The Image of Weyl Group Translations in Hecke Algebras

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Abstract

In this paper, the homomorphism $\varphi : \mathbb{C}[\tilde{B}_n] \to \mathbb{C}[B_n] \to H(S_n)$ is studied, where $\tilde{B}_n, B_n$ are the affine braid group and braid group of type A_{n-1} and $H(S_n)$ is the corresponding Hecke algebra. By interpreting elements of the affine Weyl group as elements of the affine braid group, we are able to describe a large commutative sub-algebra in $\mathbb{C}[\tilde{B}_n]$ consisting of translations in $\mathfrak{h}^*$. We investigate the image of this sub-algebra in $H(S_n)$ and show that it is the maximal commutative sub-algebra generated by the Jucys-Murphy elements. Furthermore, we touch on previous work that show that this result remains true when considering translations in $\mathbb{R}^n$, and that furthermore the standard basis maps onto the Jucys-Murphy elements directly.

1 Introduction

The purpose of this paper is to look at the representation theory and structure of the Hecke algebra of type A_n through the lens of the commutative sub-algebra generated by the Jucys-Murphy elements. In [1], Okounkov and Vershik describe a new method of studying the representation theory of the symmetric group, where the relationship between Young diagrams and irreducible representations of S_n is thoroughly motivated and comes up naturally in the description of the representations. More specifically, there is a basis for the irreducible representations of S_n called the GZ-basis, where each vector is uniquely specified by the action of the Jucys-Murphy elements $JM_1, \dots, JM_n$, for all of which these vectors are eigenvectors. Furthermore, the generated eigenvalues are exactly an order in which to construct a Young diagram of size n, where the eigenvalue is exactly the content of the box added. This leads to a natural description of a representation based on its corresponding Young diagram. As described by Isaev and Ogievetsky [3], the study of irreducible representations of Hecke algebras of type A_n can be carried out in the same manner, with some differences in the exact categorization.

The major result described in this paper appears in [5] and is the description of the image of the map $\phi' : B'_n \to H(W_n)$. More specifically, we describe the process by which one shows that

$$\phi'(T^{\epsilon_i}) = X_i$$

The paper is organized in the following manner. Section 2 offers an overview of the classification of irreducible representations of the symmetric group. Section 3 then focuses on Coxeter groups and describes such things as Weyl groups, braid groups, and Hecke algebras, as well as their affine counterparts. In particular, both their general relationship and the specific case of type A_n Coxeter groups is discussed. Finally, section 4 goes through two approaches to mapping translations in a Weyl group to the Hecke algebra $H(W_n)$ and describes how in both cases the Jucys-Murphy elements in the Hecke algebra arise in the image.

2 Irreducible Representations of the Symmetric Group

2.1 Jucys-Murphy Elements

We begin by recalling the representation theory of symmetric groups as derived in [1]. First, consider the chain of symmetric groups:

$$\{1\} = S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$$

This chain has the property of *simple branching*. That is, if V, W are irreducible representation of S_n, S_{n-1}, respectively, then the multiplicity of W in V is at most one when viewing V as a representation of S_{n-1}. This allows us to define a natural basis for the irreducible representations of S_n, called the *Gelfand-Tsetlin basis*, or *GZ-basis*. To get this basis, start with an irreducible representation V of S_n. Then, write $V = \oplus_i V_i$, where each V_i is an irreducible representation of S_{n-1}. Continuing this process yields a decomposition of V into the sum of irreducible representation of S_1, which are necessarily one-dimensional. The GZ-basis consists of a nonzero vector from each such subspace over all distinct irreducible representation of S_n.

Let $Z(n)$ denote the center of $\mathbb{C}[S_n]$. Then, we can define the *Gelfand-Tsetlin sub-algebra*, or *GZ-algebra* as

$$GZ(n) = \langle Z(1), Z(2), \ldots, Z(n) \rangle$$

By construction, this algebra is commutative. It is also the algebra of diagonal operators on the GZ-basis. To check this, note that one can write $\mathbb{C}[S_n] = \oplus_{V^\lambda}\text{End}(V^\lambda)$, where V^λ is indexed over all equivalence classes of irreducible representations of S_n. Then, given $v_T \in$ GZ-basis with $T = (\lambda_1, \ldots, \lambda_n)$, consider the product $P_T = P_{\lambda_1} \ldots P_{\lambda_n}$, where

Now, it is useful to define the *Jucys-Murphy elements* $JM_n \in \mathbb{C}[S_n]$ as follows:

$$JM_1 = 0$$
$$JM_i = (1\, i) + (2\, i) + \cdots + (i-1\, i) \text{ for } i \geq 2$$

Presented below are several facts about these elements:

1. $JM_n \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$
2. $GZ(n) = \langle JM_1, \ldots, JM_n \rangle$

The second fact is of particular importance in classifying irreducible representations of S_n. Given an element v_T of the GZ-basis, each JM_i will be act by a scalar multiple, since it is necessarily a diagonal operator. Furthermore, because $GZ(n)$ contains projectors onto each $v_{T'}$, the actions of the JM_i on v_T is enough to uniquely define it, motivating the following definition.

Definition 1. Given some v_T in the GZ-basis, define its *weight* as a vector $\alpha(v_T) = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that

$$JM_i v_T = a_i v_T$$

Furthermore, we can also write the spectrum of the JM basis

$$\text{Spec}(n) = \{\alpha(v) : v \text{ is in the GZ-basis}\}$$

Because the weight of a vector uniquely defines it up to a constant multiple, we can also write v_α as the basis vector corresponding to $\alpha \in \text{Spec}(n)$. Furthermore, we can define a further equivalence relation $\sim$ on $\text{Spec}(n)$ by writing

$$\alpha \sim \beta, \alpha, \beta \in \text{Spec}(n)$$

whenever v_α and v_β are basis elements of the same irreducible representation of S_n, up to isomorphism. From here, there are two quickly evident facts about $\text{Spec}(n)$ and

$$|\text{Spec}(n)| = \#\text{ of GZ-basis vectors}$$
$$|\text{Spec}(n)/\sim| = \#\text{ of irreducible representations of } S_n\text{, up to isomorphism}$$

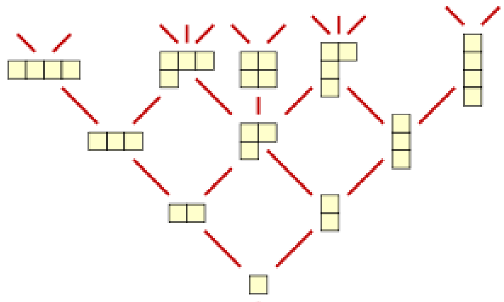

Figure 1: Young's Lattice

2.2 Young tableaus

Before continuing with the description of irreducible representations of S_n, we first describe Young tableaus in more detail. Given a partition $n = a_1 + \ldots, +a_k$, its corresponding tableau is a visualization of this partition and contains k rows of blocks, where the i-th row contains a_i blocks. Given a Young tableau with n total blocks, it is possible to construct a tableau with $n + 1$ blocks by adding a block anywhere where both its top and left side are adjacent to another block or is in the first row or column, respectively. Conversely, given a tableau with $n + 1$ blocks, it is possible to make a tableau with n simply by removing any block which has nothing to its right or bottom. All together, this process of removing and adding blocks creates a lattice, where the root is just a single box and two tableaus are connected if the removal of addition of one block is enough to get from one to the other. This lattice is called Young's lattice and is shown above.

Each block in a Young tableau also has a *content* which is equal to the difference between its x-coordinate and y-coordinate. The figure below shows the contents of different blocks in a Young tableau.

0	1	2	3	4
-1	0	1		
-2	-1	0		
-3	-2			
-4				

Figure 2: Contents of Boxes

2.3 Classifications of the Irreducible Representations

Definition 2. Given a natural number $n \in \mathbb{N}$, define the set of *content vectors* of length n as vectors

$$\alpha = (a_1, \dots, a_n) \in \mathrm{Cont}(n)$$

such that α satisfies the following conditions

1. $a_1 = 0$
2. For $q > 1$ $\{a_q \pm 1\} \cap \{a_1, \dots, a_{q-1}\} \neq \emptyset$
3. If $a_p = a_q$ for some $p < q$, then $\{a_p \pm 1\} \subset \{a_{p+1}, \dots, a_{q-1}\}$

As described in [1], each content vector corresponds to the process of building up a Young tableau block by block. Each a_i describes the content of the next box to be added, and the three rules for content vectors are exactly those that guarantee the construction of a valid Young tableau. With this motivation in mind, we can now define an equivalence relation $\approx$ on $\mathrm{Cont}(n)$, where

$$\alpha \approx \beta, \alpha, \beta \in \mathrm{Cont}(n)$$

whenever the Young tableau constructed through α and β is the same. What follows is the main result from [1].

Theorem 1. *Spec*$(n) =$ *Cont*(n) *and* $\alpha \sim \beta \Longleftrightarrow \alpha \approx \beta$.

As a result of Theorem 1, all irreducible representations of S_n correspond to some Young tableau T and have dimension equal to the number of ways to build up T one block at a time. Additionally, the branching graph of the symmetric group is exactly Young's lattice. Finally, by examining the action of the Coxeter generators $(i\ i+1)$ and their action on the basis, one can exactly describe the action of S_n on all these irreducible representations.

3 Coxeter Groups and Root Systems

3.1 Weyl group

We begin this section with a discussion of Weyl groups, a specific type of Coxeter group. Given a root system Φ, there is a corresponding group generated by reflections through the hyperplanes perpendicular to the vectors in Φ. These form a group W called the *Weyl group*. By limiting our work to the finite case, the properties of root systems guarantee that W will be a Coxeter group with generators s_α for $\alpha \in \Delta$, where Δ is some simple subsystem of Φ. Accordingly, one can also define a length function $l : W \to \mathbb{N}$ such that for $w \in W$, $l(w)$ is the length of the shortest expression of w in terms of the s_α generators.

In this paper, we are concerned solely with Weyl groups of type A_n, an example of which is worked out below.

First, let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of $\mathbb{R}^n$ and let $(\, , \,)$ be the standard inner product. For now, we will restrict ourselves to looking at $\mathfrak{h}^* = \{(a_1, \ldots, a_n) : a_1 + \cdots + a_n = 0\}$. This space contains and is spanned by the root system $\Phi = \{\epsilon_i - \epsilon_j\}_{i \neq j}$. For each $\alpha \in h^*$, there is a corresponding reflection s_α through the hyperplane perpendicular to α. In fact, $s_\alpha(x) = x - \frac{2(\alpha,x)}{(\alpha,\alpha)}\alpha = x - (\alpha, x)\alpha$. Now, for $i = 1, \ldots, n$, define $\alpha_i = \epsilon_{i+1} - \epsilon_i$. These α_i form a simple subsystem $\Delta \subset \Phi$. Let $s_i = s_{\alpha_i}$ for $i = 1, \ldots, n-1$. From this, it follows that the group of reflections generated by the s_α for $\alpha \in \Phi$ can also be expressed as a Coxeter group with generators and relations as follows:

$$W_n = \left\langle s_1, \ldots, s_{n-1} \middle| s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1 \text{ where } m_{i,j} = \begin{array}{ll} 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{array} \right\rangle$$

Notice that, in this case, $W_n \equiv S_n$, where the s_i are the standard generators $(i \; i+1)$. Furthermore, the action of W_n on $\mathfrak{h}^*$ is exactly the standard action of S_n on $\mathbb{R}^n$.

Given $\alpha \in \mathfrak{h}^*$, we can also define $T(\alpha)$, a translation by α in $\mathfrak{h}^*$. Then, we can define the *affine Weyl group* $\tilde{W}_n$ as in [6]

$$\tilde{W}_n = \langle W, \{T(\alpha)\}_{\alpha \in \mathfrak{h}^*} \rangle$$

We can give another presentation of $\tilde{W}_n$ as a Coxeter group. Define $\alpha_0 = \sum_{i=1}^n \alpha_i$ and $H_{\alpha_0,1} = \{x \in \mathfrak{h}^* | (x, \alpha_0) = 1\}$. Then, let s_0 be the reflection through the plane $H_{\alpha_0,1}$. Then, $\tilde{W}_n$ is of the form

$$W_n = \left\langle s_0, s_1, \ldots, s_{n-1} \middle| s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1 \text{ where } m_{i,j} = \begin{array}{ll} 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{array} \right\rangle$$

Here, the $|i-j|$ is interpreted modulo n, meaning that $s_0 s_{n-1} s_0 = s_{n-1} s_0 s_{n-1}$. So, we are thus able to define the Coxeter group and affine Coxeter group of type $\tilde{A}_n$.

3.2 Braid Group

Given a Coxeter group W, there is also a corresponding *braid group*, defined as follows:

$$B = \langle T_v | v \in W, T_v T_w = T_{vw} \text{ if } l(v) + l(w) = l(vw) \rangle$$

Note that, while the Weyl groups we consider are finite groups, the braid group will not be finite. In fact, a nontrivial element cannot have finite order, as any cancelling that ultimately results in the identity will need to decrease the length of elements, at which point one cannot combine terms in the braid group.

As the braid group is currently defined, there is no well-defined map from W to B, as there are multiple ways to write an element $x \in W$ that would not be interpreted as the same element in B. For example, $x = xs_\alpha^2$ for any $\alpha \in \Phi$, but $T_{s_\alpha}^2 \neq 1$.To interpret x, it is necessary to write x as a product of Coxeter generators of minimum length $x = s_{a_1} \ldots s_{a_k}$, and then to write each generator as the corresponding element of the braid group. In this example, $x = T_{s_{a_1} \ldots s_{a_k}}$. While this is a well-defined map $W \to B$, it is in general not a homomorphism and so little can be said about the map directly.

While the Weyl group can be pictured as a reflection group, it is easiest to think of the braid group as the group of different ways to “braid" n strands under composition. In this visualization, T_{s_i} is the result of pulling the i-th strand over the $i+1$-st strand, as seen below.

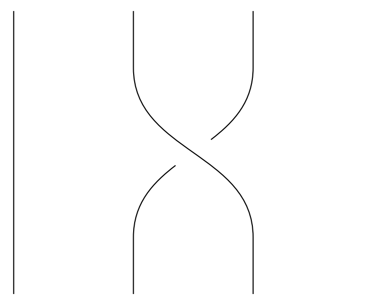

Figure 3: A visualization of s_2

We can apply this to W_n to get the braid group B_n. It has a presentation very similar to that of W_n, though things are slightly different because none of the T_{s_i} have finite order.

$$B_n = \langle s_1, \ldots, s_{n-1} | s_i s_j = s_j s_i \text{ if } i \neq j \pm 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i = 1, \ldots, n-2 \rangle$$

We can also define the affine braid group $\tilde{B}_n$ by adding a generator s_0 and taking the relations modulo n.

$$B_n = \langle s_0, s_1, \ldots, s_{n-1} | s_i s_j = s_j s_i \text{ if } i \neq j \pm 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i = 1, \ldots, n-1 \rangle$$

Similarly to $s_1, \dots, s_n$, it is possible to interpret s_0 in the "braided strand" interpretation by picturing the $n+1$ strands laid out in a circle instead of a line, and then treating s_0 as overlaying the last strand over the first. Furthermore, as in [2], there is a map $\phi : \tilde{B}_{n+1} \to B_{n+1}$ acting by the following rules

$$\begin{aligned} \phi : \quad & s_i \mapsto s_i \text{ for } i > 0 \\ & s_0 \mapsto s_n s_{n-1} \dots s_2 s_1 s_2^{-1} \dots s_{n-1}^{-1} s_n^{-1} \end{aligned}$$

To visualize this, consider the case when $n = 3$

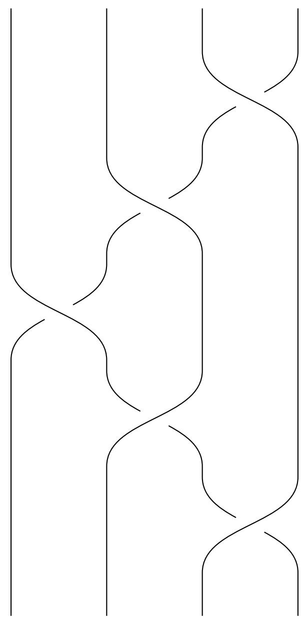

Figure 4: The braid corresponding to $\phi(s_0) = s_3 s_2 s_1 s_2^{-1} s_3^{-1}$ in B_3.

This map interprets a braid between the first and last strand through a combination of the s_i as opposed to the use of s_0. It is easy to check that this map is a group homomorphism.

3.3 Hecke Algebra

Given a Weyl group W, it is also possible to define its *Hecke algebra* in a similar manner to the braid group.

$$H(W) = \langle T_v | v \in W, T_v T_w = T_{vw} \text{ if } l(v) + l(w) = l(vw) \rangle, T_v^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) T_v + 1$$

Here, q is a formal variable. The condition that allows for simplifying these elements can be restated as $(T_v - q^{\frac{1}{2}})(T_v + q^{-\frac{1}{2}}) = 0$. This is a generalization of the Weyl group, as if $q = 1$ is set, then the original Weyl group comes out again.

As supported by the similarity between the Hecke algebra and the braid group, there is a good presentation of $H(W_n)$ that begins with the braid group B_n. In fact, $H(W_n)$ is simply the quotient of B_n by the relations $s_i^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})s_i + 1$. Consequently, we can define a homomorphism $i : B_n \to H(W_n)$ which simply sends an element to itself. From this, we can now consider the map

$$\varphi = i \circ \phi : \tilde{B}_{n+1} \to H(W_n)$$

Just as $H(W_n)$ is in some sense a generalization of the symmetric group, its representation theory also reflects this fact. As worked out in [3], the representation theory of this Hecke algebra can be developed in the same way as in Section 2, by setting up an inductive chain of algebras and defining a correspondence between irreducible representations and Young diagrams. Consequently, $H(W_n)$ also has a notion of a Jucys-Murphy element X_i. They are defined as follows

$$X_1 = 1, X_{i+1} = s_i X_i s_i \text{ for } i \geq 1$$

For $i > 1$, one can then write $X_i = s_{i-1} \ldots s_2 s_1^2 s_2 \ldots s_{i-1}$. The relations in the Hecke algebra can then be used to simplify this and get

$$X_i = 1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{k=1}^{i-1} s_k \ldots s_{i-1} \ldots s_k$$

Taking the classical limit $\frac{X_i - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ and specializing to $q = 1$, we get

$$\frac{X_i - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \sum_{k=1}^{i-1} s_k \ldots s_{i-1} \ldots s_k = JM_i$$

In this way, the Jucys Murphy elements of the symmetric group and the Hecke algebra are interconnected. Additionally, from this it is also clear that these X_i generate a large commutative sub-algebra of $H(S_{n+1})$. We now attempt to understand this sub-algebra from the perspective of the affine Weyl and braid groups.

4 Images of Translations in the Weyl Group

4.1 Translations in $\tilde{W}_n$

$\tilde{W}_n$ contains a large commutative subgroup $W_T \subset \tilde{W}_n$ of translations $T(\alpha)$ for $\alpha \in \mathfrak{h}^*$. However, as mentioned previously, writing these as elements of $\tilde{B}_n$ does not preserve the structure of this subgroup. In fact, the resulting elements do not always commute! However, as discussed by Lusztig in [4], this can be remedied by introducing the notion of the positive cone $P^+ \subset \mathfrak{h}^*$, defined as follows

$$P^+ = \{x \in \mathfrak{h}^* | (x, \alpha_i) \geq 0, i = 1, \dots, n-1\}$$

Lusztig then goes on to show that if $\alpha, \beta \in P^+$, then $T(\alpha)T(\beta) = T(\alpha + \beta)$ inside $\tilde{B}_n$. That is, all positive translations do continue to commute. As such, he then defines a map $\Theta : W_T \to \tilde{B}_n$ defined as show below.

$$\Theta(x) = T(\alpha)T(\beta)^{-1}, \alpha, \beta \in P^+, x = \alpha - \beta, T(\alpha), T(\beta) \in \tilde{B}_n$$

This map, unlike the previous map, is in fact a homomorphism from W_T to $\tilde{B}_n$. This then allows us to examine the image of this sub-algebra in $H(W_n)$.

Additionally, Lusztig also shows in his paper that given $\alpha \in P^+$ and by writing $W\alpha$ as the W_n-orbit of α,

$$\sum_{\beta \in W\alpha} \Theta(\beta) \in Z(H(W_n))$$

That is, the center of the Hecke algebra contains the sum over a W_n-orbit of any element in the positive cone. Once it is shown that the Jucys-Murphy elements are in the range of these translations, this will also mean that symmetric polynomials in the X_i will generate the center of the Hecke algebra.

Example 1. Here, we consider the case when $n = 4$.

α	$T(\alpha)$	$\varphi(T(\alpha))$
$\alpha_1 + \alpha_2 + \alpha_3$	$s_0 s_3 s_2 s_1 s_2 s_3$	X_4
$\alpha_1 + 2\alpha_2 + \alpha_3$	$(s_0 s_3 s_1 s_2)^2$	$X_4 + X_3 - X_2$
$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$(s_0 s_1 s_2 s_3 s_2)^2$	$2X_4 - X_2$
$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$(s_0 s_1 s_2 s_3)^3$	$3X_4 - X_3 - X_2$
$2\alpha_1 + 2\alpha_2 + \alpha_3$	$(s_0 s_3 s_2 s_1 s_2)^2$	$X_4 + X_3$
$3\alpha_1 + 2\alpha_2 + \alpha_3$	$(s_0 s_3 s_2 s_1)^3$	$X_4 + X_3 + X_2$

α	$\Theta(\alpha)$
α_1	X_2
$\alpha_1 + \alpha_2$	X_3
$\alpha_1 + \alpha_2 + \alpha_3$	X_4

In this example, it is apparent that, in fact, the image of these translations under the Θ map consists of linear combinations of the Jucys-Murphy elements in $H(W_4)$. Furthermore, this appears to signal that $\Theta(\sum_{i=1}^{k-1} \alpha_i) = X_k$. The next section will approach this problem from a different direction, but will come onto the same conclusion.

4.2 Translations in $\mathbb{R}^n$

Up to now, we have worked primarily with Coxeter groups and related constructions. However, there is a different approach investigated by Ram and Ramagge in [5] that begins by considering $\mathbb{R}^n$ instead of $\mathfrak{h}^*$. While this is no longer a Coxeter group, it is still similar enough that the constructions for the affine braid group and the Hecke algebra still apply. When viewed from this perspective, the map from the affine braid group to the Hecke algebra becomes much simpler.

First, let W_n' be the group generated by the s_i from before and by another element T^{ϵ_1}, corresponding to a translation by ϵ_1. This then allows us to define the corresponding braid group B_n' with presentation as follows.

$$\tilde{B}_n' = \begin{array}{ll} \langle s_1, \ldots, s_{n-1}, X^{\epsilon_1} & |s_i s_j = s_j s_i \text{ if } i \neq j \pm 1 \\ & |s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i = 1, \ldots, n-2 \\ & |T^{\epsilon_1} s_2 T^{\epsilon_1} s_2 = s_2 T^{\epsilon_1} s_2 T^{\epsilon_1} \\ & |T^{\epsilon_1} s_i = s_i T^{\epsilon_1} \text{ for } i = 3, \ldots, n-1 \qquad \rangle \end{array}$$

Now, we can define $T^{\epsilon_i} \in B_n'$ as

$$T^{\epsilon_i} s_i T^{\epsilon_{i-1}} s_i$$

These correspond exactly to the translations by ϵ_i in W_n'. Now, we can now extend this to a map $\phi' : \tilde{B}_n' \to H(W_n)$ by mapping the s_i to themselves and X^{ϵ_1} to 1. Now, to determine the image of the translations, we need only consider the X^{ϵ_i}, as they form a basis for $\mathbb{R}^n$. But it is easy to see that

$$\phi'(\epsilon_i) = s_{i-1} s_{i-2} \ldots s_1^2 s_2 \ldots s_{i-1} = X_i$$

So, the translations by the standard basis correspond exactly to the Jucys-Murphy elements, as was conjectured. Additionally, this also lines up with the work from the translations in $\mathfrak{h}^*$, as

$$\sum_{k=1}^{i-1} \alpha_i = \epsilon_i - \epsilon_1$$

However, after applying the map into the Hecke algebra, the ϵ_1 becomes trivial, indicating that in fact, $\phi(T(\sum_{k=1}^{i-1} \alpha_i)) = X_i$.

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