THE LOWER CENTRAL SERIES QUOTIENTS OF A FREE ASSOCIATIVE SUPERALGEBRA

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MENTOR: DAVID JORDAN

A *superalgebra* is an algebra along with a $(\mathbb{Z}/2\mathbb{Z})$-grading. Any associative superalgebra is a Lie superalgebra with bracket $\{a, b\} = ab - (-1)^{|a||b|}ba$. Let $A_{m,n} = \mathbb{C}\langle x_1, \ldots, x_m, y_1, \ldots, y_n\rangle$ be the free superalgebra over $\mathbb{C}$ with m even generators and n odd generators. We can consider its lower central series $L_1 \supseteq L_2 \supseteq \cdots$, defined as $L_1 = A_{m,n}$ and $L_{i+1} = \{A_{m,n}, L_i\}$. We can also consider the successive lower central series quotients defined as $B_i(A_{m,n}) = L_i/L_{i+1}$. Let $\Omega(V)$ denote the space of polynomial differential forms on the vector space V. We prove the following generalisation of a theorem by Feigin and Shoikhet:

Theorem 1. *Let $K_{m,n}$ be the space $A_{m,n}\{A_{m,n}, \{A_{m,n}, A_{m,n}\}\}A_{m,n}$. Then $A_{m,n}/K_{m,n}$ is isomorphic to $\Omega^{ev}(\mathbb{C}^{m|n})$ and $B_2(A_{m,n})$ is isomorphic to $\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$. The space $\overline{B_1}(A_{m,n})$, defined as the quotient of $B_1(A_{m,n})$ by the image of $K_{m,n}$ in $B_1(A_{m,n})$, is isomorphic to the quotient space $\Omega^{ev}(\mathbb{C}^{m|n})/\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$.*

Using the previous theorem, we derive the following formula for the Hilbert series of $B_2(A_{m,n})$ for all m, n.

Theorem 2. *The Hilbert series of $B_2(A_{m,n})$ is given by the following formula:*

$$h(B_2(A_{m,n})) = \frac{1}{4} \cdot \prod_{i=1}^{m} \frac{(1+u_i)}{(1-u_i)} \cdot \prod_{j=1}^{n} \frac{(1+v_j)}{(1-v_j)} - \frac{1}{4} - \sum_{i=1}^{m} \frac{u_i}{2(1-u_i)} - \sum_{j=1}^{n} \frac{v_j}{2(1+v_j)}.$$

Let W_n denote the Lie algebra of polynomial differential forms in n variables. Feigin and Shoikhet show that the spaces $B_k(A_{n,0})$ are W_n-modules for $k \geq 3$. Moreover, Dobrovolska and Etingof show that each $B_k(A_{n,0})$ has a finite-length Jordan-Hölder series for $k \geq 3$. The successive quotients are irreducible W_n-modules of the form $\mathcal{F}_\lambda$, where λ is a Young diagram. Building on a previous result of Arbesfeld and Jordan, we prove a bound on the number of squares in λ that do not occur in the first column of λ.

Theorem 3. *Let $k \geq 3$.*

(1) *For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_k(A_{n,0})$, we have*
$$|\bar{\lambda}| \leq 4k - 9,$$
(2) *Let n be 2 or 3. For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_k(A_{n,0})$, we have*
$$|\bar{\lambda}| \leq 2k - 5.$$

As a corollary, we can give the complete structure of $B_3(A_{m,n})$, generalising Arbesfeld and Jordan's result about the structure of $B_3(A_{n,0})$.

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HURWITZ EQUIVALENCE IN THE DIHEDRAL GROUP

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Let G be a group and G^n be the cartesian product of G with itself n times. The braid group B_n acts on G^n by Hurwitz moves. The i^{th} Hurwitz move σ_i is defined below. Let $T = (a_1, a_2, ..., a_n)$ with $a_i \in G$, then

$$\sigma_i T = (a_1, ..., a_i a_{i+1} a_i^{-1}, a_i, ..., a_n)$$

Theorem 1. The following properties of T are invariant under the action of Hurwitz moves.

(1) The product of the entries: $a_1 a_2 ... a_n$.
(2) The subgroup generated by the entries a_1 through a_n.
(3) The number of times elements from a given conjugacy class of G appear in T.

Because of this invariance, the orbit of any T consists only of tuples where (1), (2), and (3) are fixed. These properties therefore give rise to necessary conditions for Hurwitz equivalence.

Our main result involves determining the orbits under this action when G is a dihedral group and is stated below.

Main Theorem. When a tuple $T \in G^n$ consists only of reflections and G is a dihedral group, the necessary conditions mentioned in Theorem 1 serve as sufficient conditions for Hurwitz equivalence.

Our study of Hurwitz equivalence in the dihedral group was inspired by the paper [1], which gives a simple criterion for Hurwitz equivalence in the symmetric group analogous to our Main Theorem (with the added condition that the product of the entries is the identity). That paper studies tuples of transpositions in the symmetric group, which is the reason that we have chosen to restrict to reflections in the dihedral group. (Recall that the symmetric group $\mathfrak{S}_m$ acts on $\mathbb{R}^{m-1}$ in such a way that every transposition acts by a Euclidean reflection.)

References

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[2] X Hou, *Hurwitz Equivalence in Tuples of Generalized Quaternion Groups and Dihedral Groups*, The Electric Journal of Combinatorics, 2008

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ON THE SMALLNESS OF LAUMON'S RESOLUTION OF DRINFELD'S COMPACTIFICATION

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MENTOR: QIAN LIN

Given a sequence $n_1, \cdots, n_k > 0$ of integers, let $\mathbb{B}^{(n.)}$ denote the partial flag variety such that the i-th flag has dimension $n_1 + \cdots + n_i$. Let $Q_\alpha^{(n.)} = Map^\alpha(\mathbb{P}^1, \mathbb{B}^{(n.)})$ be the space of degree α maps from the projective line into the partial flag variety. There are many compactifications of this space. For instance, one can consider Drinfeld's space $D_\alpha^{(n.)}$ of quasimaps or Laumon's space $L_\alpha^{(n.)}$ of quasiflags. There is a natural surjective map $\pi : L_\alpha^{(n.)} \to D_\alpha^{(n.)}$, which turns out to be a resolution of singularities.

A. Kuznetsov proved that the map π is small, in the case of complete flag variety. In this paper, we find a suficient condition for the smallness of the natural resolution. Namely:

Theorem 1. Let $n_1, \cdots, n_k > 0$. The natural map $\pi : L_\alpha^{(n.)} \to D_\alpha^{(n.)}$ is small if

$$n_1 \leq n_2 \leq \cdots \leq n_k$$

Our result generalizes Kuznetsov theorem (which corresponds to the case $n. = (1, 1, \cdots, 1)$).

The proof of 1 is based on the stratification 2, and the computations 3,4,5.

Proposition 2. There is a finite stratification

$$D_\alpha = \coprod \mathfrak{D}_\beta^\Gamma$$

such that

$$\mathfrak{D}_\beta^\Gamma = Q_\beta \times (\mathbb{P}_1)_\Gamma^{\alpha-\beta}$$

where $(\mathbb{P}_1)_\Gamma^{\alpha-\beta}$ denotes the set of "colored divisors" (informally, a colored divisor is a collection of divisors). This allows us to identify points in Drinfeld's compactification with pairs (ϕ, D), where $\phi \in Q_\beta$ is a map and D is a colored divisor.

Proposition 3. *The codimension formula.* Let $\beta = (\beta_1, \cdots, \beta_{k-1}) \leq \alpha$, and m be the number of points in the support of the colored divisors in $(\mathbb{P}^1)_\Gamma^{\alpha-\beta}$. Then

$$\text{codim } \mathfrak{D}_\Gamma^\beta = \sum_{i=1}^{k-1} (\alpha_i - \beta_i)(n_i + n_{i+1}) - m$$

Proposition 4. *The product formula.* If $D = \gamma_1 x_1 + \cdots + \gamma_m x_m$ for distinct points $x_i \in \mathbb{P}^1$, then

$$\pi^{-1}(\phi, D) = \prod_i \pi^{-1}(\phi, \gamma_i x_i)$$

Theorem 5. *The simple fiber dimension.* If $\gamma = (d_1, d_2, \cdots, d_{k-1})$, then

$$\dim(\pi^{-1}(\phi, \gamma x)) = \sum_{i=1}^{k} (n_i - 1) d_i + \sum_{i=1}^{k-1} min(d_i, d_{i+1}) \leq \sum_{i=1}^{k} n_i d_i - 1$$

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ON THE CRITICAL GROUP OF LINE GRAPHS AND PLANAR GRAPHS

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MENTOR: HODA BIDKHORI

Let $G = (V, E)$ be a directed graph, where each edge $e \in E$ is directed from its source $s(e)$ to its target $t(e)$. The line graph $\mathcal{L}G$ of G is a graph with vertex set E, and with an edge from e to f if $t(e) = s(f)$. Let $\mathbb{Z}^V$ be the free abelian group generated by the vertices of G. We define $\Delta_v \in \mathbb{Z}^V$ for $v \in V$ as

$$\Delta_v = \sum_{e \in E,\ s(e)=v} (t(e) - v)$$

The sandpile group $K(G, r)$ with sink r is the quotient group

$$K(G, r) = \mathbb{Z}^V / (r, \Delta_v | v \in V)$$

The order of $K(G, r)$ equals the number of oriented spanning trees of G rooted at r. If G is a strongly-connected Eulerian directed graph, then the sandpile groups $K(G, r)$ are isomorphic for all vertices r. We call this group the critical group $K(G)$.

Our first result is a bijective proof of the following theorem.

Theorem 1. Let $G = (V, E)$ be a directed graph in which every vertex has indegree greater than 0. We denote the set of rooted spannings trees of a graph H by $\kappa(H)$. Then

$$\sum_{T \in \kappa(\mathcal{L}G)} \prod_{(e,f) \in T} x_f = \sum_{T' \in \kappa(G)} \prod_{e \in T'} x_e \left(\sum_{s(e)=v} x_e \right)^{\text{indeg}(v)-1}$$

As a result of this bijection, we give a bijective proof that the number of binary De Bruijn sequences of order n is $2^{2^{n-1}}$.

The Kautz graphs $\text{Kautz}_m(n)$ and the De Bruijn graphs $DB_n(m)$ are families of graphs which are important in network design. In a recent paper, Levine determined the critical groups of $DB_n(2)$ and $\text{Kautz}_n(m)$, where m is prime and $n \in \mathbb{N}$. We generalize these results, giving a complete description of the critical groups of all Kautz and De Bruijn graphs.

We also find the critical group of triangular graphs, the line graphs of the complete undirected graphs. After this work was completed, we found that this result appeared in a recent paper by Berget, Reiner, et. al.

Finally, we bound the number of generators of the critical group of a planar graph.

Theorem 2. Let $G = (V, E)$ be a connected simple planar graph. Suppose the finite dual of some embedding of G is a tree with k leaves (and possibly $m > 0$ multiedges). Then $K(G)$ can be generated by $\min(k + m - 1, 1)$ elements.

Let G be a planar graph such that the finite dual of G is a path. As an immediate corollary of theorem 2, $K(G)$ is cyclic. Our proof also shows how to compute the order of $K(G)$.

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An Exotic Map on Extended Dynkin Diagrams

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January 29, 2010

Let G be a connected and simply connected almost simple complex Lie group, and $\mathfrak{g}$ its Lie algebra. In 1969, Kac gives a classification of automorphisms of finite order of G. Gorbatsevich, Onishchik, and Vinberg points out that Kac's classification leads to a classification of elements of finite order in G. Based on these results, Drinfeld constructs in his note a bijection between $B(G)$, the set of vertices in the extended Dynkin diagram of $\mathfrak{g}$, and $A(G)$, the set of conjugacy classes of elements g such that the centralizer of g is semisimple:

$$\phi(\alpha_j) = \text{conjugacy class of } \exp_G(2\pi i \frac{1}{m_j}\check{w}_j)(j \geq 1),$$
$$\phi(\alpha_0) = 1$$

where $\alpha_0, \ldots, \alpha_n$ are regarded as the vertices of the extended Dynkin diagram corresponding to the simple roots, and $\check{w}_1, \ldots, \check{w}_n$ are the fundamental coweights. The positive integers $m_1, \ldots, m_n$ are the coefficients of the equation $\alpha_0 = \sum_{j=1}^{n} m_j \alpha_j$.

In his note, Drinfeld asks the following question: consider the map $f_p\colon A(G) \to A(G)$ defined by $f_p(g) = g^p$. What is the corresponding map $f'_p\colon B(G) \to B(G)$ defined by $f'_p = \phi^{-1} \circ f_p \circ \phi$ looks like? We solve Drinfeld's problem in this project. For each type of simple complex Lie algebra $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$, we explicitly compute the map f'_p.

For types A, C, E, F, G, we solve the problem by constructing a Lie group of matrices that is isomorphic to G. In this process, we prove some criterions which may be useful for determining whether a connected almost simple complex Lie group is simply connected:

Proposition. Let $\pi\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ be an imbedding, and $\mathfrak{l} = \pi(\mathfrak{g})$. Let L be the set of matrices of the form $e^{X_1} e^{X_2} \cdots e^{X_m}$ where $X_1, \ldots, X_m$ are elements of $\mathfrak{l}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ that contains $\pi(\mathfrak{h})$. Let Λ and Λ' be the corresponding coroot lattices. Then L is a connected Lie group, and it is simply connected if and only if $\Lambda'/\pi(\Lambda)$ is free.

Proposition. Let H be a connected almost simple complex Lie group. Then H is simply connected if and only if $\exp_H(2\pi i \frac{1}{m_j}\check{w}_j) \neq 1$ for each $j \geq 1$.

For types B and D, we solve the problem by embedding G into C_m, the complex Clifford algebra, and compute the map in C_m.

CURVES IN TROPICAL GEOMETRY

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MENTOR: JAMES PASCALEFF

The tropical semiring is a semiring in which the numbers are the real numbers and negative infinity, adding two numbers gives the larger number, and multiplying them gives what would normally be considered their sum. So, in tropical geometry, 3+5=5, 3*5=8, 9+(7*6)=13,2*(3+4)=6. Tropical equations are considered as being satisfied when the two largests terms are equal. Tropical geometry is the geometry of these equations and their solutions, and a tropical variety is basically the set of points whose coordinates satisfy some set of these equations. Since any solution to a tropical equation is also a solution to one of a finite number of classical equations defined by it, all tropical varieties are made of some set of linear components. For instance, a typical tropical line in two dimensions is the union of a central point, all points directly to the left of it, all points directly below it, and all points directly up and to the right of it at a 45^o angle. Here, I define a stable surface through a set of points as essentially a surface that goes through them all and is similar to a surface that would go through them all if any of them moved. I establish some basic properties of stable surfaces, such as the fact that for any n+1 monomials, there is a unique stable surface defined by an equation that is the sum of multiples of these monomials going through any n points. I also give a general description of what the set of stable surfaces through a set of points is like, and catagorize the possible types the set of stable cubic surfaces through 8 points in two dimensions can have. Then I note how the complete set of tropical surfaces through a set of points can be expressed as a union of subsets of stable surfaces through subsets of the points. I also attempt to make progress towards catagorizing all possible tropical curves that can be expressed as the intersection of two tropical surfaces, using the dual subdivisions of surfaces. I show that any set of triangular prisms satisfying certain properties corresponds to an apparently valid combination of subdivisions, but was neither able to detirmine exactly what graphs correspond to such a set of triangular prisms nor show that the resulting subdivisions actually correspond to tropical surfaces. I do, however, give an example of a labeled curve segment that is not topologically equivalent to a segment of any tropical curve that can be expressed as the intersection of two surfaces of sufficiently high degree and show that not all reasonable graphs match the structure of some such curve. Finally, I show that any connected second-degree tropical curve in three dimensions lies on a tropical plane.

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Explicit Examples of Strebel Differentials

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Let X be some Riemann surface, and let ω be a meromorphic quadratic differential form on X, that is, ω can be written in local coordinates as $f(z)dz^2$, for some meromorphic function f. We say that a curve $\gamma : I \to X$ is part of a horizontal leaf of ω if for each $t \in I$, we have that $f \circ \gamma(t)(\gamma'(t))^2$ is real and positive. ω is said to be a Strebel differential if it's noncompact horizontal leaves form a set of measure 0.

Now, most quadratic differential forms on any given Riemann surface are not Strebel (in fact, the space of Strebel differentials on $\mathbb{P}^1$ as a subset of the space of all quadratic differentials on $\mathbb{P}^1$ is a countable union of real lines in $\mathbb{C}$). Because of this, this paper focuses on writing down explicit examples of Strebel differentials.

Most papers that discuss Strebel differentials seem to focus on those that have only second order poles, since they define a cellular decomposition on X. However, we construct an explicit family of examples of holomorphic Strebel differentials on any hyperelliptic curve branched over the points $0, 1, \infty$, and $\frac{1}{2} + ir$, for any real r.

Later, we look at an example from a previous paper, where ω was an algebraic Strebel differential such that ω had integral residues, on a Riemann surface X defined over $\overline{\mathbb{Q}}$ that had a transcendental period. We then modify this example to obtain another algebraic Strebel differential ω', defined on an algebraic Riemann surface X', all of whose periods are transcendental.

UNDERSTANDING SHRINKING CURVES ON A SPHERE

BRAYDEN WARE
MENTOR: HOSKULDUR HALLDORSSON

In this paper, we will summarize some of the results that occur for the curve shortening flow for curves smoothly immersed in $\mathbb{R}^2$, $\mathbb{R}^3$, and on a general surface. Then we will discuss related results for the curve shortening flow on the sphere. For the plane and on a surface, the remarkable case has always been that of embedded curves (an immersed curve is embedded if it is also a bijection). However, I will discuss some new features that occur for the flow on the sphere of non-embedded curves, and conjecture stronger results for the flow of general immersed curves on the sphere.

Let $\gamma[.,t]$ be a continuous family of immersed, C^1 curves in $\mathbb{R}^2$. We say that γ is a standard solution to the curve shortening flow (CSF) if

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{N},$$

where κ is the curvature of the curve and $\mathbf{N}$ is the inward pointing normal vector to the curve. We will only be interested in the problem of finding a solution $\gamma[u,t] : S^1 \times [0,T) \to \mathbb{R}^k$ such that $\gamma[.,0] = \gamma_0$, for some specific closed curve γ_0.

For each solution $\gamma[u,t]$ of the CSF in $\mathbb{R}^3$, the scaled solutions $\gamma'[u,t] = \lambda\gamma[u, \lambda^2 t]$ also satisfy the CSF. We can continuously rescale the solutions so that circles remain at a constant size by letting $\gamma'[u,t] = \lambda[t]\gamma[u,t']$ with $\lambda[t] = \frac{1}{\sqrt{2(T-t)}}$ and $\frac{dt'}{dt} = \lambda^2$, then we see that γ' satisfies the normalized curve shortening flow (NCSF)

$$\frac{\partial \gamma'}{\partial t} = \kappa \mathbf{N} + \gamma'.$$

It turns out that not only circles but any solution of the CSF lying on a sphere corresponds to a solution of the NCSF that lies on the unit sphere and moves by the intrinsic *geodesic curvature* of the curve on the sphere.

A *figure eight* in the plane or on the sphere is a curve with one double point and zero total curvature (or geodesic curvature). For a figure eight in the plane, Grayson showed that the two loops collapse with the difference in their areas remaining constant until one loop disappears and a cusp forms, or in the case that the loops have equal area, both disappear simultaneously. On a sphere, the Gauss-Bonnet theorem constrains any figure eight to have two loops of equal area, but the area of the loops may not decrease; the area of any loop that is part of a curve satisfies $\frac{dA}{dt} = A + \alpha - 2\pi$, where α is the exterior angle of the loop, $-\pi \leq \alpha < \pi$.

Consider the special case where the double point of the figure eight is fixed under the evolution of the curve, and the two loops (necessarily congruent in this case) are convex. We show that if $\alpha \geq 0$, these curves shrink to a point, whereas if $\alpha < 0$, the curves converge to a unique twice covered geodesic (in infinite time).

For curves with more fixed double points and no other self intersections, (a strong condition on the curve, requiring it to be highly symmetric), we show that the curves converge to unique twice covered geodesic always. We conjecture that in general, any curve flowing on the sphere for infinite time will eventually behave like one of the above solutions or like a solution with at least one fixed triple or higher degree point.

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ON (CO)HOMOLOGY OF 2-GROUPS

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Recall that the nerve $\mathcal{N}C$ of a category C is a simplicial set that is constructed from the sequences of composable morphisms and used to construct the classifying space BC. Since a group G can be viewed a category with one object, one can define (co)homology of a group as the (co)homology of its classifying space BG.

A 2-group is a 2-category with one object in which all 1-morphisms and 2-morphisms are invertible. A crossed module is a group homomorphism $\mu : H \to G$ together with an action of G on H such that the homomorphism and the action agree with conjugation in both groups in a certain way. We present a detailed proof of the equivalence between the concepts of a 2-group and that of a crossed module.

We consider two different ways to assign a nerve (and subsequently a classifying space) to a 2-group $\mathcal{G}$. The first one, called the geometric nerve $\mathcal{N}\mathcal{G}$, uses the definition of a 2-group as a 2-category and is similar to the construction of the nerve of a category. The second one, called the diagonal nerve $\mathfrak{N}\mathcal{G}$, uses the definition of a crossed module $H \xrightarrow{\mu} G$. The diagonal nerve $\mathfrak{N}\mathcal{G}$ is the diagonal of the binerve, a 2-dimensional lattice of simplices with vertical and horizontal boundary maps.

The two definitions of the nerve of a 2-group are equivalent in the sense that the classifying spaces resulting from the geometric and the diagonal nerves are homotopy equivalent.

The classifying spaces of crossed modules are exactly the 2-types - spaces with trivial homotopy groups in dimension $n > 2$. In particular, the crossed modules of the form $H \to 0$ are an algebraic model of the Eilenberg-Maclane spaces $K(H, 2)$. They correspond to the 2-groups with a single 1-morphism. In this case, the geometric nerve $\mathcal{N}(H \to 0)$ is much simpler than in the general case. Since there is only one 1-morphism, we can interpret the n-simpices of $\mathcal{N}(H \to 0)$ as certain kinds of commuting diagrams of 2-morphisms. We describe in detail the n-simplices and the boundary maps between them for $n \leq 4$ and use this information to calculate some low degree (co)homologies:

a) $H_2(H \to 0) \cong H$,
b) $H^2(H \to 0, M) \cong \mathrm{Hom}(H, M)$, where H and M are arbitrary abelian groups,
c) $H_3(\mathbb{Z}_2 \to 0) = 0$,
d) $H^3(\mathbb{Z}_2 \to 0, M) \cong M/2M$.