ASYMPTOTICS AT THE EDGE OF MULTILEVEL DISCRETE β-ENSEMBLES

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Abstract. We consider a multilevel continuous time Markov chain $X(s;N) = (X_i^j(s;N) : 1 \leq i \leq j \leq N)$, which is defined by means of Jack symmetric functions. The process $X(s;N)$ describes a certain discrete interlacing particle system with local push/block interactions between the particles, which preserve the interlacing property. We study the joint asymptotic separation of the particles at the right edge of the ensemble as the number of levels tends to infinity and show that at fixed time their law converges to i.i.d negative binomial distributions.

Contents

1. Introduction

1.1. **Background on the problem.** For any $\beta > 0$, the Hermite general β-ensemble of rank N is the probability distribution on $z_1 < z_2 < ... < z_N$ with density proportional to

$$\prod_{1\le i<j\le N} (z_j - z_i)^\beta \prod_{i=1}^{N} \exp(-z_i^2/2). \tag{1}$$

For $\beta = 2$ this gives the joint distribution of the eigenvalues of an $N \times N$ random Hermitian matrix, whose entries on the diagonal are i.i.d. standard normals and the entries above the diagonal have i.i.d. normal real and imaginary parts with mean 0 and variance 1/2 (this is known as the Gaussian Unitary Ensemble or GUE). For $\beta = 1$, the density (1) gives the joint distribution of the eigenvalues of the Gaussian Orthogonal Ensemble, while $\beta = 4$ corresponds to the Gaussian Symplectic Ensemble.

There are two well-known ways to add an additional dimension to the model described above. One of them is to replace the Gaussian random variables by Brownian motions. This approach can be generalized for any β in a suitable way to give the so-called β-Dyson Brownian motion. An alternative way is by the so-called corner process. Suppose that we have an $N \times N$ random matrix from the GUE. For any $i = 1, ..., N$, we denote the eigenvalues of the upper left corner $i \times i$ matrix by $x_1^i \le ... \le x_i^i$. It is a well-known ("deterministic") property of Hermitian matrices that the interlacing condition $x_i^k \le x_i^{k-1} \le x_{i+1}^k$ holds. The joint distribution of x_i^k, $1 \le i \le k \le N$ is known as the GUE-corners process. Its generalization for general β is called the Hermite β-corner process.

In [GS1], the authors found a way to combine the two extensions described above in a single picture. In particular, they constructed a simple diffusion process that on a fixed level is given by a β-Dyson Brownian motion, while for any fixed time it is given by the Hermite β-corner process. They did this by finding a process $Y(s;N) = (Y_i^j(s;N) : 1 \le i \le j \le N)$ which can be approximated by a discrete space local interaction particle system $X(s,N) = (X_i^j(s;N) : 1 \le i \le j \le N)$ (see Definition 2.8 below).

Although a great number of results are known for $Y(s;N)$, very little is known for $X(s;N)$; however, it is expected that the asymptotic behavors of the two systems are similar. In this paper we will consider a particular result about the joint asymptotic distribution of the separation of the particles at the right edge of $Y(s;N)$, and try to find its discrete analogue for $X(s;N)$.

In [GS1] the authors show that for every $k \ge 1$,

$$\left(Y_N^N\left(\frac{2N}{\beta}+s;N\right) - Y_{N-1}^{N-1}\left(\frac{2N}{\beta}+s;N\right), ..., Y_{N-k+1}^{N-k+1}\left(\frac{2N}{\beta}+s;N\right) - Y_{N-k}^{N-k}\left(\frac{2N}{\beta}+s;N\right)\right)$$

converges as $N \to \infty$ to a continuous time stationary Markov process $(R_1(s), ..., R_k(s))$ satisfying a system of stochastic differential equations with fixed time distribution given by i.i.d. Gamma random variables. Based on this result, we expect that for any $t > 0$ one has that

$$\left(X_1^N(tN+s;N) - X_1^{N-1}(tN+s;N), ..., X_1^{N-k+1}(tN+s;N) - X_1^{N-k}(tN+s;N)\right)$$

converges to a continuous time Markov chain as $N \to \infty$. In this paper we establish a partial result in this direction by showing that for fixed s and t, the above random vectors weakly converge to a random vector $(Q_1, ..., Q_k) \in \mathbb{Z}_{\ge 0}^k$. The entries $Q_1, ..., Q_k$ are i.i.d. negative binomial random variables, which can be viewed as a discrete analog of the Gamma distribution. The exact statement we prove is as follows.

Theorem 1.1. *Let $X(s;N)$ be as in Definition 2.8 with $2\theta = \beta \ge 1$. Fix $t > 0$, $s \ge 0$ and $k \in \mathbb{N}$. Then as $N \to \infty$ the sequence*

$$\left(X_1^N(tN+s;N) - X_1^{N-1}(tN+s;N), ..., X_1^{N-k+1}(tN+s;N) - X_1^{N-k}(tN+s;N)\right)$$

converges in law to a random vector $(Q_1, ..., Q_k) \in \mathbb{Z}^k_{\geq 0}$*, where* $Q_1, ..., Q_k$ *are i.i.d. random variables with*

$$\mathbb{P}(Q_1 = n) = (1-p)^{-\theta} \frac{\Gamma(n+\theta)}{\Gamma(n+1)\Gamma(\theta)} p^n, \; n \in \mathbb{Z}_{\geq 0}, \; and \; p = \frac{\sqrt{t}}{1+\sqrt{t}}.$$

1.2. **Organization of the paper and acknowledgements.** In Section 2, we explain how the distribution of the top row $X^N = (X_1^N(s;N), ..., X_N^N(s;N))$ can be understood in the language of [BGG] as a discrete β-ensemble. This allows us to use Nekrasov's equation to find an explicit formula for the limiting empirical measure $\frac{1}{N}\sum_{i=1}^N \delta\left(\frac{X_i^N(tN;N)+(N-i+1)\theta}{N}\right)$ as $N \to \infty$. Following the work of [J], we also show that $\frac{X_1^N(tN;N)+N\theta}{N}$ converges in probability to a constant. In Section 3 we prove an important technical lemma, which forms the basis of the proof of Theorem 1.1, the latter being the content of Section 4. Section 5 describes a (conjectural) extension of our results to a dynamic limit statement.

I would like to thank Vadim Gorin for helping me find this problem and showing me interesting and useful directions to work on. Moreover, I am grateful to Evgeni Dimitrov for many fruitful discussions and comments. His advice was crucial to all parts of the paper.

2. Discrete β-ensembles and Jack measures

In this section, we define the process $X(s,N) = (X_i^j(s;N) : 1 \leq i \leq j \leq N)$, the main notion that we study in this paper, and show how it is related to the discrete β-ensemble of [BGG]. We explain how one can use the latter together with results from [J], to show that $\frac{1}{N}\sum_{i=1}^N \delta\left(\frac{X_i^N(tN;N)+(N-i+1)\theta}{N}\right)$ converges to a deterministic compactly supported measure and that $\frac{X_1^N(tN;N)+N\theta}{N}$ concentrates near the right endpoint of its support.

2.1. **Jack polynomials and Jack measures.** We will follow the approaches by [GS1] and [M]. Let Λ be the algebra of symmetric polynomials in countably many variables and Λ^N the algebra of symmetric polynomials in N variables.

A Young diagram λ with n boxes ($|\lambda| = n$) is a sequence $\lambda_1 \geq \lambda_2 \geq ... \geq 0$ of integers that sum up to n. We define $\ell(\lambda)$ to be the number of nonzero numbers in λ. For a pair (i,j) such that $\lambda_i \geq j$ (a box of the diagram) we define:

$$a(i,j) = \lambda_i - j, \; l(i,j) = \lambda'_j - i, \; a'(i,j) = j-1, \; l'(i,j) = i-1.$$

We let $J_\lambda(\cdot;\theta)$ denote the Jack polynomials, which are indexed by Young diagrams λ and a positive parameter θ. Many of the properties of these polynomials can be found in Chapter VI of [M] (the α paramter in [M] corresponds to θ^{-1} in our notation). One can view J_λ as an element of the algebra Λ or (specializing all but finitely many variables to zero) as a symmetric polynomial in Λ^N. One way to define $J_\lambda(x_1, ..., x_n;\theta)$ is as the unique symmetric polynomial in n variables, which has leading term $x_1^{\lambda_1} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$ and satisfies the following Sekiguchi differential operator eigenrelation

$$\frac{1}{\prod_{i<j}(x_i - x_j)} \det\left[x_i^{N-j}\left(x_i \frac{\partial}{\partial x_i} + (N-j)\theta + u\right)\right] J_\lambda(x_1, ..., x_n;\theta) =$$

$$\prod_{i=1}^N (\lambda_i + (N-i)\theta + u) J_\lambda(x_1, ..., x_n;\theta)$$

The dual Jack polynomial is given by $\widetilde{J}_\lambda = J_\lambda \prod_{\square \in \lambda} \frac{a(\square) + \theta l(\square) + \theta}{a(\square) + \theta l(\square) + 1}$.

The skew-Jack and dual skew-Jack polynomials are defined through the following relations

$$J_\lambda(x,y;\theta)=\sum_\mu J_\mu(x;\theta)J_{\lambda/\mu}(y;\theta)\quad \widetilde{J}_\lambda(x,y;\theta)=\sum_\mu \widetilde{J}_\mu(x;\theta)\widetilde{J}_{\lambda/\mu}(y;\theta).$$

In the above $x=(x_1,x_2,...)$, $y=(y_1,y_2,...)$ are countable sets of variables and (x,y) is their union. For simplicity, we will ignore θ in the notations.

A homomorphism ρ from the algebra of symmetric functions to the set of complex numbers is called a specialization. If ρ takes positive values on all Jack polynomials, it will be called Jack-positive. We have the following classification.

Proposition 2.1. *[KOO] For any fixed $\theta>0$, Jack positive specializations can be parametrized by triplets (α,β,γ), where α,β are sequences of real numbers with $\alpha_1\ge\alpha_2\ge...\ge 0$, $\beta_1\ge\beta_2\ge...\ge 0, \sum_i(\alpha_i+\beta_i)<\infty$ and γ is a non-negative real number. The specialization corresponding to a triplet (α,β,γ) is given by its values on Newton power sum $p_k, k\ge 1$:*
$p_1\mapsto_1(\alpha,\beta,\gamma)=\gamma+\sum_i(\alpha_i+\beta_i)$
$p_k\mapsto p_k(\alpha,\beta,\gamma)=\sum_i {\alpha_i}^k+(-\theta)^{k-1}\sum_i {\beta_i}^k, k\ge 2$

Throughout this paper we will work with two specializations from Proposition 2.1. The first is denoted by a^N and corresponds to taking $\alpha_1=\cdots=\alpha_N=a$ and all other α,β and γ parameters are set to zero. The second is the *Plancherel specialization* τ_s, which satisfies $\gamma=s$ and all other parameters set to 0. The following formulas are well-known.

Proposition 2.2. *For any Young diagram λ we have*

$$J_\lambda(a^N)=\mathbf{1}_{\{\ell(\lambda)\le N\}}a^{|\lambda|}\prod_{\square\in\lambda}\frac{N\theta+a'(\square)-\theta l'(\square)}{a(\square)+\theta l(\square)+\theta}\quad J_\lambda(\tau_s)=s^{|\lambda|}\theta^\lambda\prod_{\square\in\lambda}\frac{1}{a(\square)+\theta l(\square)+\theta},\tag{2}$$

where $\mathbf{1}_E$ is the indicator function of the set E.

Definition 2.3. Define $J_{\rho_1,\rho_2}(\lambda)=\dfrac{J_\lambda(\rho_1)\widetilde{J}_\lambda(\rho_2)}{H_\theta(\rho_1;\rho_2)}$, where ρ_1,ρ_2 are Jack-positive specializations and $H_\theta(\rho_1;\rho_2)=\exp\left(\sum_{k=1}^\infty\frac{\theta}{k}p_k(\rho_1)p_k(\rho_2)\right)$ with $p_k=\sum_i x_i^k$. This defines a probability measure on Young diagrams, provided

$$\sum_\lambda J_\lambda(\rho_1)\widetilde{J}_\lambda(\rho_2)<\infty,$$

in which case the latter sum equals $H_\theta(\rho_1;\rho_2)$ (see e.g. (10.4) in Chapter VI of [M]).

Remark 2.4. The construction of probability measures via specializations of symmetric polynomials was originally suggested by Okounkov in the context of Schur measures [Ok].

Proposition 2.5. $J_{1^N;\tau_s(\lambda)}=0$ *if λ has more than N rows. If λ has less than N rows we have:*

$$J_{1^N;\tau_s}(\lambda)=e^{-\theta sN}s^{|\lambda|}\theta^{|\lambda|}\prod_{\square\in\lambda}\frac{N\theta+a'(\square)-\theta l'(\square)}{(a(\square)+\theta l(\square)+\theta)(a(\square)+\theta l(\square)+1)}.$$

At this point, we define using Jack polynomials probability measures on the set $\mathbb{GT}^N$ of sequences $\lambda^1,...,\lambda^N$ of Young diagrams with $l(\lambda^i)\le i$ and the interlacing conditions
$\lambda_1^{i+1}\ge\lambda_1^i\ge\lambda_2^{i+1}\ge...\ge\lambda_i^i\ge\lambda_{i+1}^{i+1}$ for $i<N$ (which we denote by $\lambda^i\preceq\lambda^{i+1}$). The following definitions can be found in [GS1].

Definition 2.6. A probability distribution $\mathbb{P}$ on arrays $\lambda^1 \preceq ... \preceq \lambda^N \in \mathbb{GT}^N$ is called a Jack-Gibbs distribution, if for any $\mu \in \mathbb{GT}^N$, such that $\mathbb{P}(\lambda^n = \mu) > 0$, the conditional distribution of $\lambda^1, ..., \lambda^{N-1}$ given that $\lambda^n = \mu$ is

$$\mathbb{P}(\lambda^1, ..., \lambda^{N-1} | \lambda^N = \mu) = \frac{J_{\mu/\lambda^{N-1}}(1) J_{\lambda^{N-1}/\lambda^{N-2}}(1) ... J_{\lambda^2/\lambda^1}(1) J_{\lambda^1}(1)}{J_\mu(1^N)}.$$

Definition 2.7. Given a Jack positive specialization ρ we define the ascending Jack process:

$$J^{asc}_{\rho;N}(\lambda^1, ..., \lambda^N) = \frac{\widetilde{J}_\lambda(\rho) J_{\lambda^N/\lambda^{N-1}}(1) ... J_{\lambda^2/\lambda^1}(1) J_{\lambda^1}(1)}{H_\theta(\rho; 1^N)}.$$

We also recall the following useful formula for skew-Jack polynomials in a single variable

$$J_{\lambda^k/\lambda^{k-1}}(1) = \mathbf{1}_{\{\lambda^{k-1} \preceq \lambda^k\}} \prod_{1 \le i \le j \le k-1} \frac{f(\lambda_i^{k-1} - \lambda_j^{k-1} + \theta(j-i)) f(\lambda_i^k - \lambda_{j+1}^k + \theta(j-i))}{f(\lambda_i^{k-1} - \lambda_{j+1}^k + \theta(j-i)) f(\lambda_i^k - \lambda_j^{k-1} + \theta(j-i))},$$

where $f(z) = \frac{\Gamma(z+1)}{\Gamma(z+\theta)}$ (see e.g. (7.14') in [M] or (2.10) in [GS1]).

We are now ready to define formally the process $X(s; N), s \ge 0$ on $\mathbb{GT}^N$ constructed in [GS1].

Definition 2.8. Let $\theta > 0$ and $N \in \mathbb{N}$. The process $X(s; N) = (X_i^j(s; N) : 1 \le i \le j \le N)$ is defined as the continuous time Markov chain on $\mathbb{GT}^N$, with the following jump rates. If $X(s-; N) = \lambda^1(s-) \preceq \cdots \preceq \lambda^N(s-)$, then to each $\lambda^k(s-)$ we independently add a box $\square$ at time s at the rate

$$q(\square, \lambda^k(s-), \lambda^{k-1}(s)) = \widetilde{J}_{(\lambda^k(s-) \sqcup \square)/\lambda^k(s-)}(\tau_1; \theta) \frac{J_{\lambda^k(s-) \sqcup \square/\lambda^{k-1}(s)}(1; \theta)}{J_{\lambda^k(s-)/\lambda^{k-1}(s)}(1; \theta)}. \tag{3}$$

The latter jump rates incorporate the following push interaction: if we increase one coordinate of $X(s-; N)$, say in $\lambda^{k-1}(s-)$, in a way that results in $\lambda^{k-1}(s) \npreceq \lambda^k(s-)$, then the appropriate coordinate of $\lambda^k(s-)$ is also increased by 1 (the jump rate for this action becomes infinite), which restores the interlacing $\lambda^{k-1}(s) \preceq \lambda^k(s)$.

Proposition 2.9. *Suppose that $X(s; N), s \ge 0$ is started from a random initial condition with a Jack-Gibbs distribution. Then:*

- *The restriction of $X(s; N)$ to level k coincides with $X(s; k), s \ge 0$ started from the restriction to level k of the initial condition.*
- *The distribution of $X(s; N)$ at time $s > 0$ is again a Jack-Gibbs distribution. Moreover, if $X(0; N)$ has distribution $J^{asc}_{\rho;N}$, then $X(s; N)$ has distribution $J^{asc}_{\rho,\tau_s;N}$.*

2.2. **Discrete β-ensemble identification.** We start by giving the definition of the discrete β-ensemble as in [BGG].

Definition 2.10. We fix a parameter $\beta > 0$ and set $\theta = \beta/2$. Consider a function $w(x; N)$ which takes real values.[1] The discrete β-ensemble is the probability distribution

$$\mathbb{P}_N(l_1, ..., l_N) = \frac{1}{Z_N} \prod_{i=1}^N w(l_i; N) \prod_{1 \le i < j \le N} \frac{\Gamma(l_j - l_i + 1)\Gamma(l_j - l_i + \theta)}{\Gamma(l_j - l_i)\Gamma(l_j - l_i + 1 - \theta)} \tag{4}$$

on ordered N-tuples $l_1 < ... < l_N$ such that $l_i = \lambda_{N-i+1} + \theta i$ and $\lambda_1 \ge ... \ge \lambda_N$ are integers. l_i's will be called particles, while the state space of the above configurations $(l_1, ..., l_N)$ will be denoted by $\mathbb{W}_N^\theta$.

[1] $w(x; N)$ should decay at least as $|x|^{-2\theta(1+\epsilon)}$ for some $\epsilon > 0$ as $|x| \to \infty$.

Remark 2.11. If $l_j - l_i \to \infty$ for $1 \le i < j \le N$, the probability in (4) looks like $\prod_{i<j}(l_j - l_i)^\beta \prod_{i=1}^{N} w(l_i; N)$, which describes the general-β log-gas probability distribution.

If we set $l_i = X^N_{N-i+1}(s; N) + \theta i$ with $X(s; N)$ given in Definition 2.8, then we may combine the results of Section 2.1 to show that

$$(5)\quad \mathbb{P}(l_1, ..., l_N) = e^{-s\theta N}(s\theta)^{-\frac{N(N+1)}{2}} \frac{\Gamma(\theta)^N}{\prod_{i<N}\Gamma(i\theta)} \prod_{1\le i<j\le N} \frac{\Gamma(l_j - l_i + \theta)\Gamma(l_j - l_i + 1)}{\Gamma(l_j - l_i)\Gamma(l_j - l_i + 1 - \theta)} \prod_{i=1}^{N} \frac{(s\theta)^{l_i}}{\Gamma(l_i + 1)}$$

The above shows that the top row of $X(s; N)$ at a fixed time is described by the discrete β-ensemble of Definition 2.10 with $w(x; N) = \dfrac{(s\theta)^x}{\Gamma(x + 1)}$.

Let us also denote, $m_i = X^{N-1}_{N-i}(s; N) + \theta i$. The results of Section 2.1 allow us to express the joint distribution of $l_1, ..., l_N$ and $m_1, ..., m_{N-1}$ and we get that it is proportional to:

$$(6)\quad \prod_{i=1}^{N} \frac{(s\theta)^{l_i}}{\Gamma(l_i + 1)} \prod_{1\le i<j\le N} (l_j - l_i) \prod_{1\le i<j<N} (m_j - m_i) \prod_{1\le i\le j\le N-1} \frac{f(m_j - l_i)}{f(l_{j+1} - m_i)}, \text{ with } f(z) = \frac{\Gamma(z+1)}{\Gamma(z+\theta)}.$$

The last formula will be very useful later in this paper.

The identification of the distribution of the top row of our process at fixed time with the β-ensemble readily gives us a result about the law of large numbers of the empirical measures $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{X_i^N(tN;N)+(N-i+1)\theta}{N}\right) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{l_i}{N}\right)$. The exact statement we need is a version of Theorem 1.2 in [BGG] and we state it below

Theorem 2.12. *Suppose $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{X_i^N(tN;N)+(N-i+1)\theta}{N}\right)$ with $X(s; N)$ as in Definition 2.8. Then there exists a deterministic measure μ, such that $\mu_N \Rightarrow \mu$ as $N \to \infty$, in the sense that for any bounded continuous function f, we have the following convergence in probability:*

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x) d\mu_N(x) = \int_{\mathbb{R}} f(x) d\mu(x).$$

The idea of the proof of Theorem 2.12 is to establish a large deviations principle for the measure in (5), which would show that it is concentrated on those N-tuples $(l_1, ..., l_N)$ which maximize the probability density. Similar results are known in various contexts (see e.g. the references in the proof of Proposition 2.2 of [BGG]). In particular, Theorem 2.12 can be proved by following the same arguments as in [J]. We will not write out a complete proof; however, we recall the results of [J] below and discuss how our problem can be rephrased in that framework.

Suppose $V(x)$ is a real-valued function with $V(x) \ge (1 + \xi)\log(x^2 + 1)$ whenever $|x| >> 1$ (here $\xi > 0$). Let $\mathcal{A}_s$ denote the set of all $\phi \in L^1[0, s)$ such that $0 \le \phi \le 1$ and $\int_0^s \phi = 1$, where $1 \le s \le \infty$. For $\phi \in \mathcal{A}_s$ we define

$$k_V(x, y) := \log|x - y|^{-1} + \frac{V(x) + V(y)}{2} \text{ and } I_V[\phi] := \int_0^s \int_0^s k_V(x, y)\phi(x)\phi(y)dxdy.$$

It follows from [DS] that there exists a unique element ϕ^s_V in $\mathcal{A}_s$, which minimizes $I_V[\phi]$. The measure, which has density ϕ^s_V, is compactly supported and absolutely continuous with respect to Lebesgue mesure. The above setup is known as a variational problem, and the measure with density ϕ^s_V, minimizing the functional I_V, is called the equilibrium measure.

Fix $N \in \mathbb{N}$, $\beta > 0$ and set $\mathbb{A}_N = \frac{1}{N}$. We consider the following distribution on ordered elements $x_1 < x_2 < \cdots < x_N$, with $x_i \in \mathbb{A}_N$:

$$\mathbb{P}(x_1, ..., x_N) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N V_N(x_i)\right). \tag{7}$$

In the above, $V_N : \mathbb{R} \to \mathbb{R}$ are continuous and we assume that there exist $\xi, T, N_0 > 0$ such that for $t \le T, N > N_0$ we have $V_N(t) \ge (1+\xi)\log(t^2+1)$ and V_N converge uniformly to a function V on compact sets.

Let $X_1^s < ... < X_N^s$, have the joint distribution given by (7) conditional on $x_N \le s$, where $s \in [1, \infty]$. Under these conditions [J] shows that $\frac{1}{N}\sum_{i=1}^N \delta_{X_1^s}$ weakly converges to the equilibrium measure with density ϕ_V^s, defined above.

An additional important result in [J] is the following large deviation estimate for the position for the right-most particle X_N^∞. Let $F_N(t) = \mathbb{P}(X_N^\infty \le t)$, where $X_1^\infty < ... < X_N^\infty$ have the joint distribution given by (7). Let b_V denote the right-most endpoint of the support of ϕ_V^∞, and put $I_V[\phi_V^s] = F_V^s$ for $s < \infty$ and $I_V[\phi_V^\infty] = F_V$. We define the function $L(t) = \dfrac{F_V^t - F_V}{2}$ and $J(t) = 0$ for $t \le b_V$ and $J(t) = \inf_{\tau \le t} \int_0^\infty k_V(\tau, x)\phi_V(x)dx - F_V(x)$. Then Theorem 2.2 in [J] states

Theorem 2.13. *With the assumptions above,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log F_N(t) = -\beta L(t)$$

for any $t \ge 1$ and $L(t) > 0$ if $t < b_V$. Assume furthermore that $J(t) > 0$ for $t > b_V$. Then,

$$\lim_{N\to\infty} \frac{1}{N} \log(1 - F_N(t)) = -\beta J(t)$$

for all t. In particular, X_N^∞ converges to b_V in probability.

We turn back to our problem and set $x_i = \frac{l_i}{N}$, $s = Nt$ with $t > 0$. In view of (5) we have the following distribution

$$\mathbb{P}(x_1, ..., x_N) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} \frac{\Gamma(Nx_j - Nx_i + \theta)\Gamma(Nx_j - Nx_i + 1)}{\Gamma(Nx_j - Nx_i)\Gamma(Nx_j - Nx_i + 1 - \theta)} \exp\left(-N \log N \sum_{i=1}^N V_N(x_i)\right), \tag{8}$$

where $V_N(x) = \frac{\log \Gamma(Nx+1) - Nx\log(Nt\theta)}{N \log N}$. Using properties of the gamma function Γ, we have that

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left(1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O\left(\frac{1}{|z|^2}\right)\right),$$

for $z \to \infty$. Thus we see that for $x_i < x_j$ fixed

$$\frac{\Gamma(Nx_j - Nx_i + \theta)\Gamma(Nx_j - Nx_i + 1)}{\Gamma(Nx_j - Nx_i)\Gamma(Nx_j - Nx_i + 1 - \theta)} \sim (x_j - x_i)^{2\theta}.$$

Thus the interaction term in our problem asymptotically looks the same as the logarithmic interaction term in [J]. The potential term, represented by $\exp\left(-N\log N \sum_{i=1}^N V_N(x_i)\right)$ in (8), matches the one in (7) upto an additional logarithmic factor $\log N$. This additional factor is harmless and does not change the arguments in [J], which may be repeated to show the validity of Theorem 2.12.

We observe that over compact sets $V_N(x) = x\log x - x + O(\frac{1}{\log N})$, and so it converges to $V(x) = x\log x - x$. The measure μ to which the empirical measures μ_N of Theorem 2.12 converge is the solution to the variational problem discussed above with $V(x) = x\log x - x$. The only thing that changes is that the function space over which we take the infimum has to be replaced with $\mathcal{A}'_\infty = \{\phi \in L^1[0,\infty)$ such that $0 \le \phi \le \theta^{-1}$ and $\int_0^\infty \phi = 1\}$. This modification results from the fact that the minimal separation between our particles is $\frac{\theta}{N}$ as opposed to the $\frac{1}{N}$ considered in [J]. We remark that the arguments in [J] provide us with the concentration result of the empirical measures and classify the limit as the solution of a variational problem. An exact formula for this limit will be derived in the next section by a different technique and it will be seen that it has a continuous compactly supported density.

Finally, we remark that a result similar to Theorem 2.13 holds in our case by essentially the same arguments. The result that we will need is that the right-most particle $x_N = \frac{l_N}{N}$ concentrates near the right endpoint of the support of μ. The precise statement is given below.

Lemma 2.14. *Suppose $(l_1, ..., l_N)$ are distributed as in (5) with $s = tN$ and $t > 0$ and let b_R denote the right endpoint of the support of the measure μ from Theorem 2.12. Then we have the following convergence in probability*

$$\lim_{N\to\infty} \frac{l_N}{N} = b_R \tag{9}$$

An easy corollary of the above that will be needed in Section 3 is given below.

Corollary 2.15. *With the notation of Lemma 2.14, we have*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{N}{l_N+1}\right] = \frac{1}{\theta(\sqrt{t}+1)^2} \tag{10}$$

Proof. For now we assume that $b_R = \theta(1+\sqrt{t})^2$ is the right-most point of the support of the equilibrium measure, a result which will be proved rigorously in the next section.

We have $\dfrac{N}{l_N+1} \leq \dfrac{N}{(N-1)\theta}$, which is bounded. Moreover $\dfrac{N}{l_N+1} = \dfrac{1}{\frac{1}{N}+\frac{l_N}{N}}$ converges in probability to $\dfrac{1}{\theta(\sqrt{t}+1)^2}$ by Lemma 2.14. The Bounded convergence theorem completes the proof. $\square$

2.3. **Nekrasov's equation and limiting measure.** There is a simple way to calculate the equilibrium measure, and that is to use the so called Nekrasov's equation. This approach was followed in [BGG]. The reason why we could not use their result without proof is that they assumed compactness of the measures (which we do not have), but only discreteness of the poles and rapid convergence of the weight $w(x;N)$ near infinity are important. To be more specific, we have the following result:

Theorem 2.16. *Let $\mathbb{P}$ be a distribution on N-tuples $(l_1, ..., l_N) \in \mathbb{W}_N^\theta$ as in (5). Suppose $\phi^\pm$ are analytic and satisfy $\dfrac{w(x;N)}{w(x-1;N)} = \dfrac{\phi_N^+(x)}{\phi_N^-(x)}$, where $w(x;N) = \dfrac{(s\theta)^x}{\Gamma(x+1)}$. Define*

$$R_N(\xi) = \phi_N^-(\xi)\cdot \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^N\left(1-\frac{\theta}{\xi-l_i}\right)\right] + \phi_N^+(\xi)\cdot\mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^N\left(1+\frac{\theta}{\xi-l_i-1}\right)\right].$$

If ϕ_N^+, ϕ_N^- are polynomials of degree at most d, then so is $R_N(\xi)$.

The above is essentially Theorem 4.1 in [BGG] and the proof that we give here is very much the same as their proof. We repeat it here for reasons of completeness (as noted we do not have compactness of the support as in [BGG]).

Proof. The first step of the proof is to show that R_N is an entire function. If $Im(\xi_0) > 0$, then $\prod_{i=1}^N\left(1-\frac{\theta}{\xi-l_i}\right)$ are uniformly bounded in a neighborhood of ξ_0, so the first term of R_N is differentiable at ξ_0 (absolutely summable series of differentiable functions) and similarly for the other term. The same reasoning applies for $Im(\xi_0) < 0$.

If $\xi_0 \in \mathbb{R}$ is not on $S_N^\theta = \{0,1\} + \mathbb{W}_N^\theta$, then the fact that S_N^θ is discrete means that in a neighborhood of ξ_0 the first summand of R_N is again differentiable (absolutely summable series). From the discussion above it is clear that R_N can only possibly have simple poles at the points $l_i, l_i + 1$ (where l_i belongs to the ensemble) as singularities.

Switching from $l_i = x$ to $l_i = x - 1$ (following the proof by [BGG]) the interaction term of the density is multiplied by:

$$\frac{\Gamma(l_j - x + 1)\Gamma(l_j - x + \theta)}{\Gamma(l_j - x)\Gamma(l_j - x + 1 - \theta)} \frac{\Gamma(l_j - x + 1)\Gamma(l_j - x + 2 - \theta)}{\Gamma(l_j - x + 2)\Gamma(l_j - x + 1 + \theta)} = \frac{(l_j - x)(l_j - x + 1 - \theta)}{(l_j - x + 1)(l_j - x + \theta)}$$

for $i < j$ and for $i > j$ by:

$$\frac{\Gamma(x - l_j + 1)\Gamma(x - l_j + \theta)}{\Gamma(x - l_j + 1 - \theta)\Gamma(x - l_j)} \frac{\Gamma(x - l_j - 1)\Gamma(x - l_j - \theta)}{\Gamma(x - l_j)\Gamma(x - l_j - 1 + \theta)} = \frac{(x - l_j)(x - l_j + \theta - 1)}{(x - l_j - 1)(x - l_j - \theta)}.$$

By Fubini's theorem, the residue of R_N arising from l_i at m is:

$$-\theta \sum_{l \in \mathbb{W}_N^\theta, l_i = m} \phi_N^-(x)\mathbb{P}(l) \prod_{j \neq i} \left(1 - \frac{\theta}{m - l_j}\right) + \theta \sum_{l \in \mathbb{W}_N^\theta, l_i = m-1} \phi_N^+(x)\mathbb{P}(l) \prod_{j \neq i} \left(1 + \frac{\theta}{m - l_j - 1}\right).$$

Note that the two series are absolutely summable because of the discreteness of S_N^θ. From the definition of ϕ_N^+, ϕ_N^-, it is obvious that the two terms cancel and thus the residue at l_i is 0.

So R_N is an entire function.

Take $1 > \varepsilon > 0$. For any $R_1 > 0$ large enough such that $(R_1, 0)$ is at least at distance (in the complex plane) ε from the lattice where the ensemble lies, $|R(z)|$ is at most MR_1^d on the circle $|z| = R_1$ for some fixed constant $M = M(\varepsilon)$. Fix any $z_0 \in \mathbb{C}$. Then, by Cauchy formula we have for the $(d+1)$-th derivative of R if R_1 is large enough such that $|z - z_0| > \frac{R_1}{2}$:

$$\left|R^{(d+1)(z_0)}\right| = \left|\frac{(d+1)!}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z - z_0)^{d+2}} dz\right| \leq 2^{d+2} \frac{2\pi R_1 R_1^d M}{2\pi R_1^{d+2}} = 2^{d+2} \frac{M}{R_1}.$$

For $R_1 \to \infty$, we see that $R^{(d+1)}(z_0) = 0$. Since z_0 was arbitrary, the conclusion follows.

□

We define $G(z) = \int \frac{1}{z - x} \nu(dx)$ to be the Stieltjes transform of a measure ν. Note that $G(z)$ is analytic on the upper and lower half-planes (by differentiation under the integral sign).

We developed the tools and we are ready to calculate the equilibrium measure. After the time and space rescaling $(s, x) = (Nt, N\xi)$, we see that $\frac{w(x; N)}{w(x-1; N)} = \frac{s\theta}{x} = \frac{t\theta}{\xi}$. So we can choose $\phi_N^+(\xi) = t\theta$ and $\phi_N^-(\xi) = \xi$.

Consider $R(\xi) = \xi e^{-\theta G(\xi)} + t\theta e^{\theta G(\xi)}$, where G is the Stieltjes transform of μ. R is a polynomial of degree at most 1, as it is the $N \to \infty$ limit of $R_N(N\xi)$ defined in Nekrasov's equation. To see this, we use the approximation $1 + x \approx e^x$ for small x and the fact that

$$\frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - \frac{l_j}{N}} = \int \frac{1}{z - x} \mu_N(dx) \to \int \frac{1}{z - x} \mu(dx) \text{ as } N \to \infty$$

for all z that do not belong to S_N^θ.

We have $R(\xi)/\xi \to 1$ as $\xi \to \infty$ and $R(\xi) - \xi = \xi(e^{-\theta G(\xi)} - 1) + t\theta e^{\theta G(\xi)} \to \theta(t - 1)$. The two limits follow from the fact that $G(z) \sim \frac{1}{z}$ as $z \to \infty$.

From the results above we derive $R(z) = z + \theta(t-1)$. Solving for $G(z)$ gives:

$$\exp(\theta G(\xi)) = \frac{\xi + \theta(t-1) - \sqrt{(\xi + \theta(t-1))^2 - 4t\theta\xi}}{2t\theta} \tag{11}$$

We apply the inversion formula for the Stieltjes transform

$$f(x) = \lim_{y \to 0^+} \frac{ImG(x-iy) - ImG(x+iy)}{2\pi i}$$

to derive a formula for the density of the limiting measure. The result split into the cases $t \geq 1$ and $t \in (0,1)$.

Suppose $t \geq 1$. Then we get

$$f(x) = \begin{cases} 0 & \text{for } x < \theta(\sqrt{t}-1)^2 \text{ or } x > \theta(\sqrt{t}-1)^2, \\ \dfrac{\operatorname{arccot}\left(\frac{\sqrt{4\theta t x - [x+\theta(t-1)]^2}}{x+\theta(t-1)}\right)}{\theta\pi} & \text{otherwise.} \end{cases} \tag{12}$$

Suppose $t \in (0,1)$. Then we get

$$f(x) = \begin{cases} 0 & \text{for } x > \theta(\sqrt{t}+1)^2 \\ \frac{1}{\theta} & \text{for } x < \theta(\sqrt{t}-1)^2 \\ \dfrac{\operatorname{arccot}\left(\frac{\sqrt{4\theta t x - [x+\theta(t-1)]^2}}{x+\theta(t-1)}\right)}{\theta\pi} & \text{otherwise.} \end{cases} \tag{13}$$

We see that the above density is compactly supported in an interval with right end given by $b_R = \theta(\sqrt{t}+1)^2$. The record the following statement, which will be used later in the text

$$\lim_{\varepsilon \to 0^+} G(b_R + \varepsilon) = \frac{\log(1 + \frac{1}{\sqrt{t}})}{\theta}. \tag{14}$$

The above follows from the fact that the density behaves like $\dfrac{1}{\sqrt{x}}$ near b_R (so we can use the Dominated convergence theorem).

Remark 2.17. We will in fact need only $G(\xi)$ and not the density of μ. It was presented here just for reasons of completeness.

3. Key results

We prove some technical lemmas that we will need in Section 4. Throughout this section we denote by $\xrightarrow{\mathbb{P}}$ convergence in probability.

3.1. A useful lemma.

Lemma 3.1. *If $0 < X_n, Y_n < 1$ are random variables and $c > 0$ with $\mathbb{E}[X_n Y_n] \to c$ and for any $\eta, \xi > 0$ we have $\mathbb{P}(X_n > c + \eta) < \xi$ for n large enough, then $X_n \xrightarrow{\mathbb{P}} c$ and $Y_n \xrightarrow{\mathbb{P}} 1$.*

Proof. Since $X_n Y_n \leq X_n$, we just need to show that for any $\eta, \xi > 0$ we have $\mathbb{P}(X_n \leq c - \eta) < \xi$ and $\mathbb{P}(Y_n \leq 1 - \eta) < \xi$ for all n large enough. Suppose that for some ξ, η we have $\mathbb{P}(X_n \leq c - \eta) \geq \xi$ for infinitely many n's. Without loss of generality we assume $\mathbb{P}(X_n \leq c - \eta) \to \xi_1$ for some $\xi_1 \geq \xi$ (otherwise we take subsequential limits). Take any $\varepsilon > 0$.

We have:

$$\begin{aligned} \mathbb{E}[X_n Y_n] &= \mathbb{E}\left[X_n Y_n \mathbf{1}_{\{X_n \leq c - \eta\}}\right] + \mathbb{E}\left[X_n Y_n \mathbf{1}_{\{c - \eta < X_n \leq c + \varepsilon\}}\right] + \mathbb{E}\left[X_n Y_n \mathbf{1}_{\{c + \varepsilon < X_n\}}\right] \\ &\leq (c - \eta)\mathbb{P}(X_n \leq c - \eta) + (c + \varepsilon)\mathbb{P}(c - \eta < X_n \leq c + \varepsilon) + \mathbb{P}(c + \varepsilon < X_n). \end{aligned}$$

Taking limits on both sides of the inequality ($n \to \infty$), since $\mathbb{P}(c+\varepsilon < X_n) \to 0$, we get:

$$c \leq (c-\eta)\xi_1 + (c+\varepsilon)(1-\xi_1) = c - \eta\xi_1 + \varepsilon - \varepsilon\xi_1 \Rightarrow \xi_1 \leq \frac{\varepsilon}{\eta+\varepsilon}$$

Obviously, this cannot hold for ε arbitrarily small, so we get a contradiction. So $X_n \xrightarrow{\mathbb{P}} c$. Similarly $Y_n \xrightarrow{\mathbb{P}} 1$. The proof is completed. $\square$

3.2. **Separation of the particles on the top row.** We turn back to the notation from Section 2.2 and let l_i and m_j be distributed as in equations (6) with $s = Nt$. Intuitively one would expect from the dynamics of the problem that the two right-most particles of the top row to be far apart. Rigorously, it should be expected that $\dfrac{1}{l_N - l_{N-1}} \xrightarrow{\mathbb{P}} 0$ as $N \to \infty$. This result will be proved here. For technical reasons we will assume $\theta \geq \dfrac{1}{2}$.

Lemma 3.2. *Fix $\theta \geq 1/2$ and $t > 0$. Then as $N \to \infty$ we have*

$$i) \frac{1}{l_N - l_{N-1}} \xrightarrow{\mathbb{P}} 0, \quad ii) \sum_{j=1}^{N-1} \frac{1}{l_N - l_j} \xrightarrow{\mathbb{P}} \frac{\log(1+\frac{1}{\sqrt{t}})}{\theta}, \quad iii) \sum_{j=1}^{N-2} \frac{1}{l_N - m_j} \xrightarrow{\mathbb{P}} \frac{\log(1+\frac{1}{\sqrt{t}})}{\theta}.$$

Proof. Set $\Delta(x) = \displaystyle\prod_{j=1}^{N-1} \frac{\Gamma(x-l_j+1)\Gamma(x-l_j+\theta)}{\Gamma(x-l_j+1-\theta)\Gamma(x-l_j)}$ for $x > l_{N-1}$ and 0 otherwise.

The key relation in the proof of this lemma is: $\mathbb{E}\left[\dfrac{\Delta(l_N-1)}{\Delta(l_N)}\right] = \mathbb{E}\left[\dfrac{s\theta}{l_N+1}\right]$. This follows by taking expectations on both sides of the following relation:

$$\mathbb{E}\left[\frac{s\theta}{l_N+1}\middle| l_1, ..., l_{N-1}\right] = \mathbb{E}\left[\frac{\Delta(l_N-1)}{\Delta(l_N)}\middle| l_1, ..., l_{N-1}\right].$$

To see the latter, we use the functional equation $\Gamma(x+1) = x\Gamma(x)$, which shows that

$$\mathbb{E}\left[\frac{s\theta}{l_N+1}\middle| l_1, ..., l_{N-1}\right] = C(l_1, .., l_{N-1}) \sum_{l_N \in \theta + l_{N-1} + \mathbb{Z}_{\geq 0}} \frac{s\theta}{l_N+1} \frac{(s\theta)^{l_N}}{\Gamma(l_N+1)} \Delta(l_N) =$$

$$= C(l_1, .., l_{N-1}) \sum_{l_N \in \theta + l_{N-1} + \mathbb{Z}_{\geq 0}} \frac{(s\theta)^{l_N}}{\Gamma(l_N+1)} \Delta(l_N) \frac{\Delta(l_N-1)}{\Delta(l_N)} = \mathbb{E}\left[\frac{\Delta(l_N-1)}{\Delta(l_N)}\middle| l_1, ..., l_{N-1}\right],$$

where $C(l_1, .., l_{N-1})$ is some normalization constant. In the above we used that $\Delta(l_{N-1}+\theta-1) = 0$.

By Corollary 2.15 we get

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{\Delta(l_N-1)}{\Delta(l_N)}\right] = \frac{t}{(\sqrt{t}+1)^2}. \tag{15}$$

We observe that

$$\frac{\Delta(l_N-1)}{\Delta(l_N)} = \prod_{j=1}^{N-1} \left(1 - \frac{1}{l_N - l_j}\right)\left(1 - \frac{2\theta-1}{l_N - l_j + \theta - 1}\right) \leq X_N Y_N, \tag{16}$$

where $X_N = \exp(-\sum_{j=1}^{N-1}(\frac{1}{l_N-l_j} + \frac{2\theta-1}{l_N-l_j+\theta-1}))$, and $Y_N = \exp(-\sum_{j=1}^{N-1}(\frac{1}{(l_N-l_j)^2} + \frac{(2\theta-1)^2}{(l_N-l_j-1+\theta)^2}))$.

In what follows we want to use Lemma 3.1 to prove that X_N converges to $\frac{t}{(\sqrt{t}+1)^2}$ and Y_N to 1 in probability. We observe that $0 < X_N, Y_N < 1$.

Take $\varepsilon > 0, \delta > 0$. By Lemma 2.14 we have $\mathbb{P}\left(\frac{l_N}{N} \geq \theta(\sqrt{t}+1)^2 + \frac{\delta}{2}\right) < \varepsilon$ for N large enough. On the event $\{\frac{l_N}{N} \geq \theta(\sqrt{t}+1)^2 + \frac{\delta}{2}\}$ we have for N large enough:

$$\sum_{j=1}^{N-1} \frac{1}{l_N - l_j} + \frac{2\theta - 1}{l_N - l_j + \theta - 1} > \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{\frac{\delta}{2} + \theta(\sqrt{t}+1)^2 - \frac{l_j}{N}} + \frac{2\theta - 1}{\frac{\delta}{2} + \theta(\sqrt{t}+1)^2 + \frac{\theta-1}{N} - \frac{l_j}{N}}$$

$$> \frac{2\theta}{N} \sum_{j=1}^{N-1} \frac{1}{\delta + \theta(\sqrt{t}+1)^2 - \frac{l_j}{N}} = \frac{2\theta}{N} \sum_{j=1}^{N} \frac{1}{\delta + \theta(\sqrt{t}+1)^2 - \frac{l_j}{N}} - \frac{2\theta}{N} \frac{1}{\delta + \theta(\sqrt{t}+1)^2 - \frac{l_N}{N}}.$$

The first term in the last expression converges to $2\theta G(b_R + \delta)$ by Theorem 2.12. On the event $\{\frac{l_N}{N} \geq \theta(\sqrt{t}+1)^2 + \frac{\delta}{2}\}$ the second term is bounded by

$$\frac{2\theta}{N} \frac{1}{\delta + \theta(\sqrt{t}+1)^2 - \frac{l_N}{N}} \leq \frac{4\theta}{N\delta}.$$

The above suggests that for any $\delta, \varepsilon > 0$ we have

$$\limsup_{N\to\infty} \mathbb{P}(X_N \leq \exp(-2\theta G(b_R + \delta)) \geq 1 - \varepsilon.$$

Note that by (14) $\exp(-2\theta G(b_R+\delta)) \to \frac{t}{(\sqrt{t}+1)^2}$ as $\delta \to 0^+$, which implies that for any $\eta, \xi > 0$ we have for N large enough:

$$\mathbb{P}\left(X_N \geq \eta + \frac{t}{(\sqrt{t}+1)^2}\right) < \xi.$$

Since X_N is bounded, we get

$$\limsup_{N\to\infty} \mathbb{E}[X_N] \leq \frac{t}{(\sqrt{t}+1)^2}.$$

Combining the latter statement with $Y_N \in (0,1)$, (15) and (16) we see that

$$\lim_{N\to\infty} E[X_N Y_N] = \frac{t}{(\sqrt{t}+1)^2}. \tag{17}$$

The above work shows that X_N, Y_N satisfy the conditions of Lemma 3.1, which gives

$$\sum_{j=1}^{N-1} \frac{1}{(l_N - l_j)^2} + \frac{(2\theta-1)^2}{(l_N - l_j + \theta - 1)^2} \xrightarrow{\mathbb{P}} 0, \text{ and} \sum_{j=1}^{N-1} \frac{1}{l_N - l_j} + \frac{2\theta - 1}{l_N - l_j + \theta - 1} \xrightarrow{\mathbb{P}} 2\log\left(1 + \frac{1}{\sqrt{t}}\right). \tag{18}$$

Now it is obvious that $\frac{1}{(l_N - l_j)^2} \xrightarrow{\mathbb{P}} 0$, and so $\frac{1}{l_N - l_j} \xrightarrow{\mathbb{P}} 0$. This proves i.

We have:

$$\sum_{j=1}^{N-1} \frac{1}{l_N - l_j} + \frac{2\theta - 1}{l_N - l_j + \theta - 1} = \sum_{j=1}^{N-1} \frac{2\theta}{l_N - l_j} + (2\theta - 1)\frac{1 - \theta}{(l_N - l_j)(l_N - l_j + \theta - 1)}. \tag{19}$$

Since

$$\sum_{j=1}^{N-1} \frac{1}{(l_N - l_j)(l_N - l_j + \theta - 1)} \leq \frac{1}{l_N - l_{N-1}} \sum_{j=1}^{N-1} \frac{1}{l_N - l_j + \theta - 1} \xrightarrow{\mathbb{P}} 0,$$

we get by (18) and (19) that $2\theta \sum_{j=1}^{N-1} \frac{1}{l_N - l_j} \to 2\log\left(1 + \frac{1}{\sqrt{t}}\right)$. This proves ii.

Notice that

$$\sum_{j=1}^{N-1} \frac{1}{l_N - l_j} - \frac{1}{l_N - l_{N-1}} = \sum_{j=1}^{N-2} \frac{1}{l_N - l_j} \leq \sum_{j=1}^{N-2} \frac{1}{l_N - m_j} \leq \sum_{j=1}^{N-2} \frac{1}{l_N - l_{j+1}} \leq \sum_{j=1}^{N-1} \frac{1}{l_N - l_j}.$$

From the above it is clear that iii follows from i and ii. □

4. Analysis of the distance of the right-most particles

We will combine all the results proved in this paper for the ultimate goal of studying the distance between the particles at the right of each row of the ensemble. Specifically, we will show that the distances of the right-most particles of adjacent rows converge to independent negative binomial distributions, as noted in the introduction. The approach here closely follows ideas from [GS2], where they proved the analogous result for β-Dyson Brownian motion.

4.1. Two row analysis.

4.1. Two row analysis. First of all, we will study the separation of the right-most particles just of the first two rows.

Lemma 4.1. *Let l_i, m_j be distributed as in (6) with $s = Nt$ and $t > 0$. Then $l_N - m_{N-1} \xrightarrow{\mathbb{P}} Z$ as $N \to \infty$, where*

$$\mathbb{P}(Z = n + \theta) = (1-p)^{-\theta} \frac{\Gamma(n+\theta)}{\Gamma(n+1)\Gamma(\theta)} p^n, \ n \in \mathbb{Z}_{\geq 0}, \text{ and } p = \frac{\sqrt{t}}{1+\sqrt{t}}.$$

.

Proof. From (6), we know that $\mathbb{P}(l_N - m_{N-1} = z | l_1, ..., l_N, m_1, ..., m_{N-2})$ is proportional to

$$\frac{\Gamma(z)}{\Gamma(z+1-\theta)} \prod_{i=1}^{N-2} (l_N - z - m_i) \prod_{i=1}^{N-1} \frac{\Gamma(l_N - z - l_i + \theta)}{\Gamma(l_N - z - l_i + 1)}.$$

Put $z_N = l_N - m_{N-1} - \theta$. For some $c > 0$ (depending on $l_1, ..., l_N$ and $m_1, ..., m_{N-2}$) we have

$$\mathbb{P}(z_N = k | l_1, ..., l_N, m_1, ..., m_{N-1}) = f_N(k)$$

$$f_N(k) = c \frac{\Gamma(\theta+k)}{\Gamma(1+k)} \prod_{i=1}^{N-2} (l_N - m_i - k - \theta) \prod_{i=1}^{N-1} \frac{\Gamma(l_N - l_i - k)}{\Gamma(l_N - l_i - k + 1 - \theta)}. \tag{20}$$

We will study the ratio $\frac{f_N(k)}{f_N(0)}$ as $N \to \infty$.

$$\frac{f_N(k)}{f_N(0)} = \frac{\Gamma(\theta+k)\Gamma(1)}{\Gamma(1+k)\Gamma(\theta)} \prod_{i=1}^{N-2} \frac{l_N - m_i - k - \theta}{l_N - m_i - \theta} \prod_{i=1}^{N-1} \frac{\Gamma(l_N - l_i - k)\Gamma(l_N - l_i + 1 - \theta)}{\Gamma(l_N - l_i)\Gamma(l_N - l_i - k - \theta + 1)}$$

$$= \frac{\Gamma(\theta+k)\Gamma(1)}{\Gamma(1+k)\Gamma(\theta)} \prod_{i=1}^{N-2} \left(1 - \frac{k}{l_N - m_i - \theta}\right) \prod_{i=1}^{N-1} \frac{f(l_N - l_i - \theta)}{f(l_N - l_i - k - \theta)}, \text{ with } f(z) = \tfrac{\Gamma(z+1)}{\Gamma(z+\theta)}.$$

By Lemma 3.2, we can find a sequence of sets $D(N) \subset \mathbb{R}^{2N-2}$ satisfying the interlacing properties of the top two rows of our ensemble such that $\lim_{N\to\infty} \mathbb{P}(l_1, ..., l_N, m_1, ..., m_{N-2} \in D(N)) = 1$ and for any sequence $(x_1(N), ..., x_N(N), y_1(N), ..., y_{N-2}(N)) \in D(N)$ we have:

- $\lim_{N\to\infty} \frac{1}{x_N(N) - x_{N-1}(N)} = 0$;
- $\lim \sum_{i=1}^{N-1} \frac{1}{x_N(N) - x_i(N) - \theta} = \frac{\log(1 + \frac{1}{\sqrt{t}})}{\theta}$;

- $\lim \sum_{i=1}^{N-2} \frac{1}{x_N(N) - y_i(N) - \theta} = \frac{\log(1+\frac{1}{\sqrt{t}})}{\theta}$.

It is enough to study $f_N(k)$ on $D(N)$. For those sequences we get:

$$\lim_{N\to\infty} \frac{f_N(k)}{f_N(0)} = \frac{\Gamma(\theta+k)}{\Gamma(\theta)\Gamma(1+k)} \lim_{N\to\infty} \exp\left(-\sum_{i=1}^{N-2} \frac{k}{l_N - m_i - \theta} - \sum_{i=1}^{N-1} \frac{k(\theta-1)}{l_N - l_i - \theta} + o\left(\frac{1}{l_N - l_N - 1}\right)\right)$$

$$= \frac{\Gamma(k+\theta)}{k!\Gamma(\theta)} \exp\left(-k\log\left(1+\frac{1}{\sqrt{t}}\right)\right).$$

Thus, $\frac{f_N(k)}{f_N(0)} \to p^k \frac{\Gamma(k+\theta)}{k!\Gamma(\theta)}$, where $p = \frac{\sqrt{t}}{1+\sqrt{t}}$.

The limit above is proportional to the density of a negative binomial distribution, which is $p^k(1-p)^\theta \frac{\Gamma(k+\theta)}{k!\Gamma(\theta)}$. It remains to show that $f_N(0) \to (1-p)^\theta$.

Suppose that we had $\limsup_{N\to\infty} f_N(0) > (1-p)^\theta$. Then, for some N large enough we would have $\sum_{j\le k} \mathbb{P}(z_N = j) > 1$ for some k, contradiction. So $\limsup_{N\to\infty} f_N(0) \le (1-p)^\theta$.

It remains to show that $\liminf_{N\to\infty} f_N(0) \ge (1-p)^\theta$. Pick any $\varepsilon > 0$. For N sufficiently large and all k we have:

$$f_N(0) \ge \frac{\Gamma(\theta)\Gamma(k+1)}{\Gamma(k+\theta)} \exp\left(\sum_{i=1}^{N-2} \frac{k}{l_N - m_i - \theta} + \sum_{i=1}^{N-1} \frac{k(\theta-1)}{l_N - l_i - \theta}\right) \ge$$

$$f_N(k)\left(1+\frac{1}{\sqrt{t}} - \varepsilon\right)^k \frac{\Gamma(\theta)\Gamma(1+k)}{\Gamma(\theta+k)}.$$

The last inequality implies that for all large N:

$$\left(1+\frac{1}{\sqrt{t}} - \varepsilon\right)^{-k} f_N(0) \frac{\Gamma(\theta+k)}{\Gamma(\theta)\Gamma(1+k)} \ge f_N(k),$$

which implies

$$\liminf_{N\to\infty} f_N(0) \frac{1}{\left(1 - \frac{1}{1+\frac{1}{\sqrt{t}}} - \varepsilon\right)^\theta} \ge 1,$$

by the well-known identity $\sum_k x^{-k} \frac{\Gamma(k+\theta)}{\Gamma(\theta)\Gamma(1+k)} = \left(1 - \frac{1}{x}\right)^{-\theta}$.

For $\varepsilon \to 0$ the last inequality gives $\lim \inf_{N\to\infty} f(0) \ge (1-p)^\theta$.The proof is completed. □

4.2. **Proof of Theorem 1.1.** We proceed by induction on k. For $k = 1$, we already proved the result in Lemma 4.1 when $s = 0$. In fact, the same arguments can be repeated to prove the statement for any $s \ge 0$.

Suppose for $k < m$ the result of the theorem holds. We will prove it for $k = m$. For simplicity of notations we set $Z_j^N = X_1^{N-j+1}(Nt+s;N) - X_1^{N-j}(Nt+s;N)$ for $j = 1, ..., k$.

Take any $k_1, ..., k_m \in \mathbb{Z}_{\ge 0}$. Then, we know that

$$\mathbb{P}(Z_1^N = k_1, ..., Z_m^N = k_m) = \mathbb{P}(Z_m^N = k_m | Z_1^N = k_1, ..., Z_{m-1}^N = k_{m-1}) \mathbb{P}(Z_1^N = k_1, ..., Z_{m-1}^N = k_{m-1}).$$

By the inductive hypothesis it is enough to show that

$$\lim_{N\to\infty} \mathbb{P}(Z_m^N = k_m | Z_1^N = k_1, ..., Z_{m-1}^N = k_{m-1}) = p^{k_m}(1-p)^\theta \frac{\Gamma(k_m+\theta)}{k_m!\Gamma(\theta)}. \tag{21}$$

From Section 2.1 we have

$$\mathbb{P}(\lambda^N, ..., \lambda^{N-m}) = \frac{\widetilde{J}_{\lambda^N}(\rho) J_{\lambda^N/\lambda^{N-1}}(1) ... J_{\lambda^{N-m+1}/\lambda^{N-m}}(1) J_{\lambda^{N-m}}(1^{N-m})}{H_\theta(\rho; 1^N)}. \tag{22}$$

Let $\mathcal{F}_N$ be the σ-algebra generated by $X_i^{N-j+1}(Nt+s;N)$ for $j = 1, ..., m$ and $i = 1, ..., N-j+1$. Notice that $\mathcal{F}_N$ is a finer σ-algebra than that of $Z_1^N, ..., Z_{m-1}^N$. Equation (22) gives us that the conditional on $\mathcal{F}_N$ density of the $(m+1)$-th row is proportional to $J_{\lambda^{N-m}}(1^{N-m}) J_{\lambda^{N-m+1}/\lambda^{N-m}}(1)$. Consequently,

$$\mathbb{P}(Z_m^N = k_m | \mathcal{F}_N) = \mathbb{P}(X_1^{N-m+1}(tN; N-m) - X_1^{N-m}(tN; N-m) = k_m | X_1^{N-m+1}, ..., X_{N-m+1}^{N-m+1}).$$

From this point we can follow exactly the same proof as for $k = 1$ to conclude that

$$\lim_{N \to \infty} \mathbb{P}(Z_m^N = k_m | \mathcal{F}_N) = p^{k_m}(1-p)^\theta \frac{\Gamma(k_m + \theta)}{k_m! \Gamma(\theta)}.$$

The latter implies the validity of (21) by the Tower property for conditional expectation, which proves the theorem.

5. Dynamic limit

In this section we describe a conjectural extension of our results. As discussed in Section 1, the parallel with multilevel Dyson Brownian motion suggests that for any $t > 0$

$$\Big(X_1^N(tN+s;N) - X_1^{N-1}(tN+s;N), ..., X_1^{N-k+1}(tN+s;N) - X_1^{N-k}(tN+s;N) \Big)$$

converges to a continuous time Markov chain as $N \to \infty$. Investigating this question will be the subject of a future joint work with Evgeni Dimitrov. In what follows we describe a certain continuous time Markov process $(Q_1(s), ..., Q_k(s))$ and formulate a precise conjectural statement that extends Theorem 1.1.

Let us fix $k \geq 1$, $t > 0$ and $\theta \geq 1/2$. Suppose we have k piles of particles at locations $1, ..., k$, and at time s the i-th pile contains a non-negative integer number of particles $Q_i(s)$. In addition, we assume we have a pile with infinitely many particles at location 0 and a sink at location $k+1$. The i-th pile with $i \in \{1, ..., k\}$ has an exponential clock with parameter $\lambda_i(s) = \theta \frac{\theta + Q_i(s)}{1 + Q_i(s)}$. The clocks are independent of each other and when the i-th clock rings, a particle from the closest non-empty pile to the left jumps into pile i. The infinite pile at location 0, ensures that there is always a non-empty pile to the left and is a source for new particles to enter the system. In addition, the sink has an exponential clock with constant parameter $\lambda_{sink} = \theta \frac{1+\sqrt{t}}{\sqrt{t}}$ and when the clock rings a single particle jumps from the nearest non-empty pile to the left into the sink and disappears.

We let $Q(s) = (Q_1(s), ..., Q_k(s))$ be the Markov process defined through the dynamics in the previous paragraph and with initial distribution such that $Q_1(0), ..., Q_k(0)$ are i.i.d. random variables with

$$\mathbb{P}(Q_1(0) = n) = (1-p)^{-\theta} \frac{\Gamma(n+\theta)}{\Gamma(n+1)\Gamma(\theta)} p^n, \; n \in \mathbb{Z}_{\geq 0}, \text{ and } p = \frac{\sqrt{t}}{1+\sqrt{t}}.$$

It is easy to check that $Q(s)$ is a stationary pure jump continuous time Markov process and we view it as an element in $\mathcal{D}^n$ - the space of right continuous left limited functions from $[0, \infty)$ to $\mathbb{Z}_{\geq 0}^n$ with the topology of uniform convergence over compact sets. With this notation we formulate the following extension of Theorem 1.1.

Conjecture 5.1. Assume the same conditions as in Theorem 1.1. The distribution of the process

$$\Big(X_1^N(tN+s;N) - X_1^{N-1}(tN+s;N), ..., X_1^{N-k+1}(tN+s;N) - X_1^{N-k}(tN+s;N) \Big), \quad s \geq 0$$

on $\mathcal{D}^n$ converges weakly to that of the process $Q(s) = (Q_1(s), ..., Q_k(s))$ defined above.

The dynamics we described earlier is the dual dynamics to a certain push-TASEP interacting particle system $X(s)$ on the line that we explain here briefly. Suppose we have $k+1$ particles on $\mathbb{Z}$, whose location at time $s \geq 0$ is $X_{k+1}(s) < \cdots < X_1(s)$. Each particle i has an exponential clock with parameter λ_i and when the clock rings the particle jumps to the right by 1. If the site to the right is occupied then one finds the longest string of adjacent particles starting from that location going to the right and the entire string is pushed by 1. The clocks are independent and have rates

$$\lambda_{k+1}(s) = \theta \frac{1+\sqrt{t}}{\sqrt{t}}, \text{ and } \lambda_i(s) = \theta \frac{\theta + X_{i+1}(s-) - X_i(s-) - 1}{X_{i+1}(s-) - X_i(s-)}.$$

When $\theta = 1$ the above becomes the well-known push-TASEP. The dynamic system $Q(s) = (Q_1(s), ..., Q_k(s))$ has the same law as $(X_1(s) - X_2(s) - 1, ..., X_k(s) - X_{k+1}(s) - 1)$. In this sense $Q(s)$ describes the evolution of the gaps between particles in $X(s)$, which is sometimes called the dual process.

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