Bipartite and Euclidean Gallai-Ramsey Theory

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Abstract

In this paper, we investigate the following Gallai-Ramsey question: how large must a complete bipartite graph K_{n_1,n_2} be before any coloring of its edges with r colors contains either a monochromatic copy of $G = K_{s,t}$ or a rainbow copy of $H = K_{s,t}$? We establish upper bounds on n_1 and n_2 in the cases where $H = G = K_{2,t}$ or $K_{t,t}$. In both cases, the bound is linear in r. Furthermore, we also consider the following Euclidean Gallai-Ramsey question: given a configuration H in Euclidean space, what is the smallest n such that any r-coloring of n-dimensional Euclidean space contains a monochromatic or rainbow configuration congruent to H? Through a natural translation between edge colorings of the complete bipartite graph K_{n_1,n_2} and colorings of a subset of $(n_1 + n_2)$-dimensional Euclidean space, we prove new upper bounds on n for some simple configurations H, including simplicies, prisms, and rectangles.

1 Introduction

Given two graphs G and H and an integer r, we may ask the following question:

Question. *What is the least positive integer n such that any coloring of the edges of the complete graph K_n with r colors contains either a monochromatic copy of G (having all edges the same color) or a rainbow copy of H (having all edges different colors)?*

The answer to this question is denoted by the Gallai-Ramsey number $\text{GR}_r(H, G)$. This question is an extension of the corresponding Ramsey-type question, which asks for the minimum n such that any r-coloring of K_n contains a monochromatic copy of G; indeed, the existence of the Gallai-Ramsey number is guaranteed by the existence of the corresponding Ramsey number $R_r(G)$ [12].

Like Ramsey numbers, Gallai-Ramsey numbers are difficult to compute, and exact values are only known for a few classes of graphs. In the earliest work on this subject, Tibor Gallai studied graph colorings which contain no rainbow triangle [17]; as a result, much subsequent work has focused on the case where $H = T_3$ is a triangle. In this case, the Gallai-Ramsey number $\text{GR}_r(T_3, G)$ is known exactly for several small graphs G [5, 6, 12, 21, 24, 25, 26, 27, 29, 36]. Even when $\text{GR}_r(T_3, G)$ is not known exactly, its asymptotic behavior (as a function of r) is well understood [23]. We refer the reader to [16] for a dynamic survey of known Gallai-Ramsey results and discussion of related problems.

The landscape is different when H is not a triangle. Some work has been done on the case when H is some fixed small graph, such as a triangle with a pendant edge [15, 22], a path [32], or a star [1]. If H is allowed to be large, some bounds for $\text{GR}_r(H, G)$ are known when G is a complete graph and H is either a complete graph or a tree (of arbitrary size) [34].

In this paper, we investigate Gallai-Ramsey numbers of complete bipartite graphs. The Gallai-Ramsey number $\text{GR}_r(T_3, K_{s,t})$ is already well-studied [8, 22, 28, 35]; here, we focus on the symmetric case when

$H = G = K_{s,t}$. In addition, we examine edge colorings of the complete bipartite graph K_{n_1,n_2} rather than the complete graph K_n; this is because edge colorings of K_{n_1,n_2} translate in a natural way to colorings of subsets of $(n_1 + n_2)$-dimensional Euclidean space, as we shall explore in the next section. When $H = K_{1,p}$ and $G = K_{1,q}$ are both stars, we have the following result (which is also proven in [11]):

Theorem 1.1. *Let r be an integer, and let $n = (p - 1)(q - 1) + 1$. Then, every r-coloring of $K_{1,n}$ contains a rainbow copy of $K_{1,p}$ or a monochromatic copy of $K_{1,q}$.*

Our main contributions are the following theorems:

Theorem 1.2. *Let r be a positive integer, and $t \geq 2$. Let*

$$n_1 = (6(t-1) - 1)r + 2,$$
$$n_2 = 2(t-1)\binom{6(t-1)}{2} + 3(t-1)(r+1) + 1.$$

Then, any coloring of K_{n_1,n_2} with r colors contains a monochromatic $K_{2,t}$ or a rainbow $K_{2,t}$.

Theorem 1.3. *Let r and t be positive integers. Let $p = t^3 + 2t^2 + 2t + 1$, and let $n = (t-1)(p+1)(p+2)^t r$. Then, every r-coloring of $K_{n,n}$ contains a monochromatic $K_{t,t}$ or a rainbow $K_{t,t}$.*

Furthermore, we can ask similar questions when we consider n-dimensional Euclidean space $\mathbb{E}^n$ rather than graphs, and seek monochromatic congruent copies of configurations (i.e. finite sets of points) in the Euclidean space. We consider two configurations $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ to be congruent if there exists a bijection $\Phi : A \to B$ between these two sets such that for all $a, a' \in A$, the Euclidean distance $d(a, a')$ is equal to $d(\Phi(a), \Phi(a'))$.

As an Euclidean extension of classical Ramsey theory, Erdös, Graham, Montgomery, Rothschild, Spencer, and Straus [10] ask: for every number of colors r and configuration K, does there exists some n_0 such that every r-coloring of $\mathbb{E}^{n_0}$ contains a monochromatic configuration congruent to K? If every r-coloring of $\mathbb{E}^{n_0}$ does contain a monochromatic copy of K, we may write $\mathbb{E}^{n_0} \xrightarrow{r} K$. If such an n_0 exists for all r, we say that the configuration K is Ramsey.

The value of n_0 for some fixed small r and small configurations is known [3, 4, 7, 31, 33], and the question of which configurations are Ramsey is also well studied [13, 14]. Importantly, we note that every Ramsey set is spherical (i.e., can be embedded in an n-dimensional sphere for some n) [10]. Additionally, for two configurations $K_1 \in E^{n_1}$ and $K_2 \in E^{n_2}$, we define the Cartesian product $K_1 * K_2 \in E^{n_1+n_2}$ by

$$K_1 * K_2 = \{(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}) \mid (x_1, \ldots, x_{n_1}) \in K_1, (y_1, \ldots, y_{n_2}) \in K_2\}.$$

It is known that the product of any two Ramsey sets is itself Ramsey [13]. We refer the reader to the following surveys [18, 19, 20] for a more comprehensive understanding of known results and open problems in Euclidean Ramsey theory.

We can expand this line of questioning to include rainbow congruent copies of configurations as well. In fact, Mao, Ozeki, and Wang [30] recently introduced this topic as Euclidean Gallai-Ramsey theory, specifically asking the question:

Question. *For an integer r and configurations K and K', does there exist an integer n_0 such that for any r-coloring of the points of n-dimensional Euclidean space with $n \geq n_0$, there is a monochromatic configuration congruent to K or a rainbow configuration congruent to K'?*

Similar the the Euclidean Ramsey case, we write $\mathbb{E}^{n_0} \xrightarrow{r} (K', K)_{\text{GR}}$ if every r-coloring of E^{n_0} contains a monochromatic K or a rainbow K'. For diagonal results when $K = K'$, if such an n_0 exists for all such r, we say that K is Gallai-Ramsey. Initial work by Mao, Ozeki and Wang [30] shows some upper and lower bounds on n_0 for specific right and equilateral triangles as well as general rectangles. Further work by Cheng and Xu [9] establishes an upper bound on n_0 for any simplex with its minimum height less than its circumradius, as well as showing that the set of three colinear points with pairwise distances 1,1, and 2 is Gallai-Ramsey when restricted to only spherical colorings (such that points with the same magnitude are assigned the same color). Furthermore, they generalize bounds on n_0 for 30-60-90 right triangles presented by Mao, Ozeki, and Wang [30] to all right triangles and present some new bounds for various configurations.

In this paper, we focus on improving and finding new bounds on n_0 for the Cartesian products of regular simplices, including but not limited to triangles, prisms, and rectangles. Our Euclidean Gallai-Ramsey results are as follows:

Theorem 1.4. *For $p, q \in \mathbb{N}$ and $b \in \mathbb{R}$, let Δ_p be a p-dimensional regular simplex with side length b and Δ_q a q-dimensional regular simplex with side length b. Then for any $r \in \mathbb{N}$, $\mathbb{E}^{pq+2} \xrightarrow{r} (\Delta_p; \Delta_q)_{\text{GR}}$.*

Theorem 1.5. *For $t, r \in \mathbb{N}$ and $a, b \in \mathbb{R}$, let Q_t be the right prismatic polytope obtained by taking the Cartesian product of a line segment with length a and a $(t-1)$-dimensional regular simplex with side length b. For any dimension t and number of colors r, let $n_0(r,t) = r(9t-10) + 3(3t-2)(2t(2t-5)+7)$. Then, $\mathbb{E}^{n_0} \xrightarrow{r} (Q_t, Q_t)_{\text{GR}}$.*

Theorem 1.6. *For any $t, r \in \mathbb{N}$ and $a, b \in \mathbb{R}$, let Q_t be the Cartesian product of two $(t-1)$-dimensional regular simplices with side lengths a and b, let $p = t^3 + 2t^2 + 2t + 1$, and let $n = (t-1)(p+1)(p+2)^t r$. Then $\mathbb{E}^{2n} \xrightarrow{r} (Q_t, Q_t)_{\text{GR}}$.*

As a corollary to 1.5, we obtain bounds on n_0 for the following specific configurations:

Corollary 1.7. *Let Q be the equilateral triangular prism with height a and faces of side length b. For any number of colors r, $\mathbb{E}^{17r+273} \xrightarrow{r} (Q, Q)_{\text{GR}}$.*

Corollary 1.8. *Let Q be a rectangle with side lengths a and b. For any number of colors r, $\mathbb{E}^{8r+36} \xrightarrow{r} (Q, Q)_{\text{GR}}$.*

Previously, Mao, Ozeki, and Wang showed that $\mathbb{E}^{13r+4} \xrightarrow{r} (Q, Q)_{GR}$ when Q is a rectangle [30], so 1.8 is an improvement on this bound when $r \geq 7$.

The structure of the rest of this paper is split into three sections. In section 2, we present proofs of our bipartite graph Gallai-Ramsey results. In section 3, we utilize these bipartite graph Gallai-Ramsey results and prove our Euclidean Gallai-Ramsey results. Finally, we discuss some future work and interesting further problems in section 4.

2 Bipartite Gallai-Ramsey Numbers

In this section, we present proofs of our main results on bipartite Gallai-Ramsey numbers.

Proof of Theorem 1.1. [11] With $n = (p-1)(q-1)+1$, let $K_{1,n}$ have vertex set $U \sqcup V$ with $U = \{u\}$ and $V = \{v_1, \ldots, v_n\}$. If u is adjacent to at least p edges with distinct colors, then we have found a rainbow $K_{1,p}$. Otherwise, the edges adjacent to u have at most $p-1$ colors. By the pigeonhole principle, there must be at least q edges of the same color, and we have found a monochromatic $K_{1,q}$. $\square$

The proof of Theorem 1.2 involves a simple counting argument.

Proof of Theorem 1.2. With

$$n_1 = (6(t-1)-1)r + 2,$$
$$n_2 = 2(t-1)\binom{6(t-1)}{2} + 3(t-1)(r+1) + 1.$$

suppose for a contradiction that there exists an r-coloring of K_{n_1,n_2} with no monochromatic or rainbow $K_{2,t}$. Let the vertex set of K_{n_1,n_2} be $U \sqcup V$ with $U = \{u_1, \dots u_{n_1}\}, V = \{v_1, \dots, v_{n_2}\}$. We first have the following claim:

Claim 2.1. For any $1 \le i, j \le n_1$, at least one of u_i and u_j is adjacent to $\frac{n_2-(t-1)(r+1)}{4(t-1)}$ edges of the same color.

Proof. Fix $1 \le i, j \le n_1$. For each $1 \le k \le n_2$, consider the path $\pi_k = u_i v_k u_j$.

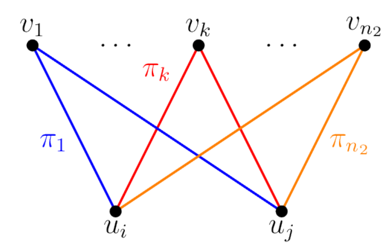


Figure 1: Visualization of the paths π_k

Among these n_2 paths, at most $(t-1)r$ of them can be monochromatic: if there were more than $(t-1)r$ monochromatic paths, then by the pigeonhole principle there would have to be t paths $\pi_{k_1}, \dots, \pi_{k_t}$ colored with the same color, but then $u_i, u_j, v_{k_1}, \dots, v_{k_t}$ would form a monochromatic $K_{2,t}$. Thus, at least $n_2 - (t-1)r$ of the paths π_k are rainbow.

Let $\mathcal{P}$ be a subset of the paths $\pi_1, \dots, \pi_{n_2}$ satisfying the following properties:

- Each $\pi_k \in \mathcal{P}$ is rainbow.
- For each pair of paths $\pi_i, \pi_j \in \mathcal{P}$, π_i and π_j share no colors.
- $\mathcal{P}$ is maximal: for each path $\pi_k \notin \mathcal{P}$, π_k shares a color with some path in $\mathcal{P}$.

The set $\mathcal{P}$ can be built greedily: as long as there's a rainbow path outside of $\mathcal{P}$ which shares no colors with the paths in $\mathcal{P}$, add it to $\mathcal{P}$. Furthermore, we claim that $|\mathcal{P}| \le t-1$: if $|\mathcal{P}|$ were at least t, then it would contain t paths $\pi_{k_1}, \dots, \pi_{k_t}$ such that no two paths share any colors; then, the vertices $u_i, u_j, v_{k_1}, \dots, v_{k_t}$ would form a rainbow $K_{2,t}$. Thus, we must have $|\mathcal{P}| \le t-1$. Since there are at least $n_2 - (t-1)r$ rainbow paths, there are at least $n_2 - (t-1)(r+1)$ rainbow paths outside of $\mathcal{P}$. See Figure 2.

The paths in $\mathcal{P}$ contain $2|\mathcal{P}|$ unique colors. Since each path outside of $\mathcal{P}$ shares a color with some path in $\mathcal{P}$, and there are at least $n_2-(t-1)(r+1)$ such paths, by the pigeonhole principle, there is some color which appears in at least $\frac{n_2-(t-1)(r+1)}{2|\mathcal{P}|}$ of these paths. Without loss of generality, suppose this color is red. Then, for each of the $\frac{n_2-(t-1)(r+1)}{2|\mathcal{P}|}$ paths containing a red edge, that edge is either connected to u_i or u_j; again by

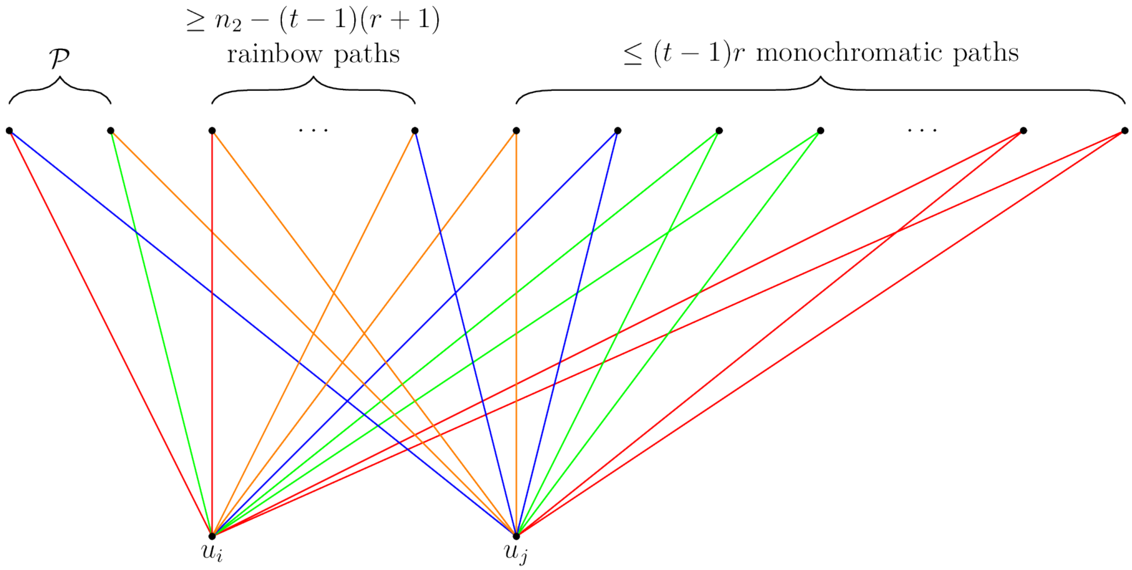


Figure 2: All the paths π_k adjacent to u_i and u_j: at most $(t-1)r$ of them are monochromatic, and at most $t-1$ of them belong to $\mathcal{P}$.

the pigeonhole principle, at least $\frac{1}{2} \cdot \frac{n-(t-1)(r+1)}{2|\mathcal{P}|}$ of these red edges are connected to the same vertex. Thus, either u_i or u_j has at least $\frac{n_2-(t-1)(r+1)}{4|\mathcal{P}|} \geq \frac{n_2-(t-1)(r+1)}{4(t-1)}$ red edges, as desired.

$\square$

This claim tells us that there is at most one vertex u_i which is not adjacent to $\frac{n_2-(t-1)(r+1)}{4(t-1)}$ edges of the same color, and so at least $n_1 - 1 = (6(t-1)-1)r + 1$ vertices u_i are adjacent to $\frac{n_2-(t-1)(r+1)}{4(t-1)}$ edges of the same color. By the pigeonhole principle, there is some color – say, red – such that at least $6(t-1)$ vertices u_i are adjacent to at least $\frac{n_2-(t-1)(r+1)}{4(t-1)}$ edges of that color. Without loss of generality, suppose that $u_1, \ldots, u_{6(t-1)}$ are each adjacent to at least $\frac{n_2-(t-1)(r+1)}{4(t-1)}$ red edges, and for $1 \leq i \leq 6(t-1)$ let $A_i = \{v_k \mid 1 \leq k \leq n, \text{edge } u_iv_k \text{ is red}\}$. If $|A_i \cap A_j| \geq t$ for any $1 \leq i, j \leq 6(t-1)$, then u_i, u_j, and any t vertices in $A_i \cap A_j$ will form a monochromatic red $K_{2,t}$, so we must have $|A_i \cap A_j| \leq t-1$ for each i, j. Using the principle of inclusion and exclusion, this tells us that

$$\begin{aligned}
\left|\bigcup_{i=1}^{6(t-1)} A_i\right| &\geq \sum_{i=1}^{6(t-1)} |A_i| - \sum_{1\leq i<j\leq 6(t-1)} |A_i \cap A_j| \\
&\geq 6(t-1) \cdot \frac{n_2-(t-1)(r+1)}{4(t-1)} - \binom{6(t-1)}{2}(t-1) \\
&= \frac{3}{2} \cdot \left[2(t-1)\binom{6(t-1)}{2} + 2(t-1)(r+1) + 1\right] - \binom{6(t-1)}{2}(t-1) \\
&= 2(t-1)\binom{6(t-1)}{2} + 3(t-1)(r+1) + \frac{3}{2} \\
&= n_2 + \frac{1}{2}.
\end{aligned}$$

But this is a contradiction, since $\left|\bigcup_{i=1}^{6(t-1)} A_i\right| \leq n_2$. Therefore, the coloring must have at least one monochromatic or rainbow $K_{2,t}$, completing the proof. $\square$

Before proceeding, we will need a result from extremal graph theory. Define the *Zarankiewicz number* $z(m, n; s, t)$ to be the maximum possible number of edges in a bipartite graph $G = (U \sqcup V, E)$, where $|U| = m$ and $|V| = n$, that does not contain $K_{s,t}$ as a subgraph. We have the following upper bound for $z(m, n; s, t)$, first established by Kővari, Sós, and Turán:

Theorem 2.2 ([2])**.** *For positive integers m, n, s, t:*

$$z(m, n; s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.$$

A slight rearrangement of this bound gives the following lemma:

Lemma 2.3. *Let H be a bipartite graph with vertex set $U \sqcup V$ such that $|U| = m$ and each vertex in U has degree at least k. If H does not contain $K_{t,t}$ as a subgraph, then*

$$|V| > \left(\frac{m}{t-1}\right)^{1/t}(k-t+1).$$

Proof. Because H does not contain $K_{t,t}$ as a subgraph, it can have at most $z(m, |V|; t, t)$ edges. But because every vertex in U has degree at least k, H must have at least mk edges. Therefore, by Theorem 2.2,

$$mk \leq z(m, |V|; t, t) < (t-1)^{1/t}(|V|-t+1)m^{1-1/t} + (t-1)m.$$

Dividing through by $m\left(\frac{t-1}{m}\right)^{1/t}$ and then rearranging, we find

$$\left(\frac{m}{t-1}\right)^{1/t}(k-t+1) < |V|-t+1 \leq |V|,$$

as desired. $\square$

With this lemma, we are prepared to prove Theorem 1.3.

Proof of Theorem 1.3. With $p = t^3 + 2t^2 + 2t + 1$ and $n = (t-1)(p+1)(p+2)^t r$, we will show that any r-coloring of $K_{n,n}$ which does not contain a monochromatic $K_{t,t}$ contains a rainbow $K_{t,t}$. Consider an r-coloring of $K_{n,n}$ with no monochromatic $K_{t,t}$, and let the vertex set of $K_{n,n}$ be $U \sqcup V$ with $U = \{u_1, \ldots, u_n\}, V = \{v_1, \ldots, v_n\}$. We begin with the following claim, which limits the number of vertices of $K_{n,n}$ that can be adjacent to many edges of the same color.

Claim 2.4. At most $(t-1)(p+2)^t r$ of the vertices in U, and at most $(t-1)(p+2)^t r$ of the vertices in V, are adjacent to at least $\frac{n}{p+1}$ edges of the same color.

Proof. Fix a color, say, red. We will show that fewer than $(t-1)(p+2)^t$ of the vertices in U can be adjacent to at least $\frac{n}{p+1}$ red edges. Indeed, let $m = (t-1)(p+2)^t$, $k = \frac{n}{p+1}$, and suppose for contradiction that the vertices $u_1, u_2, \ldots, u_m$ are each adjacent to at least k red edges. Let $\widetilde{U} = \{u_1, \ldots, u_m\}$ and $\widetilde{V} = \{v_j \in V \mid u_i v_j$ is red for some $1 \leq i \leq m\}$, and consider the subgraph H of $K_{n,n}$ whose vertex set is $\widetilde{U} \sqcup \widetilde{V}$ and whose edges are all the red edges connecting $\widetilde{U}$ to $\widetilde{V}$. Then, $|\widetilde{U}| = m$, each vertex in $\widetilde{U}$ has degree at least k, and H cannot contain $K_{t,t}$ as a subgraph, otherwise we would find a red $K_{t,t}$. Thus, by Lemma 2.3,

$$\begin{aligned}
\left|\widetilde{V}\right| &> \left(\frac{m}{t-1}\right)^{1/t}(k-t+1) \\
&= \left(\frac{(t-1)(p+2)^t}{t-1}\right)^{1/t}\left(\frac{n}{p+1}-(t-1)\right) \\
&= (p+2)\left(\frac{n}{p+1}-t+1\right) \\
&\geq (p+2)\left(\frac{n}{p+1}-\left(\frac{n}{p+1}-\frac{n}{p+2}\right)\right) \\
&= (p+2)\frac{n}{p+2} \\
&= n,
\end{aligned}$$

where the inequality $t-1 \leq \frac{n}{p+1}-\frac{n}{p+2}$ holds since $n \geq (p+1)(p+2)(t-1)$. But this is a contradiction, since $\left|\widetilde{V}\right| \leq |V| = n$. Therefore, for any color, fewer than $(t-1)(p+2)^t$ of the vertices $u_1, \ldots, u_p$ can be adjacent to more than $\frac{n}{p+1}$ edges of that color; thus, at most $(t-1)(p+2)^t r$ of the vertices in U can be adjacent to more than $\frac{n}{p+1}$ edges of any color. The proof for vertices in V is analogous. □

Let $n' = \frac{p}{p+1}n$, so $\frac{1}{p+1}\cdot n = \frac{1}{p+1}\left(\frac{p+1}{p}n'\right) = \frac{n'}{p}$. Define

$$U' = \left\{u_i \in U \;\middle|\; u_i \text{ is adjacent to fewer than } \frac{n'}{p} \text{ edges of each color}\right\},$$

$$V' = \left\{v_i \in V \;\middle|\; v_i \text{ is adjacent to fewer than } \frac{n'}{p} \text{ edges of each color}\right\}.$$

From Claim 2.4, we know that

$$\begin{aligned}
|U'| &\geq n-(t-1)(p+2)^t r \\
&= (t-1)(p+1)(p+2)^t r-(t-1)(p+2)^t r \\
&= (t-1)(p)(p+2)^t r \\
&= \frac{p}{p+1}n \\
&= n',
\end{aligned}$$

and the same holds for $|V'|$. Now, in order to find a rainbow $K_{t,t}$, we may make the following claim:

Claim 2.5. For every $1 \leq s \leq t$, there is a rainbow $K_{s,s}$ whose vertices are contained in $U' \sqcup V'$.

Proof. We will prove this by induction on s. The base case is when $s = 1$. In this case, we can choose one arbitrary vertex from U' and one from V'; these two vertices automatically form a rainbow $K_{1,1}$.

For the inductive step, suppose that there exist vertices $u_1, \ldots, u_s \in U'$ and $v_1, \ldots, v_s \in V'$ such that $u_1, \ldots, u_s, v_1, \ldots, v_s$ form a rainbow $K_{s,s}$. We wish to find two vertices u_{s+1}, v_{s+1} such that $u_1, \ldots, u_{s+1}$, $v_1, \ldots, v_{s+1}$ form a rainbow $K_{s+1,s+1}$.

The existing $K_{s,s}$ contains s^2 colors. We must choose u_{s+1} such that, for each $1 \leq i \leq s$, the color of edge $u_{s+1}v_i$ has not already been used. For each $1 \leq i \leq s$ and each of the s^2 forbidden colors, v_i is adjacent to at most $\frac{n'}{p}$ edges of that color, so there are at most $\frac{s^2 n'}{p}$ choices for u_{s+1} such that the color of $u_{s+1}v_i$ has already been used. It follows that there are at least $n' - s \cdot \frac{s^2 n'}{p} = n''$ choices for u_{s+1} such that none of the edges connecting it to the existing $K_{s,s}$ have a color which has been used. Let $n'' = \left(1 - \frac{s^3}{p}\right)n'$, and let U'' be a set containing exactly n'' such choices. Similarly, there are at least n'' choices for v_{s+1} such that none of the edges connecting it to the existing $K_{s,s}$ have a color which has been used; let V'' be a set containing exactly n'' such choices.

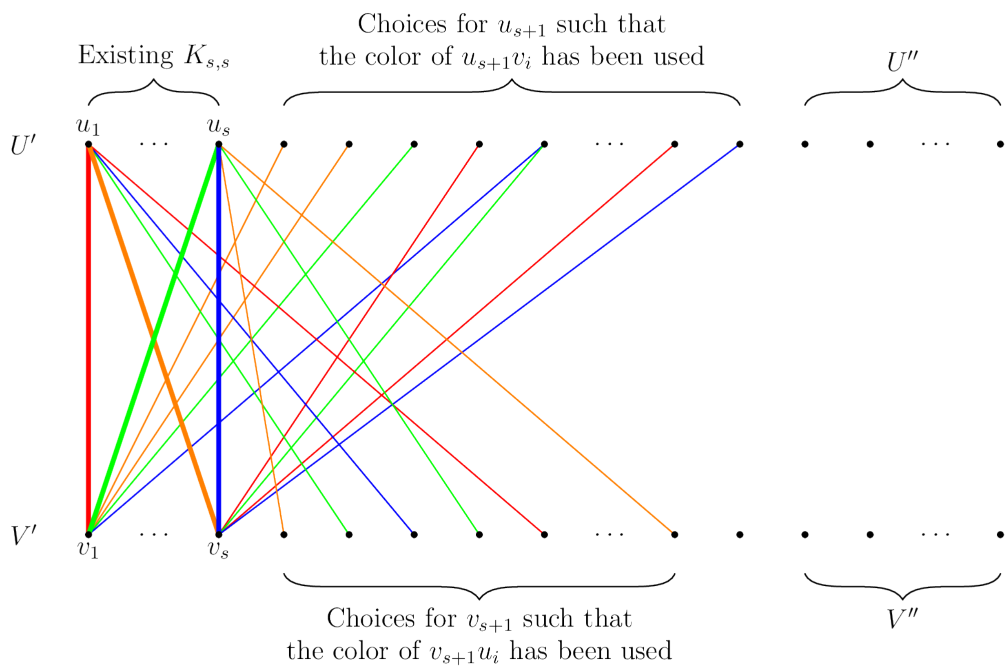
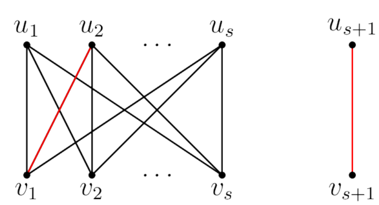


Figure 3: The construction of U'' and V''

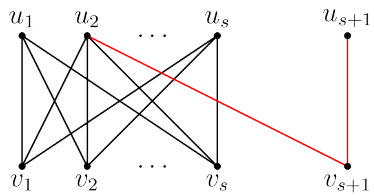
At this point, we know that we must choose $u_{s+1} \in U''$ and $v_{s+1} \in V''$, with $|U''|, |V''| = n''$. There are $(n'')^2$ ways to choose $u_{s+1} \in U''$ and $v_{s+1} \in V''$, and for each choice, there are four ways that it can fail to create a rainbow $K_{s+1,s+1}$:

Case 1. The edge u_{s+1}, v_{s+1} can be the same color as one of the colors already used in $K_{s,s}$.



For each choice of u_{s+1}, and each of the s^2 forbidden colors, there are at most $\frac{n'}{p}$ choices for $v_{s+1} \in V'$ such that $u_{s+1}v_{s+1}$ is that color. It follows that there are at most $n''s^2 \cdot \frac{n'}{p}$ choices for u_{s+1}, v_{s+1} that fail in this way.

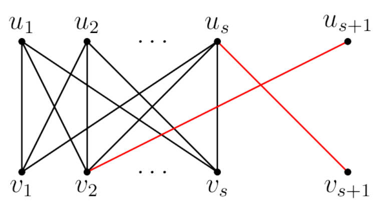
Case 2. For some $1 \leq i \leq s$, the edge $u_{s+1}v_{s+1}$ could be the same color as $u_i v_{s+1}$.



For each $1 \leq i \leq s$ and each possible choice of v_{s+1}, there are at most $\frac{n'}{p}$ choices for u_{s+1} such that $u_{s+1}v_{s+1}$ is the same color as $u_i v_{s+1}$. Thus, there are at most $sn'' \cdot \frac{n'}{p}$ choices for u_{s+1}, v_{s+1} that fail in this way

Case 3. For some $1 \leq i \leq s$, the edge $u_{s+1}v_{s+1}$ could be the same color as $u_{s+1}v_i$. This is analogous to the previous case; the number choices for u_{s+1}, v_{s+1} that fail in this way is at most $sn'' \cdot \frac{n'}{p}$.

Case 4. For some $1 \leq i, j \leq s$, the edge $u_{s+1}v_i$ could be the same color as $u_j v_{s+1}$.



For each choice of i, j and each possible choice of u_{s+1}, there's at most $\frac{n'}{p}$ choices for v_{s+1} such that $u_i v_{s+1}$ is the same color as $u_{s+1}v_i$. Thus, there are at most $s^2 n'' \cdot \frac{n'}{p}$ choices for u_{s+1}, v_{s+1} that fail in this way.

Putting all the cases together, there are at most

$$n''s^2 \cdot \frac{n'}{p} + sn'' \cdot \frac{n'}{p} + sn'' \cdot \frac{n'}{p} + s^2 n'' \cdot \frac{n'}{p} = (2s^2 + 2s) \cdot \frac{n''n'}{p}$$

ways to choose $u_{s+1} \in U''$ and $v_{s+1} \in V''$ such that $u_1, \dots, u_{s+1}, v_1, \dots, v_{s+1}$ is not a rainbow $K_{s+1,s+1}$. Thus, there are at least

$$(n'')^2 - (2s^2 + 2s) \cdot \frac{n''n'}{p}$$

ways to choose $u_{s+1} \in U''$ and $v_{s+1} \in V''$ to make a rainbow $K_{s+1,s+1}$. Since $n'' = \left(1 - \frac{s^3}{p}\right)n'$ and $p = t^3 + 2t^2 + 2t + 1$, we see that

$$\begin{aligned}
(n'')^2 - (2s^2 + 2s) \cdot \frac{n''n'}{p} &= n''\left(n'' - (2s^2 + 2s) \cdot \frac{n'}{p}\right) \\
&= n''n'\left(\left(1 - \frac{s^3}{p}\right) - \frac{2s^2 + 2s}{p}\right) \\
&= n''n'\left(1 - \frac{s^3 + 2s^2 + 2s}{p}\right) \\
&\geq n''n'\left(1 - \frac{t^3 + 2t^2 + 2t}{p}\right) \\
&= n''n'\left(1 - \frac{t^3 + 2t^2 + 2t}{t^3 + 2t^2 + 2t + 1}\right) \\
&> 0,
\end{aligned}$$

so there's at least one choice for u_{s+1} and v_{s+1} that makes a rainbow $K_{s+1,s+1}$. This completes the induction step.

□

Applying Claim 2.5 with $s = t$ completes the proof. □

3 Euclidean Gallai-Ramsey Numbers

Under a family of mappings from the edge set a complete bipartite graph to a subset of Euclidean space, we are able to obtain upper bounds on the Euclidean Gallai-Ramsey numbers for certain configurations using graph Gallai-Ramsey numbers for certain subgraphs. In this section, we present the proofs of our main results on Euclidean Gallai-Ramsey bounds using the following mapping:

Definition 3.1. For any fixed $a, b \in \mathbb{R}$, let

$$W_{s,t,a,b} = \left\{ \left(\underbrace{0, 0, \ldots, 0}_{i-1}, \frac{a}{\sqrt{2}}, \underbrace{0, 0, \ldots, 0}_{s+j-i-1}, \frac{b}{\sqrt{2}}, \underbrace{0, 0, \ldots, 0}_{t-j} \right) \in \mathbb{E}^{s+t} : 1 \leq i \leq s, 1 \leq j \leq t \right\}$$

Let $E(K_{s,t})$ be the edge set of $K_{s,t}$. We define the map $\phi_{s,t,a,b} : E(K_{s,t}) \to W_{s,t,a,b}$ where $K_{s,t}$ has vertex partition $\{v_i : 1 \leq i \leq s\} \sqcup \{u_i : 1 \leq i \leq t\}$

$$\phi_{s,t,a,b}(v_i u_j) = \left(\underbrace{0, 0, \ldots, 0}_{i-1}, \frac{a}{\sqrt{2}}, \underbrace{0, 0, \ldots, 0}_{s+j-i-1}, \frac{b}{\sqrt{2}}, \underbrace{0, 0, \ldots, 0}_{t-j} \right).$$

For concision, we omit a, b in subscripts when they are clear from context. Note that $\phi_{s,t}$ is a bijection between $W_{s,t}$ and $E(K_{s,t})$.

The map $\phi_{s,t}$ is a slight generalization of a related map found in [30]. Observe that by definition among the first s coordinates, there is exactly one nonzero coordinate with value $\frac{a}{\sqrt{2}}$ and among the last t coordinates there is exactly one nonzero coordinate with value $\frac{b}{\sqrt{2}}$. Thus, for convenience, we use $x_{i,j}$ for $i \in [s], j \in [t]$ to denote the point $\phi_{s,t}(v_i u_j)$ in $W_{s,t}$, which has all coordinates are zero except for $\frac{a}{\sqrt{2}}$ in the i-th coordinate and $\frac{b}{\sqrt{2}}$ in the $(s+j)$-th coordinate.

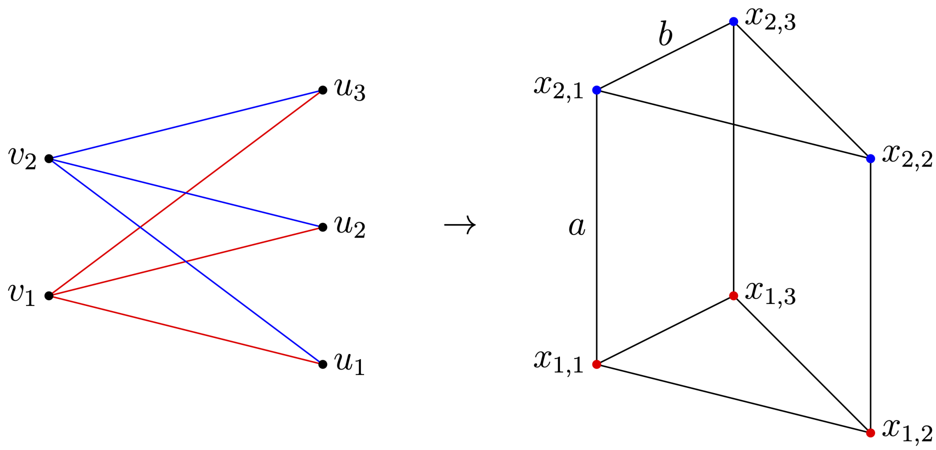


Figure 4: A visualization of $\phi_{2,3}$ from the edges of $K_{2,3}$ to an equilateral triangular prism. For example, (v_1, u_1) maps to $x_{1,1} = (\frac{a}{\sqrt{2}}, 0, \frac{b}{\sqrt{2}}, 0, 0)$. Note that the figure on the right is a 3-dimensional projection of the resulting prism, which lies in 5-dimensional space.

Recall our simple upper bound on the Gallai-Ramsey number for $K_{1,q}$ in Theorem 1.1. Using the above mapping and theorem, we can prove Theorem 1.4.

Proof of Theorem 1.4. Let a be equal to 1 (as the value of a is inconsequential to this proof). Take the set $W_{1,pq+1,a,b} \in \mathbb{E}^{pq+2}$ and the corresponding graph $K_{1,pq+1} = \phi^{-1}_{1,pq+1,a,b}(W_{1,pq+1,a,b})$ as defined above. We omit a, b in the following subscripts as they are clear from context. Consider an r-coloring of $W_{1,pq+1}$; this determines an r-coloring of $K_{1,pq+1}$ by coloring edge $v_i u_j$ with the same color as $\phi_{s,t}(v_i u_j)$. We know by Theorem 1.1 that we can always find a rainbow $K_{1,p+1}$ or a monochromatic $K_{1,q+1}$ in any r-coloring of $K_{1,pq+1}$.

In the case that we have a rainbow $K_{1,p+1}$, the corresponding $p+1$ points in $\mathbb{E}^{pq+2}$ are of the form $x_{1,i} = \left(\frac{a}{\sqrt{2}}, 0, \ldots, 0, \frac{b}{\sqrt{2}}, 0, \ldots, 0\right)$, with the $(i+1)$th coordinate equal to b, for $1 \leq i \leq pq+1$ (by the fact that every edge in a star shares a single center vertex and has a unique leaf vertex). We can check that the Euclidean distance between any pair of these points is b. Thus, these $p+1$ points in $\mathbb{E}^{pq+2}$ form a p-dimensional regular simplex, and since the corresponding $K_{1,p+1}$ is rainbow, these points have different colors. The same argument holds for an q-dimensional monochromatic regular simplex obtained from a monochromatic $K_{1,q+1}$ under $\phi_{1,pq+1}$. $\square$

As the 2-dimensional regular simplex is an equilateral triangle, we see that:

Corollary 3.2. *Any coloring of $\mathbb{E}^6$ yields a monochromatic or rainbow equilateral triangle.*

A natural next step in applying the map $\phi_{s,t}$ past $K_{1,p}$ is to examine bipartite graphs in general. With the case of $K_{2,t}$, we can prove Theorem 1.5.

Proof of Theorem 1.5. Let $n_1 = (6(t-1)-1)r+2$ and $n_2 = 2(t-1)\binom{6(t-1)}{2} + 3(t-1)(r+1)+1$. Let $n_0 = n_1 + n_2 = r(9t-10) + 3(3t-2)(2t(2t-5)+7)$.

Consider the set of points $W_{n_1,n_2,a,b} = \phi_{n_1,n_2,a,b}(E(K_{n_1,n_2})) \in \mathbb{E}^{n_0}$. We omit a, b in the following subscripts as the context is clear. As $K_{2,t}$ is exactly the union of two copies of $K_{1,t}$ with the same leaves but distinct roots, let us without loss of generality fix the root vertices as v_1 and v_2 and the shared t leaves as $\{u_1, \dots, u_t\}$. Then, $\phi_{n_1,n_2}(K_{2,t}) = \{x_{1,j} : 1 \leq j \leq t\} \cup \{x_{2,j} : 1 \leq j \leq t\})$. Note that the sets $V_1 = \{x_{1,j} : 1 \leq j \leq t\}$ and $V_2 = \{x_{2,j} : 1 \leq j \leq t\}$ are 2 congruent regular simplices with side length b, and for each $1 \leq j \leq t$, $x_{1,j}$ is an orthogonal translation of $x_{2,j}$ by the vector $x_{1,j} - x_{2,j} = (\frac{1}{\sqrt{2}}a, -\frac{1}{\sqrt{2}}a, 0, \dots, 0)$.

Thus, this configuration exactly gives us the Cartesian product of a $(t-1)$-dimensional simplex with a line segment of length $||x_{1,j} - x_{2,j}|| = a$. Any coloring of W_{n_1,n_2} corresponds to a coloring of K_{n_1,n_2} through the map ϕ_{n_1,n_2}. By Theorem 1.2, we are guaranteed to find a rainbow or monochromatic $K_{2,t}$, and so in W_{n_1,n_2}, we are guaranteed to find a rainbow or monochromatic configuration congruent to Q_t. □

This gives us a general bound for all equilateral triangular prisms stated in Corollary 1.7. In addition, we can consider a rectangle with side lengths a and b to be the Cartesian product of two one-dimensional simplices. Thus we obtain the bound presented in Corollary 1.8, an improvement on the bound of $\mathbb{E}^{13r+4}$ for general rectangles presented in [30].

In fact, it is not necessary to restrict one of the configurations in the Cartesian product to a line segment or one-dimensional simplex. In examining general bipartite graphs $K_{s,t}$, we find that under the mapping defined in 3.1, any $K_{s,t}$ corresponds to the Cartesian product of an $(s-1)$-dimensional simplex with a $(t-1)$-dimensional simplex. With this intuition and Theorem 1.3, we prove Theorem 1.6:

Proof of Theorem 1.6. See that by Theorem 1.3, any coloring of $K_{n,n}$ yields a monochromatic or rainbow $K_{t,t}$. Through the mapping $\phi_{t,t,a,b}(E(K_{n,n})) = W_{n,n,a,b}$, we see that any coloring of the set $W_{n,n,a,b}$ directly corresponds to a coloring of $K_{n,n}$. We omit a, b in the following subscripts as the context is clear. Let Y be the $(t-1)$-dimensional regular simplex with vertices y_i of the form $(0, \dots, 0, a, 0, \dots, 0)$ where the index of the nonzero coordinate is i (for all $i \in [t]$). Likewise, let Z be the $(t-1)$-dimensional regular simplex with vertices z_i of the form $(0, \dots, 0, b, 0, \dots, 0)$ where the index of the nonzero coordinate is i, again for all $i \in [t]$.

Then, as every vertex in the first partition of $K_{t,t}$ has an edge to every vertex in the second partition, and vice versa, we see that every point $x_{i,j}$ in $W_{n,n}$ can be represented as the Cartesian product of the point y_i and z_j. Thus, since we are guaranteed to find a monochromatic or rainbow $K_{t,t}$ in any coloring of $K_{n,n}$, we know through $\phi_{n,n}$ that we are also guaranteed to find a rainbow or monochromatic Q_t in $W_{n,n} \in \mathbb{E}^{2n}$. □

4 Future Work

In the process of obtaining the main results presented in this paper, the authors note a few interesting directions for further study that are beyond the scope of this paper. One such direction includes examining the distinction between Ramsey and Gallai-Ramsey configurations in Euclidean space, if such a distinction exists at all.

In Euclidean Ramsey theory, we have the following theorem:

Theorem 4.1 ([13])**.** . *If two configurations K_1 and K_2 are Ramsey, then the configuration resulting from taking the Cartesian product $K_1 * K_2$ is also Ramsey.*

We may ask if the analogous statement holds for Gallai-Ramsey configurations:

Statement 4.2. If K_1 and K_2 are Gallai-Ramsey, then the Cartesian product $K_1 * K_2$ is also Gallai-Ramsey.

In this direction, we have the following result:

Theorem 4.3. *The above statement is true if and only if all Gallai-Ramsey configurations are Ramsey.*

Proof. The first direction follows immediately from the product theorem for Ramsey configurations: suppose all Gallai-Ramsey configurations are Ramsey. Then if K_1 and K_2 are Gallai-Ramsey, they're Ramsey, so $K_1 * K_2$ is Ramsey, and hence Gallai-Ramsey. Hence, if all Gallai-Ramsey configruations are Ramsey, Statement 4.2 holds for Gallai-Ramsey configurations.

In the other direction, suppose Statement 4.2 is true for Gallai-Ramsey configurations, and let K_1 be a Gallai-Ramsey configuration. We will show that K_1 is Ramsey. For any integer r, let K_2 be a regular r-dimensional simplex with $r + 1$ vertices. Because K_2 is Ramsey, it's Gallai-Ramsey, so by assumption, $K_1 * K_2$ is Gallai-Ramsey. Thus, there's some n such that every coloring of $\mathbb{E}^n$ contains a monochromatic or rainbow copy of $K_1 * K_2$ with r colors. But $K_1 * K_2$ contains $|K_1||K_2| = |K_1|(r+1) > r$ points, so no coloring of $\mathbb{E}^n$ with r colors can contain a rainbow copy of $K_1 * K_2$. Hence, every r-coloring of $\mathbb{E}^n$ contains a monochromatic $K_1 * K_2$, and hence a monochromatic K_1. Therefore, K_1 is Ramsey. □

As this provides us an interesting and very general equivalence statement, the authors are interested in the following question:

Open Question 1. *Does there exist a configuration that is not Ramsey, but is Gallai-Ramsey?*

The converse is true for all configurations by the monochromatic condition of Gallai-Ramsey sets. Recall that any Ramsey configuration must be spherical [10]. We focus on the case of l_3, the set of three co-linear points with pairwise distances 1, 1, and 2. Notably, l_3 is not Ramsey, as it is not spherical. However, recent work [9] shows that l_3 is Gallai-Ramsey for all spherical colorings, which leads to the following question when we take away the restriction to spherical colorings:

Open Question 2. *Is l_3 Gallai-Ramsey?*

In a separate vein, we note that most of the Euclidean Gallai-Ramsey results obtained in this paper involve some linear dependency on r, the number of colors. We are curious if there exist bounds for such configurations that have no dependency on the number of colors. We suspect that if such bounds exists, they most likely cannot be obtained with the methods presented in this paper as we restrict ourselves to only considering the coloring of a subset of Euclidean space.

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