Signatures of GL_n Multiplicity Spaces

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Mrudul Thatte
Mentor: Siddharth Venkatesh
Project suggested by Pavel Etingof

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Abstract

A stable sequence of GL_n representations consists of representations with highest weights $(\lambda, 0, \ldots, 0, -\lambda')$ for fixed partitions λ, λ'. For a fixed positive integer k, the multiplicity space of restrictions of these representations to certain GL_{n-k} representations is independent of the number of 0's in the highest weights. In this paper, we compute the rational functions that occur in the ratios of norms of basis vectors of this multiplicity space.

1 Introduction

A classical goal of representation theory is to assign numerical invariants to group representations. One such invariant is the dimension of the representation. Another more refined invariant is the character of a representation which, in the case of finite groups and semisimple Lie groups, completely characterizes the representation. The key property of these invariants is that they are characters of the Grothendeick ring of the category of representations of the group. In simpler terms, these invariants are additive over direct sums and multiplicative over tensor products.

A more modern problem is to replace representations of a group by an arbitrary tensor category, i.e., a category in which it makes sense to take direct sums and tensor products. In this paper, we will compute invariants attached to pairs of irreducible objects in the Deligne category $\mathrm{Rep}GL_t$ ([1]). Objects in this category are interpolations of sequences of representations of the general linear group GL_n as n grows large. Irreducible objects are labeled by ordered pairs of partitions and the object corresponding to the pair (λ, λ'), which we denote by $V_{\lambda,\lambda'}$, should be thought of as the interpolation of the sequence of representations of GL_n with highest weight [1]

$$(\lambda, 0, \ldots, 0, -\lambda') = (\lambda_1, \ldots, \lambda_r, 0, \ldots 0, -\lambda'_s, \ldots - \lambda'_1).$$

In particular, numerical invariants of objects in $\mathrm{Rep}GL_t$ can usually be computed by computing the corresponding invariant for the sequence of representations of GL_n that the object interpolates and then replacing n with t.

The central objects in this paper are the multiplicity spaces of restrictions of objects in $\mathrm{Rep}GL_t$ to $\mathrm{Rep}GL_{t-k}$ for some fixed positive integer k. As in the classical case of GL_n representations, we have a restriction functor from $\mathrm{Rep}GL_t$ to $\mathrm{Rep}GL_{t-1}$ and, by iteration, a restriction functor from $\mathrm{Rep}GL_t$ to $\mathrm{Rep}GL_{t-k}$. Thus, we will have

$$V_{\lambda,\lambda'}|_{\mathrm{Rep}GL_{t-k}} = \oplus_{\mu,\mu'} N^{\mu,\mu'}_{\lambda,\lambda'} \boxtimes V_{\mu,\mu'}$$

where $N^{\mu,\mu'}_{\lambda,\lambda'}$ are vector spaces whose dimensions count the multiplicity of $V_{\mu,\mu'}$ in the restriction of $V_{\lambda,\lambda'}$. These multiplicity spaces are equipped with Hermitian forms that are unique up to normalization and our ultimate goal is to compute the signatures of these Hermitian forms. In this paper, however, we restrict ourselves to computing signatures when the sum of the sizes of (μ, μ') is k less than that of (λ, λ'). We can always restrict ourselves to this case, so this is the interesting part of the computation.

We do this computation via interpolation. Suppose we have a pair of partitions (λ, λ') whose sizes sum to N and a pair of partitions (μ, μ') whose

[1]Highest weights and stable sequences of representations are defined in Sections 2 and 3, respectively.

sizes sum to $N-k$ such that the latter appears in the restriction of the former with nonzero multiplicity. If this is true, then the multiplicity space of the representation of GL_n with highest weight

$$(\mu, 0, \ldots, 0, -\mu')$$

in the restriction of the representation of GL_{n-k} with highest weight

$$(\lambda, 0, \ldots, 0, -\lambda')$$

is independent of $n-N$, the number of 0's in the middle. The interpolation of this space as n grows large can then be identified with the multiplicity space in the RepGL_t restriction. Now, the multiplicity space above acquires a positive definite inner product, which is the ratio of the (unique up to scaling) invariant inner products on the GL_n and GL_{n-k} representations. If we then take an orthogonal basis for the multiplicity space and compute the ratios of the norms of the vectors, these ratios will be rational functions of n. Interpolating these ratios by replacing n with t, we will get the ratios of norms of the Deligne category theoretic multiplicity space, which determines the signature up to sign. In this paper, we compute the rational functions that appear as norm ratios.

The irreducible representations of GL_n can be extracted from the finite-dimensional irreducible representations of the Lie algebra $\mathfrak{gl}_n$. These representations are indexed by their highest weights λ and have a basis parametrized by simple combinatorial objects known as the Gelfand-Tsetlin Patterns. In Section 2, we describe these representations and construct the Gelfand-Tsetlin basis. We also give a description of the relevant multiplicity spaces and their bases. In Section 3, we define stable sequences of representations and look at the multiplicity spaces associated with these representations. Finally, in Section 4, we introduce the canonical Hermitian product on $\mathfrak{gl}_n$ representations and their multiplicity spaces. We give formulas for norms of basis vectors constructed in Section 2. We compute how the ratios of the norms of basis vectors of a multiplicity space change as we move along a stable sequence of representations.

2 $\mathfrak{gl}_n$ Representations

The general linear algebra $\mathfrak{gl}_n$ is the Lie algebra associated with the group GL_n. It consists of the complex $n \times n$ matrixes with the commutator

$$[A, B] = AB - BA.$$

For $1 \leq i, j \leq n$, let E_{ij} be the matrix with 1 in the ij^{th} position and 0 everywhere else. The n^2 matrices E_{ij} form the standard basis of $\mathfrak{gl}_n$. The matrices of the form $E_{pp}, E_{p,p+1}, E_{p+1,p}$ together generate the algebra $\mathfrak{gl}_n$.

Every finite dimensional irreducible representation of $\mathfrak{gl}_n$ contains a unique (up to scalar) nonzero vector ξ, known as the **highest weight vector**, which is a common eigenvector for $E_{11}, E_{22}, \ldots, E_{nn}$, and satisfies $E_{ij}\xi = 0$ for all $1 \leq i < j \leq n$. The n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ of the corresponding eigenvalues of $E_{11}, E_{22}, \ldots, E_{nn}$ is known as the **highest weight**. For every $1 \leq i \leq n-1$, the difference $\lambda_i - \lambda_{i+1}$ is a non-negative integer. Conversely, every n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers, for which each $\lambda_i - \lambda_{i+1}$ is a non-negative integer, is the highest weight of a unique finite dimensional irreducible representation of $\mathfrak{gl}_n$. [2]

Let V_λ be the finite dimensional irreducible representation of $\mathfrak{gl}_n$ with the highest weight λ. The restriction of V_λ to the subalgebra $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$ decomposes into irreducible representations of $\mathfrak{gl}_{n-1}$ according to the following rule:

Theorem 1 (Branching Rule)**.** *[2]*

$$V_\lambda \mid_{\mathfrak{gl}_{n-1}} = \oplus_\mu V_\mu,$$

*where the summation takes place over highest weights $\mu = (\mu_1, \ldots, \mu_{n-1})$ that satisfy the **betweenness conditions***

$$\lambda_i - \mu_i \in \mathbb{Z}_{\geq 0} \text{ and } \mu_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$$

for $1 \leq i \leq n-1$.

Note that every finite-dimensional irreducible representation of $\mathfrak{gl}_{n-1}$ occurs in $V_\lambda \mid_{\mathfrak{gl}_{n-1}}$ at most once. This observation combined with the repeated application of the Branching Rule allows us to construct a basis of V_λ parametrized by combinatorial objects known as Gelfand-Tsetlin Patterns.

Definition 1. A **Gelfand-Tsetlin Pattern** Λ associated with highest weight λ is a triangular array of complex numbers

$$\begin{array}{ccccccccc}
\lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{n,n-1} & & \lambda_{nn} \\
& \lambda_{n-1,1} & & & \cdots & & & \lambda_{n-1,n-1} & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda_{21} & & \lambda_{22} & & & \\
& & & & \lambda_{11} & & & &
\end{array}$$

[2] With this correspondence in mind, we will use the term "highest weight" to refer to any of the following:

- The eigenvalues of a highest weight vector or
- An arbitrary n-tuple $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$.

such that $\lambda_{ni} = \lambda_i$ for all $1 \leq i \leq n$, and the differences $\lambda_{pi} - \lambda_{p-1,i}$ and $\lambda_{p-1,i} - \lambda_{p,i+1}$ are non-negative integers for all $1 \leq i < p \leq n$. In other words, the top row is the highest weight λ and any two adjacent rows of satisfy the betweenness conditions described in the Branching Rule.

Theorem 2. *[2] There exists a basis $\{\xi_\Lambda\}$ of V_λ, called the **Gelfand-Tsetlin basis**, that is parametrized by all Gelfand-Tsetlin Patterns Λ associated with λ such that the action of generators of $\mathfrak{gl}_n$ is given by the following formulas:*

$$E_{pp}\xi_\Lambda = \left(\sum_{i=1}^{p} \lambda_{pi} - \sum_{i=1}^{p-1} \lambda_{p-1,i} \right) \xi_\Lambda$$

$$E_{p,p+1}\xi_\Lambda = -\sum_{i=1}^{p} \frac{\prod_{1\leq j\leq p+1}(\ell_{pi} - \ell_{p+1,j})}{\prod_{1\leq j\leq p, j\neq i}(\ell_{pi} - \ell_{pj})} \xi_{\Lambda+\delta_{pi}}$$

$$E_{p+1,p}\xi_\Lambda = \sum_{i=1}^{p} \frac{\prod_{1\leq j\leq p-1}(\ell_{pi} - \ell_{p-1,j})}{\prod_{1\leq j\leq p, j\neq i}(\ell_{pi} - \ell_{pj})} \xi_{\Lambda-\delta_{pi}}$$

Here, $\ell_{pi} = \lambda_{pi} - i + 1$. The triangular arrays $\Lambda \pm \delta_{pi}$ are obtained from Λ by replacing λ_{pi} by $\lambda_{pi} \pm 1$. If such an array is not a Gelfand-Tsetlin Pattern, then the corresponding vector $\xi_{\Lambda+\delta_{pi}}$ or $\xi_{\Lambda-\delta_{pi}}$ is considered zero.

Remark 1. The highest weight vector is the basis vector associated with the following Gelfand-Tsetlin Pattern Λ given by $\lambda_{pi} = \lambda_i$ for all $1 \leq i \leq p \leq n$.

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{n-1} & & \lambda_n \\
& \lambda_1 & & & \cdots & & & \lambda_{n-1} & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda_1 & & \lambda_2 & & & \\
& & & & \lambda_1 & & & &
\end{array}$$

The Branching Rule also gives us a basis for the multiplicity space of the form $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda)$ for the highest weights $\mu = (\mu_1, \ldots, \mu_{n-k})$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$. Note that the basis of $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda)$ is indexed by different copies of V_μ in the decomposition of $V_\lambda \mid \mathfrak{gl}_{n-k}$ into irreducible representations of $\mathfrak{gl}_{n-k}$. Using the Branching Rule, we can identify these copies of V_μ with various Galfand-Tsetlin Sub-Patterns:

Definition 2. A **Gelfand-Tsetlin Sub-Pattern** Λ_μ associated with highest weights λ and μ is a triangular array of complex numbers

$$\begin{array}{ccccccccc}
\lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{n,n-1} & & \lambda_{nn} \\
& \lambda_{n-1,1} & & & \cdots & & & \lambda_{n-1,n-1} & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda_{n-k,1} & \cdots & \lambda_{n-k,n-k} & & &
\end{array}$$

such that

$$\lambda_{ni} = \lambda_i \text{ for all } 1 \le i \le n,$$
$$\lambda_{n-k,i} = \mu_i \text{ for all } 1 \le i \le n-k,$$

and the differences $\lambda_{pi} - \lambda_{p-1,i}$ and $\lambda_{p-1,i} - \lambda_{p,i+1}$ are non-negative integers for all $1 \le i < p$ and $n-k < p \le n$. In other words, the extreme rows are the highest weights λ and μ, and any two adjacent rows of satisfy the betweenness conditions described in the Branching Rule.

Proposition 1. *There exists a basis $\{\phi_{\Lambda_\mu}\}$ of the multiplicity space $Hom_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda)$ that is parametrized by all Gelfand-Tsetlin Sub-Patterns Λ_μ associated with λ and μ. Each $\phi_{\Lambda_\mu} : V_\mu \to V_\lambda$ maps the Gelfand-Tsetlin basis vector $\xi_M \in V_\mu$ to the Gelfand-Tsetlin basis vector $\xi_\Lambda \in V_\lambda$ where Λ is obtained by merging M and Λ_μ.*

3 Stable Sequences

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ be arbitrary partitions. These are highest weights satisfying $\lambda_i, \lambda'_j \in \mathbb{Z}_{\ge 0}$. Then for $n \ge N = r + s$,

$$(\lambda, \lambda')_n = (\lambda, 0, \ldots, 0, -\lambda') = (\lambda_1, \ldots, \lambda_r, 0, \ldots, 0, -\lambda'_s, \ldots, -\lambda'_1) \in \mathbb{Z}^n$$

is another highest weight. Let $V_{(\lambda,\lambda')_n}$ denote the finite-dimensional irreducible representation of $\mathfrak{gl}_n$ with this highest weight. The sequence of representations $V_{(\lambda,\lambda')_n}$ is called as a **stable sequence** of representations.

Fix a positive integer $k < N$. Let μ, μ' be two partitions whole lengths add to $N - k$ and the multiplicity space $\text{Hom}_{\mathfrak{gl}_{N-k}}(V_{(\mu,\mu')_N}, V_{(\lambda,\lambda')_N})$ is nonzero. Then for large n, the multiplicity space

$$\text{Hom}_{\mathfrak{gl}_{n-k}}(V_{(\mu,\mu')_n}, V_{(\lambda,\lambda')_n})$$

is independent of n. We can prove this simply by looking at how the Gelfand-Tsetlin basis of this multiplicity changes with n.

Note that if the length of μ is bigger than r, the length of λ, then we may insert appropriate number of 0's in λ and μ', so that the new lengths of μ and λ are the same and the length of λ' is k bigger than the length of μ'. If the length of μ is smaller than r, we may add 0's to λ' and μ to achieve the same result. Both of these operations are equivalent to removing the first few terms of the sequence of multiplicity spaces $\text{Hom}_{\mathfrak{gl}_{n-k}}(V_{(\mu,\mu')_n}, V_{(\lambda,\lambda')_n})$. Since we only care about the nature of multiplicity spaces for large n, we may assume that the length of μ is in fact equal to r. We have

$$\mu = (\mu_1, \ldots, \mu_r) \text{ and } \mu' = (\mu'_1, \ldots, \mu'_{s-k}).$$

Now a basis vector of the multiplicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_{(\mu,\mu')_n}, V_{(\lambda,\lambda')_n})$ is indexed by a Gelfand-Tsetlin Sub-Pattern of the form

$$\begin{array}{ccccccccccc}
\lambda_1 & \dots & \lambda_r & 0 \dots 0 & -\lambda'_s & & \dots & & -\lambda'_1 \\
* & \dots & * & * \dots * & * & & \dots & & * \\
& \ddots & & \ddots & & & \iddots & & \\
* & \dots & * & * \dots * & * & & \dots & * & \\
\mu_1 & \dots & \mu_r & 0 \dots 0 & -\mu'_{s-k} & \dots & -\mu'_1 & &
\end{array}$$

From the betweenness conditions in the definitions of the Gelfand-Tsetlin Sub-Pattern, it is easy to see that the pi^{th} entries in the above Sub-Pattern must be 0 for every $n-k \le p \le n$ and $r+1 \le i \le n-N$. In other words, the entries along the diagonals running from 0's in the top row to the 0's in the bottom row are all 0. Thus, the above Sub-Pattern is obtained from a Sub-Pattern of the form

$$\begin{array}{ccccccccc}
\lambda_1 & \dots & \lambda_r & -\lambda'_s & & \dots & & -\lambda'_1 \\
* & \dots & * & * & & \dots & & * \\
& \ddots & & & & \iddots & & \\
* & \dots & * & * & & \dots & * & \\
\mu_1 & \dots & \mu_r & -\mu'_{s-k} & \dots & -\mu'_1 & &
\end{array}$$

by inserting a block of 0's. This correspondence between Sub-Patterns gives a natural bijection between the multiplicity spaces $\mathrm{Hom}_{\mathfrak{gl}_{N-k}}(V_{(\mu,\mu')_N}, V_{(\lambda,\lambda')_N})$ and $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_{(\mu,\mu')_n}, V_{(\lambda,\lambda')_n})$ that is compatible with the action of the general linear algebra. So, $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_{(\mu,\mu')_n}, V_{(\lambda,\lambda')_n})$ does not depend on n, or equivalently the number of 0's added between λ and $-\lambda'$.

However, the inner products on $V_{(\mu,\mu')_n}$ and $V_{(\lambda,\lambda')_n}$ (defined in the next section) vary with n. As a result, we get different inner products on the multiplicity space. Thus, as n increases, the norms of basis vectors of the multiplicity space change. Our goal is to compute the ratios of these norms in terms of n.

4 Norms of Basis Vectors

In Section 4.1, we define the Hermitian product on $\mathfrak{gl}_n$ representations and compute ratios of norms of vectors in the Gelfand-Tsetlin basis. In Section 4.2, we conduct a similar analysis for multiplicity spaces. In Section 4.3, we compute ratios of norms of vectors in the multiplicity space associated with stable sequences of representations.

4.1 Inner Product on Representations

Let V_λ be the finite-dimensional irreducible representation of $\mathfrak{gl}_n$ with highest weight λ. Let $\langle , \rangle$ denote the contravariant form on V_λ such that the norm of the highest weight vector, $\langle \xi, \xi \rangle$ is 1, and the operators E_{ij} and E_{ji} are adjoint for all $1 \leq i, j \leq n$. We have

Proposition 2. *[2] The Gelfand-Tsetlin basis is orthogonal with respect to* $\langle , \rangle$ *and the norms of the basis vectors are given by:*

$$\langle \xi_\Lambda, \xi_\Lambda \rangle = \prod_{p=2}^{n} \prod_{1 \leq i \leq j < p} \frac{(\ell_{pi} - \ell_{p-1,j})!}{(\ell_{p-1,i} - \ell_{p-1,j})!} \prod_{1 \leq i < j \leq p} \frac{(\ell_{pi} - \ell_{pj} - 1)!}{(\ell_{p-1,i} - \ell_{pj} - 1)!}.$$

We want to look at the ratios of norms of two basis vectors of a representation V_λ. Note that any Gelfand-Tsetlin Pattern associated with λ may be obtained from any other Pattern associated with λ by successively changing various entries of the Pattern by ± 1 such that all intermediate arrays remain Gelfand-Tsetlin Patterns (i.e. satisfy the betweenness conditions) associated with λ. Therefore, we only look at the ratios of basis vectors whose Gelfand-Tsetlin Patterns differ at a single place by ± 1 and are identical otherwise.

Proposition 3. *Let* Λ *be a Gelfand-Tsetlin Pattern associated with* λ*. We have*

$$\frac{\langle \xi_{\Lambda+\delta_{pi}}, \xi_{\Lambda+\delta_{pi}} \rangle}{\langle \xi_\Lambda, \xi_\Lambda \rangle} = -\frac{\prod_{j=1}^{p-1} (\ell_{pi} - \ell_{p-1,j} + 1)}{\prod_{j=1}^{p+1} (\ell_{pi} - \ell_{p+1,j})} \prod_{1 \leq j \leq p, j \neq i} \frac{\ell_{pi} - \ell_{pj}}{\ell_{pi} - \ell_{pj} + 1}$$

$$\frac{\langle \xi_{\Lambda-\delta_{pi}}, \xi_{\Lambda-\delta_{pi}} \rangle}{\langle \xi_\Lambda, \xi_\Lambda \rangle} = -\frac{\prod_{j=1}^{p+1} (\ell_{pi} - \ell_{p+1,j} - 1)}{\prod_{j=1}^{p-1} (\ell_{pi} - \ell_{p-1,j})} \prod_{1 \leq j \leq p, j \neq i} \frac{\ell_{pi} - \ell_{pj} - 1}{\ell_{pi} - \ell_{pj}}$$

Proof. Note that $E_{p,p+1}$ and $E_{p+1,p}$ are adjoint operators. So, we get

$$\langle E_{p,p+1} \xi_\Lambda, \xi_{\Lambda+\delta_{pi}} \rangle = \langle \xi_\Lambda, E_{p+1,p} \xi_{\Lambda+\delta_{pi}} \rangle.$$

Using the orthogonality of the Gelfand-Tsetlin basis and Theorem 2, we get

$$\langle E_{p,p+1} \xi_\Lambda, \xi_{\Lambda+\delta_{pi}} \rangle = -\frac{\prod_{1 \leq j \leq p+1} (\ell_{pi} - \ell_{p+1,j})}{\prod_{1 \leq j \leq p, j \neq i} (\ell_{pi} - \ell_{pj})} \langle \xi_{\Lambda+\delta_{pi}}, \xi_{\Lambda+\delta_{pi}} \rangle$$

and

$$\langle \xi_\Lambda, E_{p+1,p} \xi_{\Lambda+\delta_{pi}} \rangle = \frac{\prod_{1 \leq j \leq p-1} (\ell_{pi} - \ell_{p-1,j} + 1)}{\prod_{1 \leq j \leq p, j \neq i} (\ell_{pi} - \ell_{pj} + 1)} \langle \xi_\Lambda, \xi_\Lambda \rangle$$

From the above three equations, we obtain the formula for $\frac{\langle \xi_{\Lambda+\delta_{pi}}, \xi_{\Lambda+\delta_{pi}} \rangle}{\langle \xi_\Lambda, \xi_\Lambda \rangle}$. The formula for $\frac{\langle \xi_{\Lambda-\delta_{pi}}, \xi_{\Lambda-\delta_{pi}} \rangle}{\langle \xi_\Lambda, \xi_\Lambda \rangle}$ can be easily derived from this formula. □

We may prove this proposition using Proposition 2. As we change a single entry of a Gelfand-Tsetlin Pattern by ± 1, most of the factorials appearing in the norm of the corresponding basis vector remain the same. As a result, we obtain a simpler expression for the ratio $\frac{\langle \xi_{\Lambda \pm \delta_{pi}}, \xi_{\Lambda \pm \delta_{pi}} \rangle}{\langle \xi_\Lambda, \xi_\Lambda \rangle}$. In particular, the ratio contains linear factors in terms of the entries in only three rows of the Pattern Λ : the row with the altered entry, and the rows directly above and below it. Thus, when we insert a block of 0's in the Pattern, the ratio changes by a rational function of the number of 0's. We prove this computation are in Section 4.3. First, we need to introduce the Hermitian product on multiplicity spaces.

4.2 Inner Product on Multiplicity Space

Note that

$$V_\lambda = \oplus_\mu V_\mu \otimes \mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda),$$

where the summation takes place over all highest weights $\mu \in \mathbb{C}^{n-k}$. Then the Gelfand-Tsetlin vectors of V_λ are of the form

$$\xi_\Lambda = \xi_M \otimes \phi_{\Lambda_\mu}$$

where Λ is the Pattern obtained by merging M (the Pattern associated with μ) and Λ_μ (the Sub-Pattern associated with λ). Thus, the inner products on V_λ and V_μ induce an inner product $\langle , \rangle$ on the multilpicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda)$.

Definition 3. Let ϕ_{Λ_μ} be a Sub-Pattern associated with λ. For a Pattern M associated with μ and Pattern Λ obtained by merging M and Λ_μ,

$$\langle \phi_{\Lambda_\mu}, \phi_{\Lambda_\mu} \rangle = \frac{\langle \xi_\Lambda, \xi_\Lambda \rangle}{\langle \xi_M, \xi_M \rangle}.$$

Using Proposition 2 in the previous section, we get an explicit formula for the norm of ϕ_{Λ_μ} :

Proposition 4. *The basis $\{\phi_{\Lambda_\mu}\}$ is orthogonal with respect to the induced form $\langle , \rangle$ on $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_\mu, V_\lambda)$. The norms of the basis vectors are given by*

$$\langle \phi_{\Lambda_\mu}, \phi_{\Lambda_\mu} \rangle = \prod_{p=n-k+1}^{n} \prod_{1 \le i \le j < p} \frac{(\ell_{pi} - \ell_{p-1,j})!}{(\ell_{p-1,i} - \ell_{p-1,j})!} \prod_{1 \le i < j \le p} \frac{(\ell_{pi} - \ell_{pj} - 1)!}{(\ell_{p-1,i} - \ell_{pj} - 1)!}$$

where $\ell_{pi} = \lambda_{pi} - i + 1$.

Since the bottom $n-k$ rows of M and Λ are identical the expression for the norm of ϕ_{Λ_μ} only depends on the additional rows in Λ. These are precisely the entries of Λ_μ. This shows that the norm of ϕ_{Λ_μ} does not depend on the choice of M and Definition 3 is valid.

Remark 2. The formula for $\langle \phi_{\Lambda_\mu}, \phi_{\Lambda_\mu} \rangle$ is obtained by simply deleting the irrelevant terms in the formula for $\langle \xi_\Lambda, \xi_\Lambda \rangle$. The same technique may be used when we take ratios of norms of the basis vectors in the multiplicity space. So, the formulas in Proposition 3, originally derived for Gelfand-Tsetlin Patterns, also apply to Gelfand-Tsetlin Sub-Patterns.

4.3 Dependence on n

In this section, we compute the ratios of norms of basis vectors of the multiplicity space associated with a stable sequence of representation. Fix a highest weight $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ and $r \leq N$ such that $\lambda_r \geq 0 \geq \lambda_{r+1}$. [3] Fix a positive integer $k < N$ and a highest weight $\mu = (\mu_1, \ldots, \mu_{N-k}) \in \mathbb{Z}^{N-k}$ such that $\mathrm{Hom}_{\mathfrak{gl}_{N-k}}(V_\mu, V_\lambda)$ is non-zero. As explained in Section 3, we may assume that $\mu_r \geq 0 \geq \mu_{r+1}$. Consider the highest weights

$$[\lambda]_n = (\lambda_1, \ldots, \lambda_r, 0, 0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_N) \in \mathbb{C}^n$$

and

$$[\mu]_{n-k} = (\mu_1, \ldots, \mu_r, 0, 0, \ldots, 0, \mu_{r+1}, \ldots, \mu_{N-k}) \in \mathbb{C}^{n-k}.$$

Our goal is to look at the norms of the basis vectors of the multiplicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-k}}(V_{[\mu]_{n-k}}, V_{[\lambda]_n})$ associated with the stable sequence of representations $V_{[\lambda]_n}$.

4.3.1 $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$ reduction

First, we consider the case $k = 1$. The multiplicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-1}}(V_{[\mu]_{n-1}}, V_{[\lambda]_n})$ is one-dimensional with the basis vector ϕ_n corresponding to the following Gelfand-Tsetlin Sub-Pattern Λ_n :

$$\begin{array}{ccccccccccccccccc}
\lambda_1 & & \ldots & & \lambda_r & & 0 & & \ldots & & 0 & & \lambda_{r+1} & & \ldots & & \lambda_N \\
& \mu_1 & & \ldots & & \mu_r & & 0 & & \ldots & 0 & & & \mu_{r+1} & \ldots & \mu_{N-1} &
\end{array}$$

Let

$$\begin{aligned}
\ell_{pi} &= [(p,i)^{\text{th}} \text{ entry of } \Lambda_N] - i + 1, \\
\ell'_{pi} &= [(p,i)^{\text{th}} \text{ entry of } \Lambda_n] - i + 1.
\end{aligned}$$

Then the norm of ϕ_n varies with n according to the following formula:

[3] In the definition of stable sequences in Section 3, we fix two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$, and combine them into a single highest weight $(\lambda, -\lambda')$. In this section, for convenience of notation, we simply use "λ" for this combined highest weight. To keep track of the point where the partitions are merged, we use "r" with the condition "$\lambda_r \geq 0 \geq \lambda_{r+1}$."

Proposition 5.

$$\frac{\langle \phi_n, \phi_n \rangle}{\langle \phi_N, \phi_N \rangle} = \prod_{a=1}^{n-N} \prod_{i=1}^{r} \Bigg(\frac{(\ell_{N,i}+a-1)!(\ell_{N,i}+a-2)!}{(\ell_{N-1,i}+a-1)!(\ell_{N-1,i}+a-2)!} \prod_{j=r+1}^{N} \frac{(\ell_{N,i}-\ell_{N-1,j}+a)(\ell_{N,i}-\ell_{N-1,j}+a-1)}{(\ell_{N-1,i}-\ell_{N-1,j}+a)(\ell_{N-1,i}-\ell_{N-1,j}+a-1)} \Bigg).$$

Proof. Note that

$$\langle \phi_N, \phi_N \rangle = \prod_{1\le i\le j<N} \frac{(\ell_{N,i}-\ell_{N-1,j})!}{(\ell_{N-1,i}-\ell_{N-1,j})!} \prod_{1\le i<j\le N} \frac{(\ell_{Ni}-\ell_{Nj}-1)!}{(\ell_{N-1,i}-\ell_{N,j}-1)!}$$

and

$$\langle \phi_n, \phi_n \rangle = \prod_{1\le i\le j<n} \frac{(\ell'_{ni}-\ell'_{n-1,j})!}{(\ell'_{n-1,i}-\ell'_{n-1,j})!} \prod_{1\le i<j\le n} \frac{(\ell'_{ni}-\ell'_{nj}-1)!}{(\ell'_{n-1,i}-\ell'_{nj}-1)!}.$$

Using the definition of ℓ'_{pi} and Λ_n, we see that

$$\ell'_{ni}-\ell'_{n-1,j} = \begin{cases} \ell_{Ni}-\ell_{N-1,j} & \text{if } 1\le i\le j\le r \\ \ell_{Ni}+j-1 & \text{if } 1\le i\le r<j\le n-N+r \\ \ell_{Ni}-\ell_{N-1,j+N-r-n}+1 & \text{if } 1\le i\le r, n-N+r<j<n \\ \ell'_{n-1,i}-\ell'_{n-1,j} & \text{if } r<i\le n-N+r \\ \ell_{N,i+N-r-n}-\ell_{N-1,j+N-r-n} & \text{if } n-N+r<i\le j<n \end{cases}$$

Using similar computations for other terms appearing in $\langle \phi_n, \phi_n \rangle$, we prove the proposition. □

Since the multiplicity space is one-dimensional, the case $k=1$ is quite uninteresting. We quickly move to the $k=2$, where we have larger multiplicity spaces and we can look at how norms of basis vectors change as small changes are made in the respective Gelfand-Tsetlin Sub-Patterns.

4.3.2 $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-2}$ reduction

Now we consider the case $k=2$. The formulas we develop for this case can be easily generalized for bigger values of k. Note that the multiplicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-2}}(V_{[\mu]_{n-2}}, V_{[\lambda]_n})$ is spanned by basis vectors $\phi_{\omega,n}$ corresponding to the Gelfand-Tsetlin Sub-Patterns $\Lambda_{\omega,n}$ of the form

$$\begin{array}{ccccccccccc} \lambda_1 & \dots & \lambda_r & 0 & \dots & 0 & \lambda_{r+1} & & \dots & & \lambda_N \\ & \omega_1 & \dots & \omega_r & 0 & \dots & 0 & \omega_{r+1} & \dots & \omega_{N-1} & \\ & & \mu_1 & \dots & \mu_r & 0 & \dots & 0 & \mu_{r+1} \dots & \mu_{N-2} & \end{array}$$

for suitable highest weights $\omega = (\omega_1, \ldots, \omega_{N-1})$ satisfying the betweenness conditions with both λ and μ. As in the previous section, expression for the ratio $\frac{\langle \phi_{\omega,n}, \phi_{\omega,n} \rangle}{\langle \phi_{\omega,N}, \phi_{\omega,N} \rangle}$ involves products of several factorials. As we make small changes in ω, most of these factorials remain the same and this ratio changes by a rational function in n. Let

$$\ell_{pi} = [(p,i)^{\text{th}} \text{ entry of } \Lambda_{\omega,N}] - i + 1.$$

and

$$\ell'_{pi} = [(p,i)^{\text{th}} \text{ entry of } \Lambda_{\omega,n}] - i + 1.$$

Let $\omega \pm \delta_i$ denote the highest weight obtained by replacing ω_i in ω by $\omega_i \pm 1$. We have

Proposition 6. *For $n \geq N$, we have*

$$\frac{\langle \phi_{\omega+\delta_i,n}, \phi_{\omega+\delta_i,n} \rangle}{\langle \phi_{\omega,n}, \phi_{\omega,n} \rangle} = \frac{\langle \phi_{\omega+\delta_i,N}, \phi_{\omega+\delta_i,N} \rangle}{\langle \phi_{\omega,N}, \phi_{\omega,N} \rangle} f_{\omega,i}(n)$$

where $f_{\omega,i}(n)$ is a rational function of n.

Proof. By Remark 2 and Proposition 3, for $1 \leq i \leq r$, we have

$$\frac{\langle \phi_{\omega+\delta_i,n}, \phi_{\omega+\delta_i,n} \rangle}{\langle \phi_{\omega,n}, \phi_{\omega,n} \rangle} = -\frac{\prod_{j=1}^{n-2}(\ell'_{n-1,i} - \ell'_{n-2,j} + 1)}{\prod_{j=1}^{n}(\ell'_{n-2,i} - \ell'_{nj})} \prod_{1 \leq j \leq n-1, j \neq i} \frac{\ell'_{n-1,i} - \ell'_{n-1,j}}{\ell'_{n-1,i} - \ell'_{n-1,j} + 1}$$

As in the proof of Proposition 5, we write the factors appearing in the above expression in terms of ℓ_{pj}'s and explicitly compute $f_{\omega,i}$.

$$\begin{aligned} & f_{\omega,i}(n) \\ = & \prod_{j=r+1}^{N-2} \frac{\ell_{N-1,i} - \ell_{N-2,j} + n - N + 1}{\ell_{N-1,i} - \ell_{N-2,j} + 1} \prod_{j=r+1}^{N} \frac{\ell_{N-1,i} - \ell_{Nj}}{\ell_{N-1,i} - \ell_{Nj} + n - N} \times \\ & \prod_{j=r+1}^{N-1} \frac{\ell_{N-1,i} - \ell_{N-1,j} + n - N}{\ell_{N-1,i} - \ell_{N-1,j}} \times \frac{\ell_{N-1,i} - \ell_{N-1,j} + 1}{\ell_{N-1,i} - \ell_{N-1,j} + n - N + 1}. \end{aligned}$$

The same technique is used to obtain the formula for $i > r$. In this case, we get

$$\begin{aligned} & f_{\omega,i}(n) \\ = & \prod_{j=1}^{r} \frac{\ell_{N-1,i} - \ell_{N-2,j} - n + N + 1}{\ell_{N-1,i} - \ell_{N-2,j} + 1} \times \frac{\ell_{N-1,i} - \ell_{Nj}}{\ell_{N-1,i} - \ell_{Nj} - n + N} \times \\ & \frac{\ell_{N-1,i} - \ell_{N-1,j} - n + N}{\ell_{N-1,i} - \ell_{N-1,j}} \times \frac{\ell_{N-1,i} - \ell_{N-1,j} + 1}{\ell_{N-1,i} - \ell_{N-1,j} - n + N + 1}. \end{aligned}$$

□

4.3.3 $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-k}$ reduction

Note that for any $k > 1$, the multiplicity space $\mathrm{Hom}_{\mathfrak{gl}_{n-2}}(V_{[\mu]_{n-k}}, V_{[\lambda]_n})$ is spanned by basis vectors $\phi_{\Omega,n}$ corresponding to the Gelfand-Tsetlin Sub-Patterns $\Lambda_{\Omega,n}$

$$
\begin{array}{ccccccccc}
\lambda_1 & \ldots & \lambda_r & 0 \ldots 0 & \lambda_{r+1} & & \ldots & & \lambda_N \\
\omega_{N-1,1} & \ldots & \omega_{N-1,r} & 0 \ldots 0 & \omega_{N-1,r+1} & & \ldots & & \omega_{N-1,N-1} \\
& \ddots & & & \ddots & & \ldots & & \iddots \\
\omega_{N-k+1,1} & \ldots & \omega_{N-k+1,r} & 0 \ldots 0 & \omega_{N-k+1,r+1} & & \ldots & & \omega_{N-k+1,N-k+1} \\
\mu_1 & \ldots & \mu_r & 0 \ldots 0 & & \mu_{r+1} & \ldots & \mu_{N-k} &
\end{array}
$$

for suitable Sub-Patterns Ω that "fit between" λ and μ. Let

$$\ell_{pi} = [(p,i)^{\text{th}} \text{ entry of } \Lambda_{\Omega,N}] - i + 1.$$

Let $\Omega_{pi} \pm \delta_{pi}$ denote the Sub-Pattern obtain from Ω by replacing ω_{pi} by $\omega_{pi} \pm 1$. We can imitate the proof of Proposition 6 to prove the following theorem:

Theorem 3. *For $n \geq N$, we have*

$$\frac{\langle \phi_{\Omega+\delta_{pi},n}, \phi_{\Omega+\delta_{pi},n} \rangle}{\langle \phi_{\Omega,n}, \phi_{\Omega,n} \rangle} = \frac{\langle \phi_{\Omega+\delta_{pi},N}, \phi_{\Omega+\delta_{pi},N} \rangle}{\langle \phi_{\Omega,N}, \phi_{\Omega,N} \rangle} f_{\Omega,pi}(n)$$

where $f_{\Omega,pi}(n)$ is a rational function of n. For $1 \leq i \leq r$, we have

$$\begin{aligned} & f_{\Omega,pi}(n) \\ = & \prod_{j=r+1}^{p-1} \frac{\ell_{pi} - \ell_{p-1,j} + n - N + 1}{\ell_{pi} - \ell_{p-1,j} + 1} \prod_{j=r+1}^{p+1} \frac{\ell_{pi} - \ell_{p+1,j}}{\ell_{pi} - \ell_{p+1j} + n - N} \times \\ & \prod_{j=r+1}^{p} \frac{\ell_{pi} - \ell_{pj} + n - N}{\ell_{pi} - \ell_{pj}} \times \frac{\ell_{pi} - \ell_{pj} + 1}{\ell_{pi} - \ell_{pj} + n - N + 1}. \end{aligned}$$

For $i < r$, we have

$$\begin{aligned} & f_{\Omega,pi}(n) \\ = & \prod_{j=1}^{r} \frac{\ell_{pi} - \ell_{p-1,j} - n + N + 1}{\ell_{pi} - \ell_{p-1,j} + 1} \times \frac{\ell_{pi} - \ell_{p+1,j}}{\ell_{pi} - \ell_{p+1,j} - n + N} \times \\ & \frac{\ell_{pi} - \ell_{pj} - n + N}{\ell_{pi} - \ell_{pj}} \times \frac{\ell_{pi} - \ell_{pj} + 1}{\ell_{pi} - \ell_{pj} - n + N + 1}. \end{aligned}$$

By interpolating the formulas in the above theorem, we will obtain ratios of norms of vectors in the multiplicity spaces in the Deligne category RepGL_t.

References

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