mathbb{S}$ modules

$$H^*_{\tilde{T}}(Gr) \cong H^*(Gr) \otimes_{\mathbb{Q}} \mathbb{S}[t]$$

induced from fibration $(Gr \times E_{\tilde{T}})/\tilde{T} \to E_{\tilde{T}}/\tilde{T}$ with fiber Gr, we can anticipate $H^*_{\tilde{T}}(Gr)$ to be a polynomial ring. The following theorem indeed proves this speculation and specifies the generators of $H^*_{\tilde{T}}(Gr)$ as a polynomial ring under GKM description.

Theorem 5.5. *For $G_i \in \mathbb{S}^W$, $g_i \in Map(\Lambda, \tilde{\mathbb{S}})$ defined by*

$$g_i(\lambda) = \frac{\lambda G_i - G_i}{t}$$

is an element in $H^*_{\tilde{T}}(Gr)$. For a choice of algebraically independent generators $G_1, \cdots G_N$ of $\mathbb{S}^W$, $g_1, \cdots, g_N$ are algebraically independent and we have the following ring isomorphism

$$H^*_{\tilde{T}}(Gr) \cong \mathbb{S} \otimes_{\mathbb{Q}} \mathbb{Q}[g_1, \cdots, g_N].$$

Proof. Recall $\alpha^\vee = s_{\alpha,1}s_\alpha \in \tilde{W}$. For any coroot $\alpha^\vee = \frac{2\alpha}{<\alpha,\alpha>}$ of a root $\alpha \in \Phi_0$, observe $\alpha^\vee \beta = \beta + 2\frac{<\beta,\alpha>}{<\alpha,\alpha>}t$ as an action of $\tilde{W}$ on Φ by direct computation. Hence $\lambda G_i \equiv G_i \mod t$ for all $\lambda \in \Lambda$ which well defines g_i. To prove $g_i \in H^*_{\tilde{T}}(Gr)$, we only need to check GKM condition. Let $\bar{G}_i \in Map(\Lambda, \mathbb{S})$ such that $\bar{G}_i(\lambda) = \lambda G_i$. Then we have

$$\bar{G}_i(s_{\alpha,n}\lambda) = s_{\alpha,n}\lambda s_\alpha G_i = s_{\alpha,n}\bar{G}_i(\lambda)$$

from the fact that $s_\alpha G_i = G_i$. From $\bar{G}_i(s_{a,n}\lambda) \equiv G_i(\lambda) \mod \alpha + nt$, $\bar{G}_i$ satisfies the GKM condition. Therefore, g_i is an element of $H^*_{\tilde{T}}(Gr)$.

Now to prove the rest of the theorem, we suggest the following lemma.

Lemma 5.6. *Suppose $q \in \mathbb{S} \subset \mathbb{S}[t]$ is invariant under the action of $\tilde{W}$. Then $q \in \mathbb{C}$.*

Proof. Since the alcove of the affine Weyl group in V is a compact simplex, q attains maximum and minimum. If q is invariant under the action of $\tilde{W}$, then the functions q restricted to every alcoves are the same since $\tilde{W}$ acts transitively on the alcoves. Therefore, q is bounded in V which is only possible when q is a constant function in $\mathbb{C}$. □

Using the lemma above we can proceed to the following lemma.

Lemma 5.7. *Let $x_1, \cdots, x_N$ be the simple roots of Φ_0 so that $\mathbb{S}[t] = \mathbb{Q}[x_1, \cdots, x_N, t]$. Then $x_1, \cdots x_N, t, \bar{G}_1, \cdots, \bar{G}_N$ are algebraically independent in $H^*_{\tilde{T}}(Gr)$.*

Proof. Notice that $x_1, \cdots x_N, t$ is contained in $H^*_{\tilde{T}}(Gr)$ through the inclusion of $\mathbb{S}[t]$ in $H^*_{\tilde{T}}(Gr)$ given by localization theorem. Suppose $\sum_{i_0,i_1,\cdots,i_N \geq 0} q_{i_0,i_1,\cdots,i_N} t^{i_0} \bar{G}_1^{i_1} \cdots \bar{G}_N^{i_N} = 0$ where $q_{i_0,i_1,\cdots,i_N} \in \mathbb{S}$. It is sufficient to show that $q_{i_0,i_1,\cdots,i_N} = 0$ for all $i_0, i_1, \cdots, i_N \geq 0$. We proceed by induction on $\deg q = \max_{i_0,i_1,\cdots,i_N \geq 0} \deg q_{i_0,i_1,\cdots,i_N}$. Initial condition when $\deg q = 0$, is trivial from definition: $t, \bar{G}_1, \cdots, \bar{G}_N$ are algebraically independent over $\mathbb{C}$.

Notice that $\sum q_{i_0,i_1,\cdots,i_N} t^{i_0} \bar{G}_1^{i_1} \cdots \bar{G}_N^{i_N} = 0$ if and only if $\sum q t^{i_0} \lambda(G_1^{i_1} \cdots G_N^{i_N}) = 0$ for all $\lambda \in \Lambda$.(we omit the subscripts of q for brevity.) Since G_i is invariant under W and $\tilde{W} = \Lambda \rtimes W$, we have $\sum q t^{i_0} w(G_1^{i_1} \cdots G_n^{i_N}) = 0$ for all $w \in \tilde{W}$. Therefore, $\sum q t^{i_0} \bar{G}_1^{i_1} \cdots \bar{G}_N^{i_N} = 0$ if and only if $\sum (wq) t^{i_0} G_1^{i_1} \cdots G_N^{i_N} = 0$ for all $w \in \tilde{W}$. Then $\sum (D_{\alpha+nt}q) t^{i_0} G_1^{i_1} \cdots G_N^{i_N} = 0$ exploits the fact that $D_{\alpha+nt}q = 0$ for all roots $\alpha+nt \in \Phi$ by induction hypothesis. Therefore, q is invariant under the the action of $\tilde{W}$, yielding $q_{i_0,\cdots,i_N} \in \mathbb{C}$ from Lemma 5.6. This is the initial case of induction. □

Lemma 5.7 implies the algebraic independence of $x_1, \cdots, x_N, t, g_1, \cdots, g_N$. Therefore, there exists an injection

$$\mathbb{S}[t] \otimes_{\mathbb{Q}} \mathbb{Q}[g_1, \cdots, g_N] \hookrightarrow H^*_{\tilde{T}}(Gr).$$

However, the module structure

$$H^*_{\tilde{T}}(Gr) \cong H^*(Gr) \otimes_{\mathbb{Q}} \mathbb{S}[t],$$

implies that the injection above is indeed isomorphism.(For every degree, the dimension of a vector space over $\mathbb{Q}$ is identical.) Therefore, the injection is an isomorphism. □

Having the equivariant cohomology ring of affine Grassmannian Gr, we obtain the equivariant cohomology ring of affine flag variety $\tilde{Fl}$ in the following subsection.

5.2 Equivariant cohomology ring of affine flag variety $\tilde{Fl}$

The following theorem describes the ring structure of the equivariant cohomology of affine flag variety.

Theorem 5.8. *The ring homomorphism*

$$\mathcal{K} : \mathbb{S} \otimes_{\mathbb{S}^W} H^*_{\tilde{T}}(Gr) \to H^*_{\tilde{T}}(\tilde{Fl})$$

defined by $\mathcal{K}(p \otimes g)(\lambda w) = ((\lambda w)p)g(\lambda)$ *for* $\lambda \in \Lambda, w \in W$ *is an isomorphism.*

From $\tilde{W} = \Lambda \rtimes W$, any element of $\tilde{W}$ is uniquely written as the product λw.

Proof. As in the proof of Proposition 5.1, we proceed in two steps.

We first prove the surjectivity of the map

$$\mathcal{L} : \mathbb{S} \otimes_{\mathbb{Q}} H^*_{\tilde{T}}(Gr) \to H^*_{\tilde{T}}(\tilde{Fl})$$

which factors through $\mathcal{K}$. We proceed by the induction on the degree of $f \in H^*_{\tilde{T}}(\tilde{Fl})$. The initial condition is trivial. For a root of Weyl group $\alpha \in \Phi_0$, we consider the derivative $\partial_\alpha f$. Then by induction hypothesis, $\partial_\alpha f$ is contained in the image. Additionally, $\sum_{w\in W} w \cdot f = \sum_{w \in W} f(\lambda w)$ is the image of $F = \sum_{w\in W} f(\lambda w) \in H^*_{\tilde{T}}(Gr)$. Therefore f is in the image of $\mathcal{L}$ by the same logic in the proof of Lemma 5.3.

It is now sufficient to prove that the kernel of $\mathcal{L}$ is generated by $\{h \otimes 1 - 1 \otimes h \mid h \in \mathbb{S}^W\}$. Indeed if $\mathcal{L}(\sum p_i \otimes g_i) = 0$, then $\mathcal{L}(\sum D_\alpha p_i \otimes g_i) = 0$ for all $\alpha \in \Phi$. Therefore we only need to prove that $\sum(\sum_{w\in W} wp_i) \otimes g_i$ is generated by $h \otimes 1 - 1 \otimes h$ by the same logic the proof of Proposition 5.1. From $\sum_{w\in W} wp_i \in \mathbb{S}(V)^W$, we obtain what is desired. Hence the theorem is proven. □

Theorem 5.8 yields the following corollaries.

Corollary 5.9. *For an extended torus* $\tilde{T}$ *action on affine flag variety* $\tilde{Fl}$ *and affine Grassmannian* Gr*, we have the following isomorphism of rings.*

$$\begin{aligned} H^*_{\tilde{T}}(\tilde{Fl}) &\cong \mathbb{S}[t] \otimes_{\mathbb{S}[t]^W} H^*_{\tilde{T}}(Gr) \\ &\cong H^*_T(Fl) \otimes_{H^*_T(pt)} H^*_{\tilde{T}}(Gr). \end{aligned}$$

Proof. The first isomorphism is immediate from Theorem 5.8. The second isomorphism is also immediate from Proposition 5.1. □

In fact, the fibration

$$\tilde{Fl} \to Gr$$

with fiber $Fl = G/T$ directly induces the isomorphism of $\mathbb{Q}$-vector space.

$$H^*(\tilde{Fl}) \cong H^*(Fl) \otimes_{\mathbb{Q}} H^*(Gr)$$

from associated Leray spectral sequence. This classical result is compatible with our corollary.

6 Furthermore: Equivariant Cohomology of Affine Springer Fibers $\tilde{Fl}_1$ of $sl(2)$

For an affine Springer fibers $\tilde{Fl}_1$ of $sl(2)$, we have a complete description of the equivariant cohomology over the maximal torus T action as a ring. Since $\tilde{Fl}_1$ of $sl(2)$ is the chain of projective lines, fixed points of T are the elements of the Weyl group of type A_1: $\mathbb{Z} \rtimes \{+,-\}$, and one dimensional orbits are union of projective lines connecting $(n,+),(n+1,-)$ or $(n,+),(n,-)$ for all $n \in \mathbb{Z}$. Therefore using the localization theorem we can obtain the following theorem by direct computation.

Theorem 6.1. *Suppose $\tilde{Fl}_1$ is an affine Springer fiber of $sl(2)$ with maximal torus T. Then, we have*

$$H_T^*(\tilde{Fl}_1) \cong \mathbb{C}[[x_w]]_{w \in \mathbb{Z} \rtimes \{+,-\}}/(x_w x_{w'})_{w \neq w'}.$$

The proof involves computing the equivariant cohomology under GKM condition which we will not describe in detail.

Computing the equivariant cohomology of general affine Springer fiber is not quite simple as in the case of affine Springer fibers of $sl(2)$. However there seems to exist a property which connects the cohomology of affine flag variety and of affine Springer fiber. We leave this as conjecture originally proposed by Professor Bezrukavnikov.

Conjecture 6.2.

$$H^*(\tilde{Fl}) \to H^*(\tilde{Fl}_1)^{\Lambda}$$

is surjective.

Using Theorem 6.1 and the GKM description of the equivariant cohomology of affine flag variety, the conjecture is indeed true in the case of the affine Springer fibers of $sl(2)$. We would like to suggest this conjecture as a potential research project to readers.

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