Positive Mass Theorems for Asymptotically Euclidean Smooth Metric Measure Spaces

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Abstract

We prove a positive mass theorem for asymptotically Euclidean smooth metric measure spaces, which are generalizations of weighted manifolds. In a special case, we recover the weighted positive mass theorem proven by Baldauf and Ozuch. Our result is proven in two different ways: by applying spinorial techniques on certain warped products and by making a conformal change of metric. Our proof methods yield results of independent interest, including eigenvalue bounds for the Dirac operator on closed manifolds and a characterization of the Dirac operator on warped products with manifolds admitting parallel spinors.

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0 Introduction

This paper is concerned with generalizations of the *positive mass theorem* (PMT). Originating as a conjecture in general relativity, the PMT asserts that an isolated gravitational system with nonnegative local matter density must have nonnegative total mass, measured at spatial infinity. A precise mathematical formulation takes place on *asymptotically Euclidean manifolds*. An asymptotically Euclidean manifold is a Riemannian manifold (M^n, g), $n \geq 3$, that decomposes as $M = M_{\text{cpct}} \cup M_\infty$, where M_{cpct} is compact and M_∞ is diffeomorphic to $\mathbb{R}^n \backslash B_R(0)$ for some $R > 0$. Moreover, in the asymptotic coordinates, the metric g satisfies

$$g_{ij} = \delta_{ij} + \mathcal{O}(r^{-\tau}), \quad \partial_k g_{ij} = \mathcal{O}(r^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = \mathcal{O}(r^{-\tau-2}) \tag{0.1}$$

for some $\tau > \frac{n-2}{2}$, where r denotes distance to the origin in $\mathbb{R}^n$. The *(ADM) mass* of (M, g), introduced in [ADM60a], [ADM60b], and [ADM61], is then defined by

$$\mathfrak{m}(g) := \lim_{\rho \to \infty} \int_{S_\rho} \mu \lrcorner d\text{Vol}_g,$$

where $\lrcorner$ denotes the interior product and μ is the *mass-density vector field*

$$\mu = (\partial_i g_{ij} - \partial_j g_{ii}) \partial_j.$$

While $\mathfrak{m}(g)$ ostensibly depends on the diffeomorphism $M_\infty \cong \mathbb{R}^n \setminus B_R(0)$ in which (0.1) holds, Bartnik showed that it is in fact a geometric invariant [Bar86] (see also [LP87, Section 9]). Under this setup, the Riemannian positive mass theorem reads as follows.

Theorem 0.1 (PMT)**.** *Let (M^n, g), $n \geq 3$ be an asymptotically Euclidean manifold of order $\tau > \frac{n-2}{2}$. Also assume that $3 \leq n \leq 7$ or that (M, g) is spin. If the scalar curvature R_g is non-negative and integrable, then $\mathfrak{m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$.*

Physical motivations aside, the positive mass theorem played a critical role in solving the *Yamabe problem*: given a closed, smooth Riemannian manifold (M^n, g), is there a metric $\tilde{g}$ conformal to g such that $R_{\tilde{g}}$ is constant? The problem is trivial for $n = 1$, and is equivalent to the uniformization theorem when $n = 2$. For dimensions 3 and up, the combined efforts of Yamabe, Trudinger, Aubin, Schoen and Yau eventually provided an affirmative answer to this question in 1984, the final step being a proof of Theorem 0.1. For a cohesive account of the Yamabe problem (including its resolution), see [LP87].

Schoen and Yau's proof of the positive mass theorem for the case where $3 \leq n \leq 7$ uses minimal surfaces. Shortly after, Witten ([Wit81]) discovered another proof in arbitrary dimension when M is spin. Witten's proof relies on finding a so-called *Witten spinor* ψ, a spinor in the kernel of the Dirac operator with norm 1 at infinity, for which the following formula for the mass holds:

$$\mathfrak{m}(g) = 4 \int_M \left[|\nabla \psi|^2 + \frac{1}{4} R |\psi|^2 \right] d\text{Vol}_g. \tag{0.2}$$

In [BO22], Baldauf and Ozuch extend the positive mass theorem to weighted manifolds of the form (M^n, g, f), where $f \in \mathcal{C}^\infty(M)$ is in an appropriately decaying Hölder space (see Appendix A.3 for the relevant definitions) and defines the measure $e^{-f} d\text{Vol}_g$. In particular, they define the *weighted mass*

$$\mathfrak{m}_f(g) := \mathfrak{m}(g) + 2 \lim_{\rho \to \infty} \int_{S_\rho} \langle \nabla f, \vec{n} \rangle e^{-f} dA_g$$

and instead assume the weighted scalar curvature $R_f = R + 2\Delta f - |\nabla f|^2$ is non-negative:

Theorem 0.2 (Weighted PMT [BO22, Theorem 2.13])**.** *Let (M^n, g), $n \geq 3$, be an asymptotically Euclidean spin manifold of order $\tau > \frac{n-2}{2}$, and assume $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$ for some $\alpha \in (0, 1)$. If R_f is integrable and non-negative, then $\mathfrak{m}_f(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$ and $\int_{\mathbb{R}^n} (\Delta_f f) e^{-f} dx = 0$.*

Baldauf and Ozuch's proof is based on constructing *weighted Witten spinors* associated to the *weighted Dirac operator* $D_f = D - \frac{1}{2}\nabla f$ and then proving an analogue of Witten's formula (0.2) for the weighted mass.

Further widening the class of spaces under study, one arrives at *smooth metric measure spaces* (SMMS). A SMMS is a four-tuple $(M^n, g, v^m d\mathrm{Vol}_g, m)$, where $m \in \mathbb{R}$ and $v \in \mathcal{C}^\infty(M)$ is a positive function. When $m \neq 0$, we can define the function $f \in \mathcal{C}^\infty(M)$ by $v^m = e^{-f}$, making (M, g, f) a weighted manifold. Conversely, a weighted manifold (M, g, f) becomes an SMMS when one arbitrarily selects $m \in \mathbb{R} \setminus \{0\}$ and sets $v = e^{-\frac{f}{m}}$, though we are typically interested in taking $m \to +\infty$ when adopting this perspective. The impetus for studying SMMSs came from their ability to recover Perelman's entropy by taking $m \to \infty$; see [Cas12] for more details. The curvature associated to SMMSs is described by the *Bakry-Émery Ricci tensor*, which is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df.$$

Similarly, the *Bakry-Émery scalar curvature* is defined by

$$R_f^m := R_f - \frac{1}{m}|\nabla f|^2 = R + 2\Delta f - \frac{m+1}{m}|\nabla f|^2.$$

Observe that $R_f^m \to R_f$ as $m \to \infty$. When m is a positive integer, the Bakry-Émery Ricci and scalar curvatures arise naturally from certain warped products involving the base manifold (M, g). In particular, if (F^m, g_F) is an m-dimensional scalar-flat manifold, then the warped product $(M \times F, \overline{g} = g \oplus v^2 g_F)$ has scalar curvature $R_{\overline{g}}$ equal to the Bakry-Émery scalar curvature R_f^m of M, and the Ricci tensor $\mathrm{Ric}_{\overline{g}}$ of the metric $\overline{g}$ satisfies $\mathrm{Ric}_{\overline{g}}(X, X) = \mathrm{Ric}_f^m(X, X)$ on horizontal vector fields $X \in \Gamma(TM)$ (see [Cas12, section 4.1]).

For an asymptotically Euclidean SMMS $(M^n, g, v^m d\mathrm{Vol}_g, m)$ (meaning (M, g) is asymptotically Euclidean), we define its *Bakry-Émery mass* $\mathfrak{m}_{f,m}(g)$ to be the weighted mass of (M, g, f), where f is chosen so that $v^m = e^{-f}$:

$$\mathfrak{m}_{f,m}(g) = \mathfrak{m}(g) + 2\lim_{\rho\to\infty}\int_{S_\rho} \langle \nabla f, \vec{n}\rangle e^{-\frac{m+2}{m}f} dA_g.$$

In this paper, we investigate how constraints on the Bakry-Émery scalar curvature affect the Bakry-Émery mass of an asymptotically Euclidean SMMS. Our main result is the following positive mass theorem.

Theorem 0.3. *Let $(M^n, g, v^m d\mathrm{Vol}_g, m)$ be a complete, asymptotically Euclidean SMMS of order $\tau > \frac{n-2}{2}$. Assume that $3 \leq n \leq 7$, or that M is a spin manifold. Also assume $m \in \mathbb{R} \setminus [1-n, 0]$. Write $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, and suppose that R_f^m is integrable and non-negative. Then $\mathfrak{m}_{f,m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$ and f is identically* 0.

When $m \in [1-n, 0)$, the Bakry-Émery mass is also shown to be non-negative assuming a stronger lower bound on R_f^m; see Theorem 2.9 for the full statement, which contains Theorem 0.3. Note that when $m > 0$, we have $R_f \geq R_f^m$, so Theorem 0.3 recovers Theorem 0.2 and also shows the stronger result that masslessness and $R_f^m \geq 0$ imply the vanishing of f.

Theorem 0.3 and its extension Theorem 2.9 will be proved via a conformal change of metric; we elaborate further below. We also prove, using a different method which we call the *warped product method*, a special case of Theorem 0.3:

Theorem 0.4. *Let $(M^n, g, v^m d\mathrm{Vol}_g, m)$ be a complete, asymptotically Euclidean SMMS of order $\tau > \frac{n-2}{2}$. Assume that M is a spin manifold and $m \in \mathbb{N}$. Write $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, and suppose that R_f^m is integrable and non-negative. Then $\mathfrak{m}_{f,m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$ and f is identically* 0.

The warped product method. Here, we take $m \in \mathbb{N}$ and M to be spin. Let T^m be the flat m-dimensional torus of unit volume. From the preceding discussion, the warped product metric $\bar{g} = g \oplus v^2 g_{T^m}$ on $M \times T^m$ has scalar curvature $R_{\bar{g}}$ equal to R_f^m. One can define a mass for the warped product $(M \times F, \overline{g})$ analogously to the ordinary mass, and we prove the following intriguing fact in Section 2.1:

Theorem 0.5. *The mass of* $(M \times T^m, \overline{g})$ *is precisely the weighted mass of* (M^n, g) *with respect to* f*, i.e. the Bakry-Émery mass of* $(M^n, g, v^m d\mathrm{Vol}_g, m)$*.*

To prove Theorem 0.4, it suffices to prove that $R_{\overline{g}}$ being integrable and non-negative implies $(M \times T^m, \overline{g})$ has non-negative mass. Similar remarks apply for the rigidity statement. This is, of course, a positive mass theorem on $M \times T^m$, but Theorem 0.1 does not apply because $M \times T^m$ is not asymptotically Euclidean. Our approach is to adapt Witten's proof to get a PMT on $M \times T^m$, which at its core comes down to understanding the Dirac operator on $M \times T^m$.

For a warped product $(M \times F, \overline{g})$ of spin manifolds with M even-dimensional, the spinor bundle $\Sigma(M \times F)$ splits as the tensor product $\pi_1^*(\Sigma M) \otimes \Sigma V$ between the pullback spinor bundle of M, and the spinor bundle ΣV on the vertical distribution $V = \ker(d\pi_2)$. When F admits parallel spinors, we prove the following result, based on the discussion in [Roo20].

Theorem 0.6. *Let* $(M^n \times F^m, \overline{g})$ *be the warped product manifold with metric* $\overline{g} = g \oplus v^2 g_F$*, and let* $v^m = e^{-f}$*. Assume* (F, g_F) *admits parallel spinors and* M *has even dimension. Let* ϕ *be a section of* ΣM *and* ν *a parallel section of* ΣV*. We have* $D_{M\times F}(\phi \otimes \nu) = (D_f \phi) \otimes \nu$*, where* $D_{M\times F}$ *is the Dirac operator of* $(M \times F, \overline{g})$ *and* D_f *is the weighted Dirac operator on* M*.*

An analogous result holds when n is odd; see Theorem 1.2 for details. We will use Theorem 0.6 and the construction of weighted Witten spinors on M from [BO22] to construct Witten spinors on $M \times T^m$, leading to the desired positive mass theorem on $M \times T^m$. In fact, this construction of Witten spinors applies more generally to warped products $(M \times F, \overline{g})$ where (F, g_F) admits parallel spinors, and allows simplified proofs of the positive mass theorems in [Dai04] for warped product manifolds $(M \times F, \overline{g})$ that are asymptotically unwarped on the end $M_\infty \times F$.

The presence of the weighted Dirac operator in Theorem 0.6 also sparks independent interest. Since all manifolds with special holonomy except quaternionic Kähler manifolds admit parallel spinors (see [Dai04, Section 1]), Theorem 0.6 provides many situations in which the weighted Dirac operator arises naturally. Moreover, Theorem 0.6 enables one to reprove some formulas from [BO22], namely the weighted Ricci identity, weighted Lichnerowicz formula, and weighted Witten's formula, by working directly on the warped product $M \times F$. We also give a bound on the eigenvalues of the Dirac operator on M in terms of R_f^m when M is compact. These results are organized in Table 1.

The conformal metric method. This method proves Theorem 0.3, and its extension Theorem 2.9, in their stated forms. To this end, we introduce the *Bakry-Émery conformal metric* $g_{f,m} := v^{\frac{2m}{n-1}} g$, and show that the ordinary mass with respect to $g_{f,m}$ equals the Bakry-Émery mass with respect to g (Lemma 2.12). We then show that the non-negativity (vanishing) of $R_{g_{f,m}}$ is equivalent to the non-negativity (vanishing) of R_f^m (Lemma 2.14), and we use this to deduce Theorem 2.9 from the ordinary PMT applied to $(M^n, g_{f,m})$. The quantity $F_f^m = \frac{m+n-1}{m(1-n)} |\nabla f|^2$ appears naturally in the computations. We refer to F_f^m as the *Bakry-Émery barrier function*, and its behavior leads to the exclusion of the interval $[1-n, 0]$ in Theorem 0.3 (formulated as a 'dichotomy' in Theorem 2.9).

We also generalize Baldauf and Ozuch's weighted PMT (Theorem 0.2) by showing that non-negativity of weighted mass still holds if $R_f \geq -\frac{1}{n-\varepsilon} |\nabla f|^2$ for some $\varepsilon < 1$ (cf. Corollary 2.10). We subsequently show that there exists a one-parameter family $\{f_m\}_{m \in (-\infty, 1-n)}$ of appropriately decaying functions satisfying $\frac{1}{m} |\nabla f|^2 \leq R_{f_m} < 0$ (cf. Theorem 2.18), proving it is indeed possible for the weighted mass to be strictly positive when the weighted scalar curvature is strictly negative.

The Bakry-Émery conformal metric is also related to the weighted Dirac operator. In particular, it corresponds to the conformal change which recovers the weighted Dirac operator in an appropiate sense (see Remark 1.4). From our analysis of the scalar curvature of the Bakry-Émery conformal metric, we use this to prove the following generalization of the classical fact that on a compact spin manifold with positive scalar curvature, the Dirac operator has trivial kernel (see Corollary 2.20 for details):

	Riemannian	Weighted	Bakry-Émery
Ricci curvature	Ric	$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}(f)$	$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{m} df \otimes df$
Scalar curvature	R	$R_f = R + 2\Delta f - \lvert\nabla f\rvert^2$	$R_f^m = R + 2\Delta f - \frac{m+1}{m}\lvert\nabla f\rvert^2$
Dirac operator*	D	$D_f = D - \frac{1}{2}\nabla f$	$D_{M\times F}$*
Lichnerowicz formula*	$D^2 = -\Delta + \frac{1}{4}R$	$D_f^2 = -\Delta_f + \frac{1}{4}R_f$	$D_{M\times F}^2 = -\Delta_{M\times F} + \frac{1}{4}R_f^m$*
Ricci identity*	$[D, \nabla_X] = \frac{1}{2}\mathrm{Ric}(X)$	$[D_f, \nabla_X] = \frac{1}{2}\mathrm{Ric}_f(X)$	$[D_{M\times F}, \nabla_X] = \frac{1}{2}\mathrm{Ric}_f^m(X)$*
Eigenvalue bound*	$\lambda(D)^2 \geq \frac{n}{4(n-1)} \min R$	$\lambda(D)^2 = \lambda(D_f)^2 \geq \frac{n}{4(n-1)} \min R_f$	$\lambda(D_{M\times F})^2 \geq \frac{n+m}{4(n+m-1)} \min R_f^m$*
ADM Mass*	$\mathfrak{m} = \lim_{\rho\to\infty} \int_{S_\rho} \mu \lrcorner d\mathrm{Vol}_g$	$\mathfrak{m}_f := \mathfrak{m} + 2\lim_{\rho\to\infty} \int_{S_\rho} \langle \nabla f, \vec{n}\rangle e^{-f} dA_g$	$\mathfrak{m}_{f,m} := \mathfrak{m} + 2\lim_{\rho\to\infty} \int_{S_\rho} \langle \nabla f, \vec{n}\rangle e^{-\frac{m+2}{m}f} dA_g$*
Witten formula*	$\mathfrak{m} = 4\int_M (\lvert\nabla\psi\rvert^2 + \frac{1}{4}R\lvert\psi\rvert^2) dV$	$\mathfrak{m}_f = 4\int_M (\lvert\nabla\psi\rvert^2 + \frac{1}{4}R_f\lvert\psi\rvert^2) e^{-f} dV$	$\mathfrak{m}_{f,m} = 4\int_{M\times F} \left[\lvert\nabla^{M\times F}\psi\rvert_{\overline{g}}^2 + \frac{1}{4}R_f^m\lvert\psi\rvert_{\overline{g}}^2\right] d\mathrm{Vol}_{M\times F}$*

Table 1: Classical vs. weighted vs. Bakry-Émery quantities. Contributions from this paper are labeled with an asterisk (*).

Corollary 0.7. *Let M be a closed spin manifold. If $R_f^m \geq F_f^m$, and $R_f^m > F_f^m$ at some point for some $m \in \mathbb{R}$ and some $f \in C^\infty(M)$, then M admits no nontrivial harmonic spinors. In particular, if $m \in \mathbb{R} \setminus [1-n, 0]$, M admits no harmonic spinors if $R_f^m \geq 0$ and $R_f^m > 0$ at some point.*

Corollary 0.7 leads to a discussion of the question of solving for f so that R_f^m is constant.

Structure of the paper. In Section 1, we describe the product spin structure, Dirac operator, and spinor norm on warped products $(M \times F, \overline{g})$ with manifolds (F, g_F) admitting parallel spinors. We prove Theorem 0.6 and discuss applications. In Section 2.1, we use the warped product method to prove Theorem 0.4 and recover the weighted Witten's formula in [BO22]. In Section 2.2, we use the conformal metric method to prove Theorem 0.3 and provide some examples of functions with negative (weighted) scalar curvature and positive (weighted) mass. A number of routine computations are relegated to Appendices A.1 and A.2. In Appendix A.3, we provide the relevant definitions and rudimentary results on weighted Hölder spaces assumed throughout the paper.

Notation and conventions. We use Einstein summation notation throughout the paper. Given a Riemannian manifold (M^n, g), ∇ denotes the Levi-Civita connection of (M^n, g). Our convention for the Laplacian is

$$\Delta = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^i} \left[\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x^j} \right],$$

which agrees with the convention used in [BO22]. For a fixed radius $\rho > 0$, we shall denote by $d\mathrm{Vol}_g$ and dA_g the volume forms of (M^n, g) and $(S_\rho^M, \tilde{g})$, respectively, where $\tilde{g}$ is the metric induced on S_ρ^M by M. Additionally, $\vec{n}$ denotes the outward normal of S_ρ.

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1 The Bakry-Émery Dirac operator

Let $(M^n, g, v^m d\mathrm{Vol}_g, m)$, $m \in \mathbb{N}$, be a smooth metric measure space with $e^{-f} = v^m$ and $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, so that M is a complete spin manifold. Let (F, g_F) be an m-dimensional Riemannian manifold admitting parallel spinors with respect to some spin structure, and consider the warped product $(M \times F, g \oplus v^2 h) = (M \times F, \overline{g})$ of M and F. By the Ricci identity, F is Ricci flat, hence by the discussion in [Cas12] the scalar curvature of $(M \times F, \overline{g})$ is given by the Bakry-Emery scalar curvature R^m_f.

In this section, we show in Theorem 1.2 that in an appropriate sense the Dirac operator of $(M \times F, \overline{g})$ associated to the product spin structure on $M \times F$ is given by the weighted Dirac operator discussed in [BO22]. This result will be applied in the Positive Mass Theorems proven in section 2, and allows us to show the weighted Ricci Identity and weighted Lichnerowicz formulas in [BO22] from the ordinary Ricci Identity and Lichnerowicz formulas on the warped product $M \times F$. In Corollary 1.7, we discuss applications to eigenvalue bounds for the Dirac operator on M. Examples of manifolds with parallel spinors include all manifolds with special holonomy except quaternionic Kähler manifolds. In particular, a complete, simply connected, irreducible Riemannian spin manifold admits parallel spinors if and only if its holonomy group is one of $SU(m), Sp(m), Spin(7), G_2$ by results in [Wan89]. The results in this section thus show that the weighted Dirac operator D_f arises in many geometric contexts. We assume the reader is familiar with basic notions from spin geometry; for a thorough treatment of these matters we defer to [LM89, BHM+15].

1.1 The spin structure on the warped product $M \times F$

Fix a spin structure on M, and fix the spin structure on F for which parallel spinors exist. Denote by π_1, π_2 the projection maps $\pi_1 : M \times F \to M, \pi_2 : M \times F \to F$. The spin structures on M, F induce spin structures on the bundles $ker(d\pi_1) \cong \pi_1^*(TM), V = ker(d\pi_2)$ as follows. Note that $ker(\pi_1)$ is isometric to $\pi_1^*(TM)$ with the pullback metric, since the metric on each copy $M \times \{q\}, q \in F$ is identical to the metric on M, thus the spin structure on $ker(\pi_1)$ is given by the covering map $\pi_1^*(P_{spin}(TM)) \to \pi_1^*(P_{SO}(TM))$. In contrast, the metric on each copy $\{m_0\} \times F, m_0 \in M$ is different from that on F, but there is a bundle isometry $T : \pi_2^*(TF) \to V$ given by scaling by $v^{-1} = e^{\frac{f}{m}}$, which induces an isomorphism $T : \pi_2^*(P_{SO}(TF)) \to P_{SO}(V)$, where $P_{SO}(TF) \cong F \times SO(m)$. The spin structure on V is then given by the composition $\pi_2^*(P_{spin}(TF)) \to \pi_2^*(P_{SO}(TF)) \xrightarrow{T} P_{SO}(V)$.

Since $T(M \times F) = ker(d\pi_1) \oplus ker(d\pi_2) \cong \pi_1^*(TM) \oplus V$, the spin structures on $\pi_1^*(TM), V$ induce a unique spin structure on $T(M \times F)$ by Proposition 1.15 in [LM89], which can be constructed as follows. If $a \in H^1(P_{SO}(\pi_1^*(TM)), \mathbb{Z}/2), a' \in H^1(P_{SO}(V), \mathbb{Z}/2)$ correspond to the spin structures on M, V respectively, then the spin structure on $T(M \times F)$ is the pullback $\Delta^* b$ along the diagonal map

$$\Delta : P_{SO}(\pi_1^*(TM) \oplus V) \to P_{SO}(\pi_1^*(TM) \times V)$$

of the unique class $b \in H^1(P_{SO}(\pi_1^*(TM) \times V), \mathbb{Z}/2)$ which pulls back to the class $a' \times 1 + 1 \times a'' \in H^1(P_{SO}(\pi_1^*(TM)) \times P_{SO}(V), \mathbb{Z}/2)$ under the inclusion $P_{SO}(\pi_1^*(TM)) \times P_{SO}(V) \subset P_{SO}(\pi_1^*(TM) \times V)$. This is a general phenomenon, spin structures on two vector bundles E, E' always induce a unique spin structure on their sum $E \oplus E'$ in this way.

In the case where M^n is an asymptotically flat manifold with $n > 2$ and $F = T^m$ is a flat torus, the asymptotic end $M^n \setminus M_{cpct} \cong \mathbb{R}^n \setminus B_R(0)$ is simply connected, hence the spin structure on M restricts on $M \setminus K$ to the unique trivial spin structure, which induces a trivialization of the associated spinor bundle. By uniqueness, it follows that the chosen spin structure on $M \times T$ restricts to the trivial spin structure over the end $(M \setminus M_{cpct}) \times T$, hence its spinor bundle is also trivialized on the asymptotic end. We next describe this spinor bundle explicitly using some general facts about Riemannian submersions.

1.2 General facts about spin geometry on Riemannian submersions

The following discussion is based on [Roo20, §3.2]. We first recall some facts from the representation theory of the complex Clifford algebras $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}(\mathbb{R}^n)$. Let Σ_n be a complex vector space of dimension $2^{\lfloor \frac{n}{2} \rfloor}$. When n is even, there is a unique irreducible complex representation $\chi_n : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma_n)$, and when n is odd there exist two non-isomorphic irreducible representations $\chi_n^{\pm} : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma_n)$. In the latter we case we label the representations by $\Sigma_n^{\pm}$, though the underlying vector spaces are just Σ_n.

Denote by $(e_i)_{i=1}^n$ the standard basis for $\mathbb{R}^n$ and let $\omega_n^{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n \in \mathbb{C}\mathrm{l}_n$ be the *complex volume element*. When n is even we have $(\omega_n^{\mathbb{C}})^2 = 1$, so there is a splitting $\Sigma_n = \hat{\Sigma}_n^+ \oplus \hat{\Sigma}_n^-$ into the ± 1 eigenspaces. From this decomposition we construct a *complex conjugation* map

$$\Sigma_n \to \Sigma_n, \phi = \phi_+ + \phi_- \mapsto \overline{\phi} = \phi_+ - \phi_-.$$

The representations $\hat{\Sigma}_n^+, \hat{\Sigma}_n^-$ restrict to irreducible, non-isomorphic representations of the spin group $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n$. When n is odd, both representations χ_n^+, χ_n^- become isomorphic when restricted to $\mathrm{Spin}(n)$, and $\omega_n^{\mathbb{C}}$ acts by ± 1 on $\Sigma_n^{\pm}$ respectively. The spin representations are defined by

$$\theta_n = \begin{cases} \chi_n|_{\mathrm{Spin}(n)} & n \text{ even} \\ \chi_n^+|_{\mathrm{Spin}(n)} & n \text{ odd,} \end{cases}$$
$$\theta_n^- = \chi_n^-|_{\mathrm{Spin}(n)} \quad \text{for odd } n.$$

For odd n, Clifford multiplication by $x \in \mathbb{R}^n$ in the representation χ_n^- is given by $-x\cdot$, where $\cdot$ denotes the Clifford action for χ_n^+. This distinction is important for describing how the Clifford multiplication behaves on a product. The following isomorphisms hold as Clifford representations of $\mathbb{C}\mathrm{l}_{n+k}$:

$$\Sigma_{n+k} \cong \begin{cases} \Sigma_n \otimes \Sigma_k & \text{one of } n,k \text{ even} \\ (\Sigma_n^+ \oplus \Sigma_n^-) \otimes \Sigma_k & n,k \text{ odd.} \end{cases} \tag{1.1}$$

For $(x,v) \in \mathbb{R}^n \times \mathbb{R}^k$, the Clifford multiplication on elementary tensors is given by

$$(x,v)\cdot(\phi \otimes \nu) = \begin{cases} (x \cdot \phi) \otimes \nu + \overline{\phi} \otimes (v \cdot \nu) & n \text{ even} \\ (x \cdot \phi) \otimes \overline{\nu} + \phi \otimes (v \cdot \nu) & k \text{ even} \\ (x \cdot \phi^+ \oplus -x \cdot \phi^-) \otimes \nu + (\phi^- \oplus \phi^+) \otimes (v \cdot \nu) & n,k \text{ odd.} \end{cases} \tag{1.2}$$

When n,k are both even, both possibilities for Clifford Multiplication are isomorphic. In this case, we use the first multiplication in 1.2 by convention. Using these facts we are in a position to describe the spinor bundle and spin connection for a general Riemannian submersion, following the discussion in [Roo20]. Let $f : (N^{n+k}, g) \to (B^k, h)$ be a Riemannian submersion. Recall this means that $df : \ker(df)^{\perp} \to TB$ is an isometry; we henceforth identify these two bundles. We will refer to $V := \ker(df)$ as the *vertical distribution*. This is a rank-k vector bundle over N. For a vector $w \in TN$, we denote by w^H, w^V the projections of v to TB, V respectively. From the Riemannian connection ∇ of N we obtain the following connections and tensors which describe the decomposition $TN = TB \oplus V$.

$$\begin{aligned} T(X,Y) &:= (\nabla_{X^V} Y^V)^H + (\nabla_{X^V} Y^H)^V \\ A(X,Y) &:= (\nabla_{X^H} Y^V)^H + (\nabla_{X^H} Y^H)^V \\ \nabla_X^Z Y &:= (\nabla_{X^V} Y^V)^V \\ \nabla_X^V Y &:= (\nabla_{X^H} Y^V)^V. \end{aligned}$$

We view ∇^V, ∇^Z as connections on V. Assume now that N is a spin manifold with a fixed spin structure. The submersion f induces local spin structures on B and V, so we have locally defined spinor bundles $\Sigma B, \Sigma V$. By (1.1), the spinor bundle on N is

$$\Sigma N = f^*(\mathbf{\Sigma} B) \otimes \Sigma V, \quad \text{where} \quad \mathbf{\Sigma} B = \begin{cases} \Sigma B, & \text{one of } n,k \text{ even} \\ \Sigma^+ B \oplus \Sigma^- B & n,k \text{ odd,} \end{cases} \tag{1.3}$$

with the Clifford multiplication as in (1.2). In particular, any section of ΣN is locally a $C^\infty(N)$-linear combination of sections of the form $f^*(\phi) \otimes \nu$, where ν is a section of ΣV and ϕ is a section of ΣB or $\Sigma^+ B \oplus \Sigma^- B$ depending on the parity of n,k. The connections ∇^V, ∇^Z on V induce connections on ΣV, also denoted ∇^V, ∇^Z. We let ∇^B denote the connection on $\mathbf{\Sigma} B$.

From now on we work in an orthonormal frame $\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_k$ so that the ζ_a are sections of V and the ξ_α are the horizontal lifts of an orthonormal frame for B. We will also identify sections of $\Sigma B, f^*(\Sigma B)$ for simplicity. The following identities are from [Roo20, Lemma 3.9], and they completely describe the spin connection and Dirac operator on N, when applied to $\varphi = \phi \otimes \nu$.

Lemma 1.1. *Let $f : N \to B$ be a Riemannian Submersion with N a spin manifold and let $\nabla^V, \nabla^Z, \nabla^B$ denote the associated connections. The following identities hold:*

$$
\begin{aligned}
\nabla^N_{\xi_\alpha}\varphi &= (\nabla^B_{\xi_\alpha}\phi) \otimes \nu + \phi \otimes (\nabla^V_{\xi_\alpha}\nu) + \frac{1}{2}\sum_{\beta=1}^{n} \xi_\beta \cdot A(\xi_\alpha, \xi_\beta) \cdot \varphi \\
&:= \nabla^\tau_{\xi_\alpha}\varphi + \frac{1}{2}\sum_{\beta=1}^{n} \xi_\beta \cdot A(\xi_\alpha, \xi_\beta) \cdot \varphi, \\
\nabla^N_{\zeta_a}\varphi &= \phi \otimes \nabla^Z_{\zeta_a}\nu + \frac{1}{2}\sum_{b=1}^{m} \zeta_b \cdot T(\zeta_a, \zeta_b) \cdot \varphi + \frac{1}{4}\sum_{\alpha=1}^{n} \xi_\alpha \cdot A(\xi_\alpha, \zeta_a) \cdot \varphi \\
&:= \nabla^Z_{\zeta_a}\varphi + \frac{1}{2}\sum_{b=1}^{m} \zeta_b \cdot T(\zeta_a, \zeta_b) \cdot \varphi + \frac{1}{4}\sum_{\alpha=1}^{n} \xi_\alpha \cdot A(\xi_\alpha, \zeta_a) \cdot \varphi, \qquad (1.4) \\
D^N\varphi &= \sum_{\alpha=1}^{n} \xi_\alpha \cdot \nabla^\tau_{\xi_\alpha}\varphi + \sum_{a=1}^{m} \zeta_a \cdot \nabla^Z_{\zeta_a}\varphi - \frac{1}{2}\sum_{a=1}^{m} T(\zeta_a, \zeta_a) \cdot \varphi + \frac{1}{2}\sum_{\alpha,\beta=1,\alpha<\beta}^{n} A(\xi_\alpha, \xi_\beta) \cdot \xi_\alpha \cdot \xi_\beta \cdot \varphi \\
&\quad + \frac{1}{2}\sum_{a,\alpha} \zeta_a \cdot \xi_\alpha \cdot A(\xi_\alpha, \zeta_a) \cdot \varphi \\
&:= D^\tau\varphi + D^Z\varphi - \frac{1}{2}\sum_{a=1}^{k} T(\zeta_a, \zeta_a) \cdot \varphi + \frac{1}{2}A \cdot \varphi + \frac{1}{2}\sum_{a,\alpha} \zeta_a \cdot \xi_\alpha \cdot A(\xi_\alpha, \zeta_a) \cdot \varphi.
\end{aligned}
$$

1.3 The Dirac operator on the warped product $M \times F$

In this subsection we use the identities (1.4) to describe the Dirac operator on the warped product $(M^n \times F^m, \overline{g}) = (M^n \times F^m, v^2 h)$ introduced at the beginning of §1. Using the notation of §1.2, we take $N = M \times F$, $B = M$ and $f = \pi_1 : M \times F \to M$. The spinor bundles $\Sigma M, \Sigma V$, which are now globally defined, induced from the spin structure on $M \times F$ are then precisely those associated to the fixed spin structure on M and the fixed trivial spin structure on F. For simplicity, we will work with the case when n is even. By (1.3), we have $\Sigma(M \times F) \cong \pi_1^*(\Sigma M) \otimes \Sigma V$. We identify sections ϕ of ΣM with their pullbacks $\pi_1^*(\phi)$ to sections of $\pi_1^*(\Sigma M)$. We will refer to sections of ΣV which are identified with the pullbacks of parallel sections of ΣF under the isomorphism $T : \Sigma V \to \pi_2^*(\Sigma F)$ as *parallel* sections of ΣV.

Recall that $v^m = e^{-f}$. In [BO22], the weighted Dirac operator $D^M_f = D_M - \frac{1}{2}\nabla f$ on a manifold M with the weighted volume $e^{-f}d\mathrm{Vol}_g$, introduced by Perelman, was discussed as a natural generalization of the standard Dirac operator D_M on M. The following result establishes a relationship between the weighted Dirac operator D^M_f and the standard Dirac operator $D_{M\times F}$ of the warped product. We denote by ∇^M the spin connection on M.

Theorem 1.2. *Let ϕ be a section of ΣM and ν a parallel section of ΣV, with M of even dimension. We have $D_{M\times F}(\phi \otimes \nu) = (D^M_f \phi) \otimes \nu$.*

We will prove Theorem 1.2 by first computing the connection and spin connection on $M \times F$ in an appropiate coordinate system. We work near a point (m_0, f_0), and choose coordinates $\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m$ for $M \times F$ so that $\xi_1, \dots, \xi_n$ are geodesic normal coordinates at a point $m_0 \in M$ and $\zeta_1 = e^{\frac{f}{m}}\partial_1, \dots, \zeta_m = e^{\frac{f}{m}}\partial_m$ are orthonormal at $f_0 \in F$, with ∂_i geodesic normal coordinates at $f_0 \in F$. The following lemma is proved in Appendix A.1:

Lemma 1.3. *The connection of $M \times F$ is given in the coordinates ξ_α, ζ_a by*

$$\begin{aligned}
\nabla_{\xi_\alpha}\xi_\beta &= 0\\
\nabla_{\xi_\alpha}\zeta_a &= 0\\
\nabla_{\zeta_a}\xi_\alpha &= -\frac{1}{m}\xi_\alpha(f)\zeta_a\\
\nabla_{\zeta_a}\zeta_b &= \frac{1}{m}\delta_{ab}\nabla f
\end{aligned}$$

Let ψ be a section of the spinor bundle $\Sigma(M\times F)$. The spin connection of $M\times F$ is given by

$$\begin{aligned}
\nabla_{\xi_a}\psi &= \xi_\alpha(\psi)\\
\nabla_{\zeta_a}\psi &= \zeta_a(\psi) + \frac{1}{2m}\zeta_a\cdot\nabla f\cdot\psi
\end{aligned}$$

Proof of Theorem 1.2. Fix $\psi = \phi\otimes\nu$ for a section ϕ of ΣM and a constant section ν of ΣV. Note that the assumption on ν implies that $\zeta_a(\nu)=0$ for all a. By Lemma 1.3,

$$\begin{aligned}
D_{M\times F}\psi &= \sum_{\alpha=1}^{n}\xi_\alpha(\phi)\otimes\nu - \frac{1}{2}(\nabla f\cdot\phi)\otimes\nu\\
&= D_f^M(\phi)\otimes\nu,
\end{aligned}$$

where the Clifford multiplications are implemented according to (1.2). When M^n has odd dimension, there are some minor differences with the Clifford multiplication when applying Lemma 1.3 as described in Equation 1.2. When m is even and n is odd, we still have $\Sigma(M\times F)\cong\pi_1^*(\Sigma M)\otimes\Sigma V$, but the Clifford multiplication yields the following modified expression for $D_{M\times F}\varphi = D_{M\times F}(\phi\otimes\nu)$

$$\begin{aligned}
D_{M\times F}\varphi &= \sum_{\alpha=1}^{n}\xi_\alpha\cdot\nabla^\tau_{\xi_\alpha}\varphi - \frac{1}{2}\nabla f\cdot\varphi\\
&= \sum_{\alpha=1}^{n}(\xi_\alpha\cdot\nabla^M_{\xi_\alpha}\phi)\otimes\overline{\nu} - \frac{1}{2}(\nabla f\cdot\phi)\otimes\overline{\nu}\\
&= D_f^M(\phi)\otimes\overline{\nu}
\end{aligned}$$

This is identified with the expression in the even case when ν is a section of the positive subbundle $\hat{\Sigma}V^+$. When both n,m are odd, we have $\Sigma(M\times F)\cong\pi_1^*(\Sigma M^+\oplus\Sigma M^-)\otimes\Sigma V$. For a section $\varphi=\Phi\otimes\nu$ with $\Phi=\phi^+\oplus\phi^-$ and ν constant, the Clifford multiplication yields the following expression for $D_{M\times F}\varphi$

$$\begin{aligned}
D_{M\times F}\varphi &= \sum_{\alpha=1}^{n}\xi_\alpha\cdot\nabla^\tau_{\xi_\alpha}\varphi - \frac{1}{2}\nabla f\cdot\varphi\\
&= \sum_{\alpha=1}^{n}(\xi_\alpha\cdot\nabla^M_{\xi_\alpha}\Phi)\otimes\nu - \frac{1}{2}(\nabla f\cdot\Phi)\otimes\nu\\
&= (D_f^M(\phi^+)\oplus -D_f^M(\phi^-))\otimes\nu
\end{aligned}$$

This also becomes identified with the expression in the even case when ϕ is a section of the bundle ΣM^+.

Remark 1.4. Theorem 1.2 shows that in a sense, the Dirac operator on the spinor bundle for a metric conformal to (M,g) 'is' the Dirac operator on a warped product $M\times F$ with a manifold F admitting parallel spinors. To make this precise, consider a conformal change of metric $\overline{g}=e^{2u}g$. Let $\Sigma M,\Sigma\overline{M}$ be the spinor bundles associated to $g,\overline{g}$ respectively, and denote their Dirac operators by $D,\overline{D}$. The conformal change induces a natural isometry of spinor bundles $\sigma:\Sigma M\to\Sigma\overline{M}$ (see [BHM+15, §2.3.5], and the Dirac operators are related by

$$(e^u\sigma^{-1}\overline{D}\sigma)(\psi) = D\psi + \frac{(n-1)}{2}\nabla u\cdot\psi,\quad \psi\in\Gamma(\Sigma M). \tag{1.5}$$

The operator $e^u\sigma^{-1}\overline{D}\sigma$ may then be seen as a perturbed Dirac operator D, with the perturbation depending on the conformal factor u. Now suppose F is m-dimensional and admits parallel spinors. Construct the warped product $(M\times F, g\oplus v^2 g_F)$ with warping factor $v=e^{\frac{(n-1)u}{m}}$. Applying Theorem 1.2 and (1.5) to this warped product (recall that $v^m=e^{-f}$), we see that on spinors of the form $\phi\otimes\nu$ where ν is parallel,

$$D_{M\times F}(\phi\otimes\nu)=((e^u\sigma^{-1}\overline{D}\sigma)\phi)\otimes\nu.$$

This shows that the effect of a conformal change, insofar as spinors are concerned, can be detected in two ways: (i) a perturbation of the Dirac operator as in (1.5), and (ii) a change in spinor bundle, namely by considering that of a warped product where the degree of the warping depends on the conformal factor.

We note down a consequence of (1.5) for later: if (M,g,f) is a weighted manifold, then setting $u=-\frac{f}{n-1}$, we have

$$D_f\psi=(e^{-\frac{f}{n-1}}\sigma^{-1}\bar{D}\sigma)(\psi). \tag{1.6}$$

There is also a relationship with spin$^{\mathbb{C}}$ Dirac operators. If M is given the trivial spin$^{\mathbb{C}}$ structure, then the spin$^{\mathbb{C}}$ Dirac operator for the connection $\frac{(n-1)}{2}du$ is given by the same expression as (1.5). More generally, if M is a spin$^{\mathbb{C}}$ manifold with a closed spin$^{\mathbb{C}}$ connection, such a connection is locally the differential of some function, hence the spin$^{\mathbb{C}}$ Dirac operator locally agrees with the Dirac operator on a warped product in the sense of Theorem 1.2.

Since both F and the weight v^m depend on m, one might find it surprising that there is no dependence on m in $D_{M\times F}\varphi$. It turns out that all such dependence is accounted for by the spinor Laplacian $-\Delta_{M\times F}$ as we will now show. Recall that the scalar curvature of $(M\times F,\overline{g})$ is given by the Bakry-Emery scalar curvature $R_f^m=R+2\Delta f-\frac{(m+1)}{m}|\nabla f|^2$. Using that ν is constant and Lemma 1.3, we have

$$\begin{aligned}
\Delta^{M\times F}\varphi &= \sum_\alpha(\nabla_{\xi_\alpha}\nabla_{\xi_\alpha}\varphi-\nabla_{\nabla_{\xi_\alpha}\xi_\alpha}\varphi)+\sum_a(\nabla_{\zeta_a}\nabla_{\zeta_a}\varphi-\nabla_{\nabla_{\zeta_a}\zeta_a}\varphi)\\
&=\sum_\alpha\nabla_{\xi_\alpha}\nabla_{\xi_\alpha}\varphi+\sum_a(\nabla_{\zeta_a}\nabla_{\zeta_a}\varphi-\nabla_{A(\zeta_a,\zeta_a)}\varphi)\\
&=\sum_\alpha(\nabla^M_{\xi_\alpha}\nabla^M_{\xi_\alpha}\phi)\otimes\nu+\sum_a\phi\otimes(\nabla^Z_{\zeta_a}\nabla^Z_{\zeta_b}\nu)+\frac{1}{4m^2}\sum_{a,b}\zeta_a\zeta_b|\nabla f|^2-\nabla_{\nabla f}\varphi\\
&=\sum_\alpha(\nabla^M_{\xi_\alpha}\nabla^M_{\xi_\alpha}\phi)\otimes\nu-\frac{1}{4m}|\nabla f|^2-\nabla_{\nabla f}\varphi
\end{aligned}$$

By Proposition 1.8 in [BO22], one has $(D_f^M)^2\phi=-\Delta_f\phi+\frac14 R_f\phi$, where $R_f=R+2\Delta f-|\nabla f|^2$ is the $m\to\infty$ limit of the Bakry-Emery scalar curvature, and Δ_f is the weighted Laplacian $\Delta_f\phi=\Delta_M\phi-\nabla_{\nabla f}\phi$. Using this we have

$$\begin{aligned}
-\Delta^{M\times F}\varphi+\frac14 R_f^m\varphi &=-\sum_\alpha(\nabla^M_{\xi_\alpha}\nabla^M_{\xi_\alpha}\phi)\otimes\nu+\frac{1}{4m}|\nabla f|^2+\nabla_{\nabla f}\varphi+\frac14(R+2\Delta f-\frac{(m+1)}{m}|\nabla f|^2)\varphi\\
&=(-\Delta_f^M\phi+R_f\phi)\otimes\nu\\
&=D^2_{M\times F}\varphi
\end{aligned}$$

This verifies the Lichnerowicz Formula for $M\times F$. Alternatively, this gives a proof of the weighted Lichnerowicz formula $(D_f^M)^2\phi=-\Delta_f\phi+\frac14 R_f\phi$ in [BO22] using the standard Lichnerowicz formula for $D_{M\times F}$. We will apply this same method to give a proof of the weighted Ricci identity [BO22, Proposition 1.15]:

$$[D_f,\nabla_X]\phi=\frac12\mathrm{Ric}_f(X)\cdot\phi \tag{1.7}$$

for $X\in TM,\phi\in\Gamma(\Sigma M)$. Here Ric_f is the P-scalar curvature $\mathrm{Ric}+\mathrm{Hess}_f$. In fact, we will prove the following m-dependent version of the weighted Ricci identity, from which (1.7) is recovered by taking $m\to\infty$. Note that we are viewing $\mathrm{Ric}_f^m(X)$ as a tangent vector to M.

Proposition 1.5. *Let $X\in TM$ and $\varphi\in\Gamma(\Sigma(M\times F))$ be of the form $\varphi=\phi\otimes\nu$, where $\phi\in\Gamma(\Sigma M)$ and $\nu\in\Gamma(V)$ is parallel. Then*

$$([D_f^M,\nabla_X]\phi)\otimes\nu=[(\mathrm{Ric}_f^m(X)+\frac1m X(f)\nabla f)\cdot\phi)]\otimes\nu.$$

Proof. The usual Ricci identity (see e.g. [BHM+15, Corollary 2.8]) applied to $M \times F$ yields

$$\frac{1}{2}\mathrm{Ric}_{M\times F}(X)\cdot(\phi\otimes\nu) = \sum_\alpha \xi_\alpha \cdot (\nabla_{\xi_\alpha}\nabla_X - \nabla_X\nabla_{\xi_\alpha} - \nabla_{[\xi_\alpha,X]})(\phi\otimes\nu)$$
$$+\sum_a \zeta_a \cdot (\nabla_{\zeta_a}\nabla_X - \nabla_X\nabla_{\zeta_a} - \nabla_{[\zeta_a,X]})(\phi\otimes\nu).$$

From the calculations in the last section, we have $\nabla_{\xi_\alpha}\xi_\beta = \nabla_{\xi_\alpha}\zeta_a = \nabla_{\zeta_a}\zeta_b = 0$, $\nabla_{\zeta_a}\xi_\alpha = -\frac{1}{m}\xi_\alpha(f)\zeta_a$. and the following identities

$$\nabla_X(\phi\otimes\nu) = (\nabla_X\phi)\otimes\nu$$
$$\nabla_{\zeta_a}(\phi\otimes\nu) = \frac{1}{2m}(\nabla f\cdot\phi)\otimes(\zeta_a\cdot\nu)$$

These imply that

$$\begin{aligned}\frac{1}{2}\mathrm{Ric}_{M\times F}(X)\cdot(\phi\otimes\nu) &= [D^M,\nabla_X](\phi\otimes\nu) + \sum_a(\zeta_a\nabla_{\zeta_a}\nabla_X - \nabla_X(\zeta_a\nabla_{\zeta_a}))(\phi\otimes\nu)\\ &\quad+\sum_a \frac{1}{m}X(f)\nabla_{\zeta_a}(\phi\otimes\nu)\\ &= [D^M_f,\nabla_X](\phi\otimes\nu) - \frac{1}{2m}X(f)\nabla f\cdot(\phi\otimes\nu)\\ &= ([D^M_f,\nabla_X]\phi)\otimes\nu - \frac{1}{2m}(X(f)\nabla f\cdot\phi)\otimes\nu.\end{aligned}$$

Note that $\mathrm{Ric}_{M\times F}(X)$ is a horizontal vector, since $\mathrm{Ric}_{M\times F}(X,Y)=0$ for all vertical vectors $Y\in V$ (see, e.g. [Bes07, Proposition 9.106]). Thus the above becomes

$$([D^M_f,\nabla_X]\phi)\otimes\nu = [(\mathrm{Ric}_{M\times F}(X) + \frac{1}{m}X(f)\nabla f)\cdot\phi)]\otimes\nu.$$

The proposition follows from this once it is proved that $\mathrm{Ric}_{M\times F}(X) = \mathrm{Ric}^m_f(X)$. Using that $g=\bar g$ and $\mathrm{Ric}_{M\times F} = \mathrm{Ric}^m_f$ on horizontal vectors, as well as the definitions of $\mathrm{Ric}_{M\times F}, \mathrm{Ric}^m_f$, we see that

$$g(\mathrm{Ric}_{M\times F}(X),X') = \bar g(\mathrm{Ric}_{M\times F}(X),X') = \mathrm{Ric}_{M\times F}(X,X') = \mathrm{Ric}^m_f(X,X') = g(\mathrm{Ric}^m_f(X),X')$$

for all horizontal vectors $X'\in TM$. Thus $\mathrm{Ric}_{M\times F}(X) = \mathrm{Ric}^m_f(X)$.

□

Theorem 1.2 also allows us to show an m-dependent bound on the eigenvalues of the Dirac Operator of M when M is compact. Recall that a SMMS $(M^n, g, v^m dVol_g, m)$ is *Quasi-Einstein* if there exists $\lambda\in\mathbb{R}$ such that $Ric^m_f = \lambda g$ (see [Cas12, Definition 4.6]). In [BO22], the following weighted Frederich Inequality was shown.

Lemma 1.6. *(Theorem 1.23 in [BO22]) Suppose (M^n,g) is closed, let $f\in C^\infty(M)$ and λ an eigenvalue of the Dirac Operator D of M. Then $\lambda^2 \geq \frac{n}{4(n-1)}\min R_f$, with equality if and only if f is constant and (M,g) admits a Killing spinor φ for which $\nabla_X\varphi = -\frac{\lambda}{n}X\cdot\varphi$, in which case (M,g) is Einstein.*

We have the following result

Corollary 1.7. *Let $D, D_{M\times F}$ be the Dirac Operators on $(M^n,g),(M^n\times F^m,\bar g)$ respectively.*

(a) Any eigenvalue of D is an eigenvalue of $D_{M\times F}$.

(b) When M,F are closed, any eigenvalue $\lambda(D_{M\times F})$ of $D_{M\times F}$ satisfies $\lambda(D_{M\times F})^2 \geq \frac{n+m}{4(n+m-1)}\min R^m_f$. If equality is reached, then M is Quasi-Einstein.

(c) If M,F are closed, and $\lambda(D)^2 = \frac{n+m}{4(n+m-1)}\min R^m_f$ for an eigenvalue $\lambda(D)$ of D, then $\lambda(D)=0, f$ is constant and M is Ricci Flat and admits a parallel spinor. Conversely, if f is constant and M is Ricci Flat admitting a harmonic spinor, then equality holds with $0=\lambda(D)^2 = R^m_f$ and the harmonic spinor is parallel.

(d) If M is closed and $R_f^m \geq 0$ with $R_f^m > 0$ at some point in M, then M admits no nonzero harmonic spinors.

Proof. (a) By unitary equivalence of the weighted Dirac Operator D_f on M and the ordinary Dirac operator D (see [BO22, Proposition 1.20]), it is sufficient to show that any eigenvalue of D_f is an eigenvalue of $M \times F$. Let φ be an eigenspinor of D_f for the eigenvalue λ, and let ν be a parallel section of ΣV. Suppose first that n is even. We have $D_{M\times F}(\varphi \otimes \nu) = D_f(\varphi) \otimes \nu = \lambda(\varphi \otimes \nu)$ by Theorem 1.2. Suppose next that n is odd and m is even. We can assume that the parallel spinor ν is in the positive subbundle $\hat{\Sigma}V^+$ by taking the positive part of the decomposition $\nu = \nu^+ \oplus \nu^-$, which we can assume is nonzero , since the splitting $\Sigma F = \hat{\Sigma}F^+ \oplus \hat{\Sigma}F^-$ is preserved by the spin connection on F ([LM89, Corollary 4.12]). We then have $D_{M\times F}(\varphi \otimes \nu) = \lambda(\varphi \otimes \overline{\nu}) = \lambda(\varphi \otimes \nu)$. When n, m are both odd, we also have $D_{M\times F}(\varphi \otimes \nu) = \lambda(\varphi \otimes \nu)$ after pulling back φ to a section of the bundle $\Sigma M^+ \subset \mathbf{\Sigma} M$.

(b) When M, F are closed, $M \times F$ is also closed, thus since $R_{\overline{g}} = R_f^m$, Friedrich's Inequality ([BHM+15, Theorem 5.3]) for $(M \times F, \overline{g})$ implies that $\lambda(D_{M\times F}) \geq \frac{n+m}{4(n+m-1)} \min R_f^m$, and equality implies that $(M \times F, \overline{g})$ is Einstein with Einstein constant some $\lambda \in \mathbb{R}$. For tangent vector fields $X, X' \in \Gamma(TM)$, one has $Ric_{\overline{g}}(X, X') = Ric_f^m(X, X')$, thus in the equality case we have

$$\lambda g(X, X') = \lambda \overline{g}(X, X') = Ric_{\overline{g}}(X, X') = Ric_f^m(X, X')$$

hence M is Quasi-Einstein.

(c) Suppose $\lambda(D)^2 = \frac{n+m}{4(n+m-1)} \min R_f^m$. Since $m \in \mathbb{N}$, we have $R_f \geq R_f^m$ and $\frac{n}{4(n-1)} > \frac{n+m}{4(n+m-1)}$, thus $\frac{n}{4(n-1)} \min R_f \geq \frac{n+m}{4(n+m-1)} \min R_f^m$ with equality if and only if $\min R_f = \min R_f^m = 0$. Lemma 1.6 implies

$$\frac{n+m}{4(n+m-1)} \min R_f^m = \lambda(D)^2 \geq \frac{n}{4(n-1)} \min R_f \geq \frac{n+m}{4(n+m-1)} \min R_f^m$$

Thus $\lambda(D) = 0, \min R_f = 0$. By Lemma 1.6, it follows that f is constant and M admits a parallel spinor. For the converse, note that when f is constant and M is Ricci Flat, we have $\min R_f^m = \min R = 0$. On a scalar flat closed manifold, any harmonic spinor is automatically parallel by [LM89, Corollary 8.10].

(d) If M had some nonzero harmonic spinor, then by (c), we must have $R_f^m = 0$, a contradiction. This can also be proven directly by integrating the Lichnerowicz formula for $D_{M\times F}$ as in [LM89, Corollary 8.9].

□

The bound from part b of Corollary 1.7 is weaker than the bound in Lemma 1.6 even in the limit $m \to \infty$, but has an interesting proof based on the geometry of warped products. In Section 2.2.4, we discuss stronger results for $m < 0$.

1.4 The spinor norm on $\Sigma(M \times F)$

We end this section with a description of the spinor norm on $\Sigma(M \times F)$, which will be used in the proof of the Bakry–Emery positive mass theorem in Section 2. Recall that an irreducible complex representation Σ_n of the complex Clifford algebra $\mathbb{C}l_n$ can be given a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}l_n}$ for which $\langle v \cdot \varphi_1, v \cdot \varphi_2 \rangle_{\mathbb{C}l_n} = \langle \varphi_1, \varphi_2 \rangle_{\mathbb{C}l_n}$ for all $v \in \mathbb{R}^n$ with $|v| = 1$. Such an inner product is constructed by choosing an orthonormal frame $e_1, \ldots, e_n$ for $\mathbb{R}^n$ and averaging an arbitrary inner product over the multiplicative group generated by $e_1, \ldots, e_n$ in $\mathbb{C}l_n$; see [LM89, Proposition 5.16] for details. The inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}l_n}$ is unique up to a positive constant, as any such inner product yields an isomorphism of irreducible representations $\Sigma_n \to \overline{\Sigma}_n^*$ between Σ_n and its conjugate dual representation, and any two such isomorphisms differ by a constant by the Schur Lemma.

Applying this fact globally, any spin manifold (N, g) admits an inner product $\langle \cdot, \cdot \rangle_N$ on its complex spinor bundle ΣN for which multiplication by unit tangent vectors is unitary. Over a local trivialization of TN, this inner product can be obtained by averaging the restriction of an arbitrary inner product on ΣN over an orthonormal frame and then scaling by some positive function. Consider now the warped product manifold $(M \times F, \overline{g})$ with the spin structure described at the beginning of Section 1. Recall that $\Sigma(M \times F) \cong \pi_1^*(\mathbf{\Sigma} M) \otimes \Sigma V$, where $\mathbf{\Sigma} M$ is given by

$$\mathbf{\Sigma} M = \begin{cases} \Sigma M & \text{one of } n, k \text{ even} \\ \Sigma^+ M \oplus \Sigma^- M & n, k \text{ odd} \end{cases}$$

We will identify sections of $\mathbf{\Sigma}M$ with their pullbacks to $\pi_1^*(\mathbf{\Sigma}M)$. We fix an inner product $\langle , \rangle_{M\times F}$ on $\Sigma(M\times F)$ for which Clifford multiplication by unit vectors in $T(M\times F)$ is unitary, and describe this inner product in a trivialization of $\Sigma(M\times F)$. We work now over a trivialization of $T(M\times F)$ in an orthonormal frame $\xi_1,\dots\xi_n,\zeta_1,\dots,\zeta_n$ following the conventions in subsection 1.2.

Suppose first that one of n,k is even. Let $\langle , \rangle_M, \langle , \rangle_V$ be inner products on $\mathbf{\Sigma}M=\Sigma M, \Sigma V$ for which Clifford multiplication by unit vectors in TM, V is unitary respectively. We construct $\langle , \rangle_M$ by pulling back an invariant metric on M, and $\langle , \rangle_V$ by averaging a constant inner product (this makes sense since ΣV is trivial) over the group generated by the orthonormal frame ζ_i which are each constant along the F directions, so that $\langle , \rangle_V$ is also constant along the F directions. Define an inner product $\langle , \rangle$ on $\Sigma(M\times F)$ on elementary tensors $\phi\otimes\nu$ by

$$\langle \phi\otimes\nu, \phi'\otimes\nu'\rangle = \langle\phi,\phi'\rangle_M\langle\nu,\nu'\rangle_V$$

and extending bilinearly. Averaging $\langle , \rangle$ over the orthonormal frame yields an inner product $\langle , \rangle^2_{M\times F}$ for which clifford multiplication by unit vectors in $T(M\times F)$ is unitary. We claim that $\langle , \rangle^2_{M\times F}$ agrees with $\langle , \rangle$ on elementary tensors. Recall the expressions for the Clifford multiplication on $\Sigma(M\times F)$ in 1.2. When n is even, we have

$$\begin{aligned}\langle \xi_\alpha\cdot\phi\otimes w, \xi_\alpha\cdot\phi'\otimes w'\rangle &= \langle\xi_\alpha\cdot\phi, \xi_\alpha\cdot\phi'\rangle\langle w,w'\rangle = \langle\phi,\phi'\rangle\langle w,w'\rangle\\ \langle \zeta_a\cdot\phi\otimes w, \zeta_a\cdot\phi'\otimes w'\rangle &= \langle\overline{\phi},\overline{\phi'}\rangle\langle\zeta_a\cdot w,\zeta_a\cdot w'\rangle = \langle\phi,\phi'\rangle\langle w,w'\rangle\end{aligned}$$

where we use that complex conjugation preserves inner products, which can be seen by working in an orthonormal basis of ΣM which respects the $\langle , \rangle_M$-orthogonal splitting $\Sigma_n = \hat{\Sigma}_n^-\oplus\hat{\Sigma}_n^+$ into eigenspaces of the complex volume element. The computation for when k is even is entirely analogous, except that w,w' are conjugated instead of ϕ,ϕ'. Thus up to scaling $\langle , \rangle_{M\times F}$, we can assume it is given on elementary tensors by Equation 1.4.

Suppose next that n,k are both odd. Let $\langle , \rangle_V, \langle , \rangle_M$ be as in the last case, and let $\langle , \rangle_{\mathbf{\Sigma}M}$ be a Hermitian inner product on $\mathbf{\Sigma}M$ given by taking the direct sum of $\langle , \rangle_M$ with itself. For sections $\Phi=\phi^+\oplus\phi^-$, $\Phi'=(\phi^+)'\oplus(\phi^-)'$ of $\mathbf{\Sigma}M=\Sigma^+M\oplus\Sigma^-M$, we have $\langle\Phi,\Phi'\rangle_{\mathbf{\Sigma}M}=\langle\phi^+,\phi^+\rangle_M+\langle\phi^-,\phi^-\rangle_M$. Define an inner product $\langle , \rangle$ on $\Sigma(M\times F)$ on elementary tensors $\Phi\otimes\nu$ by

$$\langle\Phi\otimes\nu,\Phi'\otimes\nu'\rangle = \langle\Phi,\Phi'\rangle_{\mathbf{\Sigma}M}\langle\nu,\nu'\rangle_V$$

Let $\langle , \rangle^3_{M\times F}$ denote the inner product obtained by averaging $\langle , \rangle$ over the chosen orthonormal frame. Note that

$$\begin{aligned}\langle\xi_\alpha\cdot\Phi,\xi_\alpha\cdot\Phi'\rangle_{\mathbf{\Sigma}M} &= \langle\xi_\alpha\cdot\phi^+,\xi_\alpha\cdot(\phi^+)'\rangle_M + \langle -\xi_\alpha\cdot\phi^-, -\xi_\alpha\cdot(\phi^-)'\rangle_M\\ &= \langle\phi^+,(\phi^+)'\rangle_M + \langle\phi^-,(\phi^-)'\rangle_M\\ &= \langle\Phi,\Phi'\rangle_{\mathbf{\Sigma}M}\end{aligned}$$

We have

$$\begin{aligned}\langle\xi_\alpha\cdot\Phi\otimes w,\xi_\alpha\cdot\Phi'\otimes w'\rangle &= \langle\xi_\alpha\cdot\Phi,\xi_\alpha\cdot\Phi'\rangle\langle w,w'\rangle = \langle\Phi,\Phi'\rangle\langle w,w'\rangle\\ \langle\zeta_a\cdot\Phi\otimes w,\zeta_a\cdot\Phi'\otimes w'\rangle &= \langle\Phi,\Phi'\rangle\langle\zeta_a\cdot w,\zeta_a\cdot w'\rangle = \langle\Phi,\Phi'\rangle\langle w,w'\rangle\end{aligned}$$

Thus $\langle , \rangle^3_{M\times F}$ agrees with $\langle , \rangle$ on elementary tensors, and up to scaling $\langle , \rangle_{M\times F}$ we can assume it is given on elementary tensors by Equation 1.4.

2 The Bakry-Émery Positive Mass Theorems

2.1 The warped product method

2.1.1 Some consequences of the results in [Dai04]

In [Dai04], a positive mass theorem is shown for manifolds N^{n+m} which outside of a compact set N_{cpct} are diffeomorphic to the product $(\mathbb{R}^n \setminus B_R(0)) \times F$ for (F^m, g_F) a compact simply connected manifold admitting parallel spinors, so that the metric over the end $(\mathbb{R}^n \setminus B_R(0)) \times F$ is asymptotic to the product metric $\mathring{g} = g_{\mathbb{R}^n} \oplus g_F$ on $\mathbb{R}^n \times F^m$. In particular, the metric on M_∞ is required to satisfy the following asymptotic conditions, where $\mathring{\nabla}$ is the Levi-Civita connection of the product metric $\mathring{g}$, and r is the distance on the $\mathbb{R}^n$ factor induced from the asymptotic coordinates:

$$g = \mathring{g} + h,\ h = O(r^{-\tau}),\ \mathring{\nabla} h = O(r^{-\tau-1}),\ \mathring{\nabla}\mathring{\nabla} h = O(r^{-\tau-2})$$

The mass of N is defined by the following integral in the coordinates induced from the diffeomorphism $N \setminus N_{cpct} \cong (\mathbb{R}^n \setminus B_R(0)) \times F$:

$$\mathfrak{m}(g) = \int_{S_r \times F} (\partial_\alpha g_{\alpha\beta} - \partial_\beta g_{ii}) \partial_\beta \lrcorner d\text{Vol}_{\overline{g}}$$

We refer to the vector field $(\partial_\alpha g_{\alpha\beta} - \partial_\beta g_{ii})\partial_\beta$ as the *mass density field* $\overline{\mu}$ of N. Note that we use a different convention for the mass than that in [Dai04], but it is the same mass up to a positive constant. Following the conventions in Section 1, the indexes α, β run over the Euclidean factor, while the i index runs over the full range of N. The following analogue of the positive mass theorem holds for N satisfying these conditions.

Lemma 2.1. *(Theorem 0.2 in [Dai04]). Let N be a complete spin manifold as above with order $\tau > \frac{n-2}{2}$ and $n \geq 3$. If N has non-negative scalar curvature, then $m(g) \geq 0$ with equality if and only if $M = \mathbb{R}^n \times F$.*

Lemma 2.1 applies in the following setting, where the results of Section 1 are especially relevant.

Definition 2.2. *Let $(M^n, g, v^m d\text{Vol}_g, m)$, $m \in \mathbb{N}$, be an asymptotically flat smooth metric measure space of order τ with $e^{-f} = v^m$ and $f \in \mathcal{C}^{2;\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, and let F be a closed manifold. A metric $\tilde{g}$ on $M \times F$ is asymptotically warped if on the end $(M \setminus M_{cpct}) \times F$ it is of the form $\overline{g} + h$, where $\overline{g}$ is the warped product metric $g \oplus v^2 g_F$ and h satisfies the following asymptotic conditions*

$$h = O(r^{-\tau}),\ \mathring{\nabla} h = O(r^{-\tau-1}),\ \mathring{\nabla}\mathring{\nabla} h = O(r^{-\tau-2})$$

where $\mathring{\nabla}$ is the Levi-Civita connection on the product $\mathbb{R}^n \times F$. We refer to $(M \times F, \tilde{g})$ as an asymptotically warped manifold.

An immediate consequence of Lemma 2.1 is the following result:

Theorem 2.3. *Let $(M \times F, \tilde{g})$ be an asymptotically warped manifold, with M, F complete spin manifolds and F a closed manifold admitting parallel spinors with respect to some spin structure. If the scalar curvature $R_{\tilde{g}}$ is non-negative, then $\mathfrak{m}(\tilde{g}) \geq 0$ with equality if and only if $(M \times F, \tilde{g})$ is isometric to $\mathbb{R}^n \times F$.*

Proof. We first show that the warped product manifold $(M \times F, \overline{g})$ satisfies the conditions required to apply Lemma 2.1. Since the metric g itself already satisfies each asymptotic condition in, it suffices to show that $v^2 g_{F,ij}$ satisfies these conditions. Since $v^m = e^{-f}$ and $f \in \mathcal{C}^{2;\alpha}_{-\tau}(M)$, $f = \mathcal{O}(r^{-\tau})$, so there is a constant C_1 such that $-f \leq C_1 r^{-\tau}$, where $r = |x|$. Then

$$\begin{aligned} v^2 &= e^{-2f/m} \\ &= (e^{-f})^{2/m} \\ &\leq e^{(2/m)C_1 r^{-\tau}}. \end{aligned}$$

Using the Taylor series expansion for e^x, it follows that

$$\begin{aligned} e^{(2/m)C_1 r^{-\tau}} &= 1 + \frac{2C_1}{m} r^{-\tau} + \frac{2C_1^2}{m^2} r^{-2\tau} + \cdots \\ &= 1 + \mathcal{O}(r^{-\tau}). \end{aligned}$$

Similarly, there is a constant C_2 such that $C_2 r^{-\tau} \leq -f$ and $e^{(2/m)C_2 r^{-\tau}} = 1 + \mathcal{O}(r^{-\tau})$. Then

$$v^2 = 1 + \mathcal{O}(r^{-\tau}),$$

hence

$$v^2 g_{F,ij} = g_{F,ij} + \mathcal{O}(r^{-\tau}),$$

from which the first asymptotic property of 2.1.1 follows. As for the second property, we have

$$\partial_k(v^2 g_{F,ij}) = (2v)v_k g_{F,ij}.$$

Since $v = 1 + \mathcal{O}(r^{-\tau})$, $v_k = \mathcal{O}(r^{-\tau-1})$. Then

$$\begin{aligned}\partial_k(v^2 g_{F,ij}) &= (2 + \mathcal{O}(r^{-\tau}))\mathcal{O}(r^{-\tau-1})g_{F,ij} \\ &= \mathcal{O}(r^{-\tau-1}) + \mathcal{O}(r^{-2\tau-1}) \\ &= \mathcal{O}(r^{-\tau-1}).\end{aligned}$$

The verification of the third property is analogous. This shows that the asymptotically warped manifold satisfies the correct decay conditions outside of the compact set $M_{cpct} \times F$, but we can not yet apply Lemma [Dai04] because we are not assuming the factor F is simply connected. However, the simply connected assumption on F is only necessary in the proof in [Dai04] to ensure that the spin structure on N restricts to the product spin structure on the asymptotic end $(\mathbb{R}^n \setminus B_R(0)) \times F$. Since in our case $N = M \times F$ is globally a product, we may give it the product spin structure as in subsection 1.1 to guarantee this is the case. □

Theorem 2.3 is of interest because the warped product $(M \times F^m, \overline{g})$ has scalar curvature equal to the Bakry-Emery scalar curvature R_f^m of the SMMS M. Moreover, by Theorem 1.2, as long as the projection onto the first factor $\pi_1 : M \times F \to M$ remains a Riemannian submersion, the Dirac Operator of $(M \times F, \tilde{g})$ can be described as a perturbation of the weighted Dirac Operator on M using the formulas in 1.4.

2.1.2 The Warped Bakry-Émery Positive Mass Theorem

In the case of manifolds that are actually warped, the results of Section 1 apply directly to give a simplified proof of Theorem 2.3, and the Weighted Positive Mass Theorem ([BO22, Theorem 2.13]) can then be seen as a special case of Theorem 2.3. In particular, when the manifold with parallel spinors F is a flat torus T^m normalized so that $\mathrm{Vol}(T^m) = 1$, one recovers the Bakry-Émery mass, equal to the weighted mass introduced in [BO22], and the weighted Witten's formula from the mass and Witten's formula for the warped product $M \times F$. Motivated by this, we will refer to the following special case of Theorem 2.3 as the *warped Bakry-Émery positive mass theorem.*

Theorem 2.4 (Warped Bakry-Émery positive mass theorem)**.** *Suppose $(M^n, g, v^m d\mathrm{Vol}_g, m)$, $m \in \mathbb{N}$, is an asymptotically Euclidean smooth metric measure space of order $\tau > \frac{n-2}{2}$, where (M, g) is a complete spin manifold. Let (F^m, g_F) be a complete, closed spin manifold admitting parallel spinors with respect to some spin structure, and let $\overline{g} = g \oplus v^2 g_F$ be the warped product metric on $M \times F$. Assume $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, $R_f^m \geq 0$, and $R_f^m \in L^1(M, g)$. Then $\mathfrak{m}(\overline{g}) \geq 0$, with equality if and only if $(M \times F, \overline{g})$ is isometric to $\mathbb{R}^n \times F$ and f is identically 0.*

We first show that, when F is a flat torus with $g_F = \delta_{ij}$, the following surprising relationship holds between the mass of the warped product and the weighted mass with respect to f.

Theorem 2.5. *The mass of $(M \times F, \overline{g})$ is precisely the weighted mass of (M^n, g) with respect to f, i.e. the Bakry-Émery mass of $(M^n, g, v^m d\mathrm{Vol}_g, m)$.*

Proof. Note that

$$\overline{g}_{ij} = \begin{cases} g_{ij} & i, j \leq n \\ 0 & i \leq n,\ n+1 \leq j \leq n+m \ \text{ or } \ n+1 \leq i \leq n+m,\ j \leq n\,. \\ v^2 h_{ij} & n+1 \leq i, j \leq n+m \end{cases} \tag{2.1}$$

Then since g_{ij} and v only depend on the M coordinates, we have by (2.1)

$$(\partial_i \overline{g}_{ij} - \partial_j \overline{g}_{ii})\partial_j = \begin{cases} (\partial_i g_{ij} - \partial_j g_{ii})\partial_j & \text{if } i, j \leq n, \\ -\partial_j (v^2)\partial_j & \text{if } i > n, j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \bar{\mu} &= \sum_{i,j=1}^{n+m} (\partial_i \overline{g}_{ij} - \partial_j \overline{g}_{ii})\partial_j \\ &= \sum_{i,j=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii})\partial_j - m \sum_{j=1}^{n} \partial_j (v^2)\partial_j \\ &= \mu - 2mv \sum_{j=1}^{n} v_j \partial_j. \end{aligned} \tag{2.2}$$

This last expression is constant along each fiber. Using (2.2) and the relation $v^m = e^f$, $\overline{\mu}$ can be written as

$$\begin{aligned} \overline{\mu} &= \mu + 2e^{-\frac{2f}{m}} \sum_{j \leq n} (\partial_j f)\partial_j \\ &= \mu + 2e^{-\frac{2f}{m}} \nabla f, \end{aligned}$$

where μ is the mass-density vector field of (M^n, g). Consequently, the mass of $(M \times F, \overline{g})$ may be written as

$$\begin{aligned} \mathfrak{m}(\overline{g}) &= \lim_{\rho \to \infty} \int_{S_\rho^M \times F} (\mu + 2e^{-\frac{2f}{m}} \nabla f) \lrcorner d\mathrm{Vol}_{\overline{g}} \\ &= \underbrace{\lim_{\rho \to \infty} \int_{S_\rho^M \times F} \mu \lrcorner d\mathrm{Vol}_{\overline{g}}}_{=I_1} + \underbrace{2 \lim_{\rho \to \infty} \int_{S_\rho^M \times F} e^{-\frac{2f}{m}} \nabla f \lrcorner d\mathrm{Vol}_{\overline{g}}}_{=I_2} \end{aligned} \tag{2.3}$$

We compute

$$\begin{aligned} d\mathrm{Vol}_{v^2 h} &= \sqrt{\det(v^2 h)} dx_{n+1} \wedge \cdots \wedge dx_{n+m} \\ &= \sqrt{v^{2m} \det(h)} dx_{n+1} \wedge \cdots \wedge dx_{n+m} \\ &= v^m d\mathrm{Vol}_h. \end{aligned}$$

We may now use Fubini's theorem to write I_1 as

$$\begin{aligned} I_1 &= \lim_{\rho \to \infty} \int_F \left(\int_{S_\rho^M} \mu \lrcorner v^m d\mathrm{Vol}_g \right) d\mathrm{Vol}_h \\ &= \lim_{\rho \to \infty} \int_{S_\rho^M} \mu \lrcorner v^m d\mathrm{Vol}_g, \end{aligned}$$

where we used the fact that $\mathrm{Vol}(F) = 1$. Since $v^m = e^{-f} = 1 + \mathcal{O}(r^{-\tau})$ and $\mu = \mathcal{O}(r^{-\tau-1})$, it follows that

$$\begin{aligned} I_1 &= \lim_{\rho \to \infty} \int_{S_\rho^M} \mu [1 + \mathcal{O}(r^{-\tau})] \lrcorner d\mathrm{Vol}_g \\ &= \lim_{\rho \to \infty} \int_{S_\rho^M} \mu \lrcorner d\mathrm{Vol}_g. \end{aligned} \tag{2.4}$$

Similarly, since $\nabla f = \mathcal{O}(r^{-\tau-1})$,

$$\begin{aligned} I_2 &= \lim_{\rho\to\infty}\int_F \left(\int_{S^M_\rho} e^{-\frac{2f}{m}}\nabla f \lrcorner e^{-f} d\mathrm{Vol}_g\right) d\mathrm{Vol}_h \\ &= \lim_{\rho\to\infty}\int_{S_\rho} \nabla f[1+\mathcal{O}(r^{-\tau})]\lrcorner e^{-f} d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty}\int_{S^M_\rho} \nabla f \lrcorner e^{-f} d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty}\int_{S^M_\rho} \langle \nabla f, \vec{n}\rangle e^{-f} dA_g \end{aligned} \tag{2.5}$$

Plugging (2.4) and (2.5) into (2.3), we obtain

$$\begin{aligned} \mathfrak{m}(\overline{g}) &= \mathfrak{m}(g) + \lim_{\rho\to\infty}\int_{S^M_\rho} \langle \nabla f, \vec{n}\rangle e^{-f} dA_g \\ &= \mathfrak{m}_f(g). \end{aligned}$$

□

Theorem 2.5 shows that the weighted positive mass theorem in [BO22] can be seen as a positive mass theorem for a warped product when $R^m_f \geq 0$. In fact, our proof of theorem 2.4 will show that $\mathfrak{m}(\overline{g}) \geq 0$ given only that $R_f \geq 0$, though the conclusion that f vanishes identically only follows if $R^m_f \geq 0$ for some $m \in \mathbb{N}$.

2.1.3 Proof of the Warped Bakry-Émery Positive Mass Theorem

We now give a self contained proof of Theorem 2.4 in the case when F is a flat torus, based on arguments that also simplify the proof in [Dai04] in the general case. In particular, our proof relies on Theorem 1.2 and Lemma A.3 in [BO22] for the construction of harmonic spinors that tend to a constant spinor of norm 1 at infinity, as opposed to the fibered boundary calculus of Mazzeo and Melrose used for the more general case in [Dai04]. We let $(M \times F, \overline{g})$ be as in the hypothesis of Theorem 2.4. We work with the spin structure on $M \times F$ constructed in subsection 1.1 and will use the description of the spinor norm on $\Sigma(M \times F)$ in subsection 1.4. In particular, let $\langle,\rangle_M, \langle,\rangle_V$ be Clifford invariant and connection compatible inner products on $\mathbf{\Sigma} M, \Sigma V$ respectively. The inner product $\langle,\rangle$ defined on elementary tensors as in 1.4 by $\langle \phi\otimes\nu, \phi'\otimes\nu'\rangle = \langle\phi,\phi'\rangle_M\langle\nu,\nu'\rangle_V$ is Clifford Invariant, and compatible with the spin connection.

Lemma 2.6. *Suppose $R_f \geq 0$. There exists a $D_{M\times F}$-harmonic spinor ψ on $(M \times F, \overline{g})$ which is asymptotic to a constant spinor ψ_0 at infinity in the sense that $\psi - \psi_0 \in O(r^{-\tau})$, where $r = |x|$.*

Proof. Choose a constant section φ_0 with $|\varphi_0| \to 1$ of ΣM defined over the trivialization of ΣM on the asymptotic end $M \setminus K$ and extend φ_0 smoothly over all of M. Since $R^m_f \in L^1(M,g)$ $|\nabla f|^2 = \mathcal{O}(r^{-2\tau-2})$, and $-2\tau-2 < -n$, it follows that $R_f \in L^1(M,g)$ as well. Theorem 2.5 in [BO22] then provides a D^M_f harmonic spinor φ on M with $\varphi_0 - \varphi \in C^{2,\alpha}_{-\tau}(M)$. Let ν be a parallel section of ΣV with $|\nu|_V = 1$, where $|\nu|_V$ denotes the norm of ν with respect to a fixed invariant metric on ΣV as in subsection 1.4. Note such a ν exists by hypothesis on F. Consider the sections $\psi = \pi_1^*(\varphi)\otimes\nu, \psi_0 = \pi_1^*(\varphi_0)\otimes\nu$ of $\Sigma(M \times F)$, where we view φ, φ_0 as sections of $\mathbf{\Sigma} M = \Sigma M$ when one of n, m is even, and $\Sigma^+ M$ when n, m are both odd. From the results of Section 1, we have $D_{M\times F}\psi = (D^M_f\varphi)\otimes\nu = 0$ in the case when n is even or both n, m are odd, and $D_{M\times F}\psi = (D^M_f\varphi)\otimes\overline{\nu} = 0$ when n is odd and m is even. From the discussion of the spinor norm in subsection 1.4 we have

$$|\psi - \psi_0| = |\pi_1^*(\varphi-\varphi_0)\otimes\nu| = |\varphi-\varphi_0|_M|\nu|_V = |\varphi-\varphi_0|$$

It follows that $\psi - \psi_0 = \mathcal{O}(r^{-\tau})$, as desired.

□

In the case when F is a flat torus, we now show explicitly that the harmonic spinor constructed in the previous lemma is in fact a Witten spinor for $M \times F$. This is proven for general F in [Dai04, Lemmas 2.1,4.2].

Lemma 2.7 (Bakry–Émery Witten formula)**.** *Let T^m be the m-dimensional flat torus of unit volume. The harmonic spinor ψ constructed in Lemma 2.6 satisfies Witten's formula for the mass of $(M \times T, \overline{g})$:*

$$\mathfrak{m}(\overline{g}) = 4 \int_{M \times F} \left[|\nabla_{\overline{g}} \psi|^2_{\overline{g}} + \frac{1}{4} R^m_f |\psi|^2_{\overline{g}} \right] d\mathrm{Vol}_{\overline{g}}. \tag{2.6}$$

Proof. By the choice of spin structure on $M \times F$, the spinor bundle $\Sigma(M \times F)$ is trivialized over the asymptotic end $(M \setminus K) \times F$. With respect to this trivialization, the Dirac operator is given by

$$D_{M \times F} \psi = e^i \cdot \partial_i \psi - \frac{1}{8} (\partial_k \overline{g}_{ij}) e^i \cdot [e^j, e^k] \psi + \mathcal{O}(r^{-2\tau - 1}) \psi. \tag{2.7}$$

With ψ, ψ_0 as in Lemma 2.6, we define $\xi := \psi_0 - \psi$. Integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_{B^M_\rho \times F} |D_{M \times F} \psi|^2_{\overline{g}} d\mathrm{Vol}_{\overline{g}} \\ &= \int_{B^M_\rho \times F} |\nabla_{\overline{g}} \psi|^2 - \mathrm{Re} \int_{S^M_\rho \times F} \langle \psi, \nabla_{e_i} \psi \rangle_{\overline{g}} \iota_{e_i} d\mathrm{Vol}_{\overline{g}} + \frac{1}{4} \int_{B^M_\rho \times F} R^m_f |\psi|^2_{\overline{g}} d\mathrm{Vol}_g, \end{aligned}$$

hence

$$\int_{B^M_\rho \times F} \left[|\nabla_{\overline{g}} \psi|^2_{\overline{g}} + \frac{1}{4} R^m_f |\psi|^2_{\overline{g}} \right] d\mathrm{Vol}_{\overline{g}} = \mathrm{Re} \int_{S^M_\rho \times F} \langle \psi, \nabla_{e_i} \psi \rangle_{\overline{g}} \iota_{e_i} d\mathrm{Vol}_{\overline{g}}. \tag{2.8}$$

By the definition of ψ,

$$\begin{aligned} \langle \psi, \nabla_{e_i} \psi \rangle_{\overline{g}} &= \langle \psi_0 - \xi, \nabla_{e_i} (\psi_0 - \xi) \rangle_{\overline{g}} \\ &= \langle \psi_0, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} - \langle \psi_0, \nabla_{e_i} \xi \rangle_{\overline{g}} - \langle \xi, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} + \langle \xi, \nabla_{e_i} \xi \rangle_{\overline{g}}. \end{aligned} \tag{2.9}$$

We note that $[e^j, e^k]$ is skew-Hermitian. To see this, since $\langle e^i \phi, e^i \phi' \rangle_{\overline{g}} = \langle \phi, \phi' \rangle_{\overline{g}}$ for any two spinors ϕ and ϕ', it follows that

$$-\langle e^i \phi, \phi' \rangle_{\overline{g}} = \langle e^i \phi, e^i \cdot e^i \cdot \phi' \rangle_{\overline{g}} = \langle \phi, e^i \phi' \rangle_{\overline{g}}.$$

Then

$$\begin{aligned} \langle \phi, [e^j, e^k] \phi' \rangle_{\overline{g}} &= \langle \phi, 2 e^j e^k \phi' \rangle_{\overline{g}} \\ &= -2 \langle e^j \phi, e^k \phi' \rangle_{\overline{g}} \\ &= 2 \langle e^k e^j \phi, \phi' \rangle_{\overline{g}} \\ &= -\langle [e^j, e^k] \phi, \phi' \rangle_{\overline{g}}, \end{aligned} \tag{2.10}$$

as claimed. Letting $\phi = \phi' = \psi_0$, it follows that

$$\begin{aligned} \langle \psi_0, [e^j, e^k] \psi_0 \rangle_{\overline{g}} &= -\langle [e^j, e^k] \psi_0, \psi_0 \rangle_{\overline{g}} \\ &= -\overline{\langle \psi_0, [e^j, e^k] \psi_0 \rangle_{\overline{g}}}, \end{aligned}$$

hence $\langle \psi_0, [e^j, e^k] \psi_0 \rangle_{\overline{g}}$ is purely imaginary. Since ψ_0 is a constant spinor, $\partial_i \psi_0 = 0$, hence

$$\begin{aligned} \mathrm{Re} \langle \psi_0, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} &= \left\langle \psi_0, -\frac{1}{8} \mathrm{Re}(\partial_k g_{ij}) \mathrm{Re}([e^j, e^k] \psi_0) + \mathcal{O}(r^{-2\tau - 1}) \right\rangle_{\overline{g}} \\ &= -\frac{1}{8} \mathrm{Re}(\partial_k g_{ij}) \mathrm{Re} \langle \psi_0, [e^j, e^k] \psi_0 \rangle_{\overline{g}} + \mathcal{O}(r^{-2\tau - 1}) |\psi_0|^2_{\overline{g}} \\ &= \mathcal{O}(r^{-2\tau - 1}) |\psi_0|^2_{\overline{g}}. \end{aligned}$$

Since $\partial_k \overline{g}_{ij} = \mathcal{O}(r^{-\tau - 1})$ and $|\psi_0|_{\overline{g}} \to 1$ as $r = |x| \to \infty$, it follows that $\mathrm{Re} \langle \psi_0, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} = \mathcal{O}(r^{-\tau - 1})$. Similarly,

$$\mathrm{Re} \langle \xi, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} = \mathrm{Re} \langle \partial_i \xi, \nabla_{e_i} \psi_0 \rangle_{\overline{g}} - \frac{1}{8} \mathrm{Re}(\partial_k \overline{g}_{ij}) \langle \xi, [e^j, e^k] \nabla_{e_i} \psi_0 \rangle_{\overline{g}} + \mathcal{O}(r^{-2\tau - 1}) \langle \xi, \nabla_{e_i} \psi_0 \rangle_{\overline{g}}. \tag{2.11}$$

Since $\nabla_{e_i}\psi_0$ is a linear combination of $\partial_i\psi_0 = 0$ for each $i \le n+m$ and the Christoffel symbols with respect to the orthonormal frame, which are $\mathcal{O}(r^{-\tau-1})$, it follows that $\nabla_{e_i}\psi_0 = \mathcal{O}(r^{-\tau-1})$. Since $\xi = \mathcal{O}(r^{-\tau})$, we have $\partial_i\xi = \mathcal{O}(r^{-\tau-1})$. Thus the first term of (2.11) is $\mathcal{O}(r^{-2\tau-2})$, the second term is $\mathcal{O}(r^{-3\tau-2})$, and the third term is $\mathcal{O}(r^{-2\tau-1})$, so overall, $\mathrm{Re}\langle\xi, \nabla_{e_i}\psi_0\rangle_{\overline{g}} = \mathcal{O}(r^{-2\tau-1})$. Similarly, $\mathrm{Re}\langle\xi, \nabla_{e_i}\xi\rangle_{\overline{g}} = \mathcal{O}(r^{-2\tau-1})$, so the first, third, and fourth terms of (2.9) vanish as $r = |x| \to \infty$.

It remains to analyze the second term of (2.9). To do this, for each $i \le n+m$, we define the operator L_i by

$$L_i := \nabla_{e_i} + e^i \cdot D_{M\times F}.$$

We note that

$$\delta_{ij} + e^i \cdot e^j = \frac{1}{2}[e^i, e^j]. \tag{2.12}$$

Indeed, if $i = j$, then the left-hand side is 0 and $[e^i, e^j] = 0$ as well. If $i \ne j$, then the left-hand side is $e^i \cdot e^j$ and the right-hand side is $\frac{1}{2}(2e^i \cdot e^j) = e^i \cdot e^j$. Thus

$$\begin{aligned} L_i &= \nabla_{e_i} + \sum_{j\le n+m} e^i \cdot e^j \cdot \nabla_{e_j} \\ &= \sum_{j\le n+m} (\delta_{ij}\nabla_{e_j} + e^i \cdot e^j \cdot \nabla_{e_j}) \\ &= \sum_{j\le n+m} (\delta_{ij} + e^i \cdot e^j)\nabla_{e_j} \\ &= \frac{1}{2}\sum_{j\le n+m} [e^i, e^j]\nabla_{e_j}. \end{aligned} \tag{2.13}$$

We now define the $(n-2)$-form

$$\alpha := \sum_{j\le n+m} \langle [e^i, e^j]\psi_0, \xi\rangle_{\overline{g}} e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}.$$

Then

$$d\alpha = -2\sum_{j\le n+m} (\langle [e^i, e^j]\nabla_{e_j}\psi_0, \xi\rangle_{\overline{g}} + \langle [e^i, e^j]\psi_0, \nabla_{e_j}\xi\rangle_{\overline{g}}) e_i \lrcorner d\mathrm{Vol}_{\overline{g}}. \tag{2.14}$$

We defer the proof of (2.14) to Appendix A.2. It now follows from (2.13) that

$$\sum_{j\le n+m} \langle [e^i, e^j]\nabla_{e_j}\psi_0, \xi\rangle_{\overline{g}} = 2\langle L_i\psi_0, \xi\rangle_{\overline{g}}, \tag{2.15}$$

and by (2.10),

$$\begin{aligned} \sum_{j\le n+m} \langle [e^i, e^j]\psi_0, \nabla_{e_j}\xi\rangle_{\overline{g}} &= -\sum_{j\le n+m} \langle \psi_0, [e^i, e^j]\nabla_{e_j}\xi\rangle_{\overline{g}} \\ &= -2\langle \psi_0, L_i\xi\rangle_{\overline{g}}. \end{aligned} \tag{2.16}$$

Plugging (2.15) and (2.16) into (2.14), we obtain

$$d\alpha = -4[\langle L_i\psi_0, \xi\rangle_{\overline{g}} - \langle \psi_0, L_i\xi\rangle_{\overline{g}}] e_i \lrcorner d\mathrm{Vol}_{\overline{g}}.$$

It now follows from Stokes' theorem that

$$\begin{aligned} \int_{S^M_\rho \times F} [\langle L_i\psi_0, \xi\rangle_{\overline{g}} - \langle \psi_0, L_i\xi\rangle_{\overline{g}}] e_i \lrcorner d\mathrm{Vol}_{\overline{g}} &= -\frac{1}{4}\int_{S^M_\rho \times F} d\alpha \\ &= -\frac{1}{4}\int_{\partial S^M_\rho \times F} \alpha \\ &= 0, \end{aligned}$$

hence

$$\int_{S_\rho \times F} \langle L_i\psi_0, \xi\rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}} = \int_{S_\rho \times F} \langle \psi_0, L_i\xi\rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}}. \tag{2.17}$$

Using the fact that $e^i \cdot D_{M\times F} - L_i = -\nabla_{e_i}$ and $D_{M\times F}(\psi_0 - \xi) = 0$, (2.17) implies

$$\begin{aligned}
-\mathrm{Re}\int_{S_\rho^M \times F} \langle \psi_0, \nabla_{e_i}\xi\rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}} &= \mathrm{Re}\int_{S_\rho^M \times F} \langle \psi_0, (e^i \cdot D_{M\times F} - L_i)\xi\rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}} \\
&= \mathrm{Re}\int_{S_\rho^M \times F} [\langle \psi_0, e^i \cdot D_{M\times F}\xi\rangle_{\overline{g}} - \langle \psi_0, L_i\xi\rangle_{\overline{g}}] e_i \lrcorner d\mathrm{Vol}_{\overline{g}} \\
&= \mathrm{Re}\int_{S_\rho^M \times F} [\langle \psi_0, e^i \cdot D_{M\times F}\psi_0\rangle_{\overline{g}} - \langle \psi_0, L_i\xi\rangle_{\overline{g}}] e_i \lrcorner d\mathrm{Vol}_{\overline{g}} \\
&= \mathrm{Re}\int_{S_\rho^M \times F} [\langle \psi_0, e^i \cdot D_{M\times F}\psi_0\rangle_{\overline{g}} - \langle L_i\psi_0, \xi\rangle_{\overline{g}}] e_i \lrcorner d\mathrm{Vol}_{\overline{g}}.
\end{aligned} \tag{2.18}$$

Since $\nabla_{e_i}\psi_0 = \mathcal{O}(r^{-\tau-1})$, $D_{M\times F}\psi_0 = \mathcal{O}(r^{-\tau-1})$, and $\xi = \mathcal{O}(r^{-\tau})$, it follows that $\langle L_i\psi_0, \xi\rangle_{\overline{g}} = \mathcal{O}(r^{-2\tau-1})$. As for the first term, by (2.7) and the fact that $\partial_i\psi_0 = 0$, we have

$$\begin{aligned}
e^i \cdot D\psi_0 &= -\frac{1}{8}\sum_{j,k,\ell\le n+m} (\partial_k \overline{g}_{\ell j} + \mathcal{O}(r^{-2\tau-1})) e^i \cdot e^\ell \cdot [e^j, e^k]\psi_0 \\
&= -\frac{1}{4}\sum_{j,k,\ell\le n+m} (\partial_k \overline{g}_{\ell j} + \mathcal{O}(r^{-2\tau-1})) e^i \cdot e^\ell \cdot (\delta^{jk} + e^j \cdot e^k)\psi_0
\end{aligned} \tag{2.19}$$

Moreover, it follows from relabeling indices that

$$\begin{aligned}
\sum_{j,k,\ell\le n+m} \partial_k \overline{g}_{\ell j} e^i \cdot e^\ell \cdot \delta^{jk}\psi_0 &= \sum_{k,\ell\le n+m} \partial_k \overline{g}_{\ell k} e^i \cdot e^\ell \psi_0 \\
&= \sum_{k,\ell\le n+m} \partial_\ell \overline{g}_{k\ell} e^i \cdot e^k \psi_0 \\
&= \sum_{j,k\le n+m} \partial_j \overline{g}_{kj} e^i \cdot e^k \psi_0.
\end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned}
\sum_{j\ne\ell,k\le n+m} \partial_k \overline{g}_{\ell j} e^i \cdot e^\ell \cdot e^j \cdot e^k\psi_0 &= -\sum_{j\ne\ell,k\le n+m} \partial_k \overline{g}_{\ell j} e^i \cdot e^j \cdot e^\ell \cdot e^k\psi_0 \\
&= -\sum_{j\ne\ell,k\le n+m} \partial_k \overline{g}_{j\ell} e^i \cdot e^\ell \cdot e^j \cdot e^k\psi_0 \\
&= -\sum_{j\ne\ell,k\le n+m} \partial_k \overline{g}_{\ell j} e^i \cdot e^\ell \cdot e^j \cdot e^k\psi_0,
\end{aligned}$$

hence

$$\sum_{j\ne\ell,k\le n+m} \partial_k \overline{g}_{\ell j} e^i \cdot e^\ell \cdot e^j \cdot e^k\psi_0 = 0. \tag{2.21}$$

Analogously,

$$\sum_{j\ne\ell,k\le n+m} \mathcal{O}(r^{-2\tau-1}) e^i \cdot e^\ell \cdot e^j \cdot e^k\psi_0 = 0. \tag{2.22}$$

It follows from (2.21) and (2.22), respectively, that

$$\begin{aligned}\sum_{j,k,\ell\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0 &= \sum_{j=\ell,k\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0+\sum_{j\neq\ell,k\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0\\ &= \sum_{j=\ell,k\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0\\ &= \sum_{j,k\leq n+m}\partial_k\overline{g}_{jj}e^i\cdot e^j\cdot e^j\cdot e^k\psi_0\\ &= -\sum_{j,k\leq n+m}\partial_k\overline{g}_{jj}e^i\cdot e^k\psi_0\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}\sum_{j,k,\ell\leq n+m}\mathcal{O}(r^{-2\tau-1})e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0 &= \sum_{j=\ell,k\leq n+m}\mathcal{O}(r^{-2\tau-1})e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0\\ &= -\sum_{k\leq n+m}\mathcal{O}(r^{-2\tau-1})e^i\cdot e^k\psi_0.\end{aligned}\tag{2.24}$$

Plugging (2.20), (2.23), and (2.24) into (2.19), we obtain

$$\begin{aligned}&e^i\cdot D_{M\times F}\psi_0\\ &= -\frac{1}{4}\sum_{j,k,\ell\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot\delta^{jk}\psi_0+\frac{1}{4}\sum_{j,k,\ell\leq n+m}\partial_k\overline{g}_{\ell j}e^i\cdot e^\ell\cdot e^j\cdot e^k\psi_0\\ &\quad -\frac{1}{4}\mathcal{O}(r^{-2\tau-1})e^i\sum_{k\leq n+m}\mathcal{O}(r^{-2\tau-1})e^i\cdot e^k\psi_0\\ &= -\frac{1}{4}\sum_{j,k\leq n+m}(\partial_j\overline{g}_{kj}-\partial_k\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))e^i\cdot e^k\psi_0.\end{aligned}$$

It now follows from (2.12) that

$$\begin{aligned}e^i\cdot D_{M\times F}\psi_0 &= -\frac{1}{4}\sum_{j,k\leq n+m}(\partial_j\overline{g}_{kj}-\partial_k\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))e^i\cdot e^k\psi_0\\ &= -\frac{1}{4}\sum_{j,k\leq n+m}(\partial_j\overline{g}_{kj}-\partial_k\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))\left(\frac{1}{2}[e^i,e^k]-\delta^{ik}\right)\psi_0\\ &= -\frac{1}{8}\sum_{j,k\leq n+m}(\partial_j\overline{g}_{kj}-\partial_k\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))[e^i,e^k]\psi_0\\ &\quad+\frac{1}{4}\sum_{j\leq n+m}(\partial_j\overline{g}_{ij}-\partial_i\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))\psi_0.\end{aligned}\tag{2.25}$$

Plugging (2.25) into (2.18) and using the fact that $[e^i,e^k]$ is skew, we obtain

$$-\mathrm{Re}\int_{S^M_\rho\times F}\langle\psi_0,\nabla_{e_i}\xi\rangle_{\overline{g}}e_i\lrcorner d\mathrm{Vol}_{\overline{g}}=\frac{1}{4}\sum_{j\leq n}\int_{S^M_\rho\times F}(\partial_j\overline{g}_{ij}-\partial_i\overline{g}_{jj}+\mathcal{O}(r^{-2\tau-1}))|\psi_0|^2_{\overline{g}}e_i\lrcorner d\mathrm{Vol}_{\overline{g}}.$$

Summing over $i\leq n+m$, it follows that

$$-\mathrm{Re}\int_{S^M_\rho\times F}\langle\psi_0,\nabla_{e_i}\xi\rangle_{\overline{g}}e_i\lrcorner d\mathrm{Vol}_{\overline{g}}=\frac{1}{4}\int_{S^M_\rho\times F}\overline{\mu}|\psi_0|^2_{\overline{g}}\lrcorner d\mathrm{Vol}_{\overline{g}}+\mathcal{O}(r^{-2\tau-1}).\tag{2.26}$$

Plugging (2.26) into (2.8) and being conscious of the asymptotically decaying terms (i.e. only the second term of (2.9) contributes to the limit), we obtain

$$\begin{aligned}\int_{M\times F}\left[|\nabla_{\overline{g}}\psi|^2_{\overline{g}}+\frac{1}{4}R^m_f|\psi|^2_{\overline{g}}\right]d\mathrm{Vol}_{\overline{g}}&=\lim_{\rho\to\infty}\int_{B^M_\rho\times F}\left[|\nabla_{\overline{g}}\psi|^2_{\overline{g}}+\frac{1}{4}R^m_f|\psi|^2_{\overline{g}}\right]d\mathrm{Vol}_{\overline{g}}\\&=-\lim_{\rho\to\infty}\mathrm{Re}\int_{S^M_\rho\times F}\langle\psi_0,\nabla_{e_i}\xi\rangle_{\overline{g}}e_i\lrcorner d\mathrm{Vol}_{\overline{g}}\\&=\frac{1}{4}\lim_{\rho\to\infty}\int_{S^M_\rho\times F}\overline{\mu}|\psi_0|^2_{\overline{g}}\lrcorner d\mathrm{Vol}_{\overline{g}}.\end{aligned}$$

Since $|\psi_0|_{\overline{g}}\to 1$ as $r\to\infty$, it follows that

$$\begin{aligned}\int_{M\times F}\left[|\nabla_{\overline{g}}\psi|^2_{\overline{g}}+\frac{1}{4}R^m_f|\psi|^2_{\overline{g}}\right]d\mathrm{Vol}_{\overline{g}}&=\frac{1}{4}\lim_{\rho\to\infty}\int_{S^M_\rho\times F}\overline{\mu}\lrcorner d\mathrm{Vol}_{\overline{g}}\\&=\frac{1}{4}\mathfrak{m}(\overline{g}),\end{aligned}$$

from which the Bakry-Émery Witten formula (2.6) follows. □

Using the discussion in subsection 1.4, we can describe Witten's formula for the mass $\mathfrak{m}(\overline{g})$ more explicitly. Let $\psi=\pi_1^*(\varphi)\otimes\nu$ be the harmonic spinor constructed in Lemma 2.6. The formula for the Bakry-Émery mass in Lemma 2.7 yields

$$\mathfrak{m}_{f,m}(g)=4\int_{M\times F}\left[|\nabla^{M\times F}(\varphi\otimes\nu)|^2_{\overline{g}}+\frac{1}{4}R^m_f|\varphi\otimes\nu|^2_{\overline{g}}\right]d\mathrm{Vol}_{M\times F}.$$

The norms here are all with respect to the spinor norm on $M\times F$, and we identify sections of ΣM with their pullbacks to sections on $\pi_1^*(\Sigma M)$. From the calculation of the spin connection on $M\times F$, we have

$$\begin{aligned}|\nabla^{M\times F}(\varphi\otimes\nu)|^2&=|(\nabla^M\varphi)\otimes\nu|^2+\sum_a\left\langle\frac{1}{2m}\zeta_a\cdot\nabla f\cdot(\varphi\otimes\nu),\frac{1}{2m}\zeta_a\cdot\nabla f\cdot(\varphi\otimes\nu)\right\rangle\\&=|(\nabla^M\varphi)\otimes\nu|^2+\sum_a\frac{1}{4m^2}|(\nabla f\cdot\varphi)\otimes\nu|^2\\&=|(\nabla^M\varphi)\otimes\nu|^2+\frac{1}{4m}|(\nabla f\cdot\varphi)\otimes\nu|^2.\end{aligned}$$

This lets us cancel terms when $|\nabla f|\neq 0$. We have

$$\frac{1}{4m}|(\nabla f\cdot\varphi)\otimes\nu|^2=\frac{|\nabla f|^2}{4m}\left\langle\frac{\nabla f}{|\nabla f|}\cdot\varphi\otimes\nu,\frac{\nabla f}{|\nabla f|}\cdot\varphi\otimes\nu\right\rangle=\frac{|\nabla f|^2}{4m}|\varphi\otimes\nu|^2$$

This cancels the extra term in $\frac{1}{4}R_{m,f}|\psi\otimes\nu|^2$, hence we obtain

$$\mathfrak{m}_{f,m}(g)=4\int_{M\times F}\left[|(\nabla^M\varphi)\otimes\nu|^2+\frac{1}{4}R_f|\varphi\otimes\nu|^2\right]d\mathrm{Vol}_{M\times F}$$

From the discussion in subsection 1.4,

$$\begin{aligned}\mathfrak{m}_{f,m}(g)&=4\int_{M\times F}\left[|(\nabla^M\varphi)\otimes\nu|^2+\frac{1}{4}R_f|\varphi\otimes\nu|^2\right]d\mathrm{Vol}_{M\times F}\\&=4\int_{M\times F}\left[|\nabla^M\varphi|^2_M+\frac{1}{4}R_f|\varphi|^2_M\right]d\mathrm{Vol}_{M\times F}.\end{aligned}$$

We will show next that the general Witten's formula 2.1.3 reduces to the weighted Witten's formula in the case when $F=T^m$ is a flat torus of unit volume. The norm $|\cdot|_M$ is the pullback norm on the spinor bundle $\pi^*(\Sigma M)$ of an invariant norm on ΣM, hence it is constant along the T directions. Similarly, the norm $|\cdot|_V$ is also constant along the T directions, hence we can write $\mathfrak{m}_{f,m}(g)$ as

$$\mathfrak{m}_{f,m}(g)=4\int_M\left[|\nabla^M\varphi|^2_M+\frac{1}{4}R_f|\varphi|^2_M\right]e^{-f}d\mathrm{Vol}_M.$$

Recall from Theorem 2.5 in [BO22] that the weighted mass $\mathfrak{m}_f$ of (M, g, f) is given by

$$\mathfrak{m}_f(g) = 4 \int_M \left[|\nabla^M \varphi|_M^2 + \frac{1}{4} R_f |\varphi|_M^2 \right] e^{-f} d\mathrm{Vol}_M.$$

Thus we recover the weighted Witten's formula from the Witten's formula for $M \times T$. Under the additional assumptions that $R \geq 0, R \in L^1(M, g)$ (see [BO22, pp.11] and [LP87, pp.87–90]), one can take $\varphi = e^{\frac{f}{2}} \overline{\varphi}$, where $\overline{\varphi}$ is an unweighted Witten spinor satisfying the following standard Witten's formula for the ordinary mass:

$$\mathfrak{m}(g) = 4 \int_M \left[|\nabla^M \overline{\varphi}|_M^2 + \frac{1}{4} R_f |\overline{\varphi}|_M^2 \right] d\mathrm{Vol}_M.$$

Thus, under these additional assumptions, one can obtain the spinor ψ using the harmonic spinors on M constructed for the standard proof of the positive mass theorem.

Proof of Theorem 2.4. It follows from Lemma 2.7 that $\mathfrak{m}(\overline{g}) \geq 0$. When $\mathfrak{m}(\overline{g}) = 0$, formula 2.1.3 implies that $\varphi \otimes \nu$ is parallel, and the identity

$$|\nabla^{M \times F}(\varphi \otimes \nu)|^2 = |(\nabla^M \varphi) \otimes \nu|^2 + \frac{1}{4m} |(\nabla f \cdot \varphi) \otimes \nu|^2$$

then implies that φ is a parallel spinor on M and $\nabla f = 0$. In particular, it follows that f is constant and M is Ricci flat by the Ricci identity. Since $f = O(r^{-\tau})$, it follows that $f = 0$ identically. We conclude by noting that any Ricci flat, asympotically flat, complete Riemannian manifold is isometric to Euclidean space by the Bishop-Gromov volume comparison theorem (see [CLN06, Corollary 1.134]). $\square$

Remark 2.8. One can alternatively prove Lemma 2.4 using the reasoning from [Dai04] as follows. It follows from Witten's formula that $\mathfrak{m}(\overline{g}) \geq 0$. If $\mathfrak{m}(\overline{g}) = 0$, then $\nabla_{\overline{g}} \psi = 0$, so ψ is a parallel spinor. By the Ricci identity,

$$e_i \cdot R(e_i, X)\psi = -\frac{1}{2} \mathrm{Ric}(X)\psi,$$

it follows that $M \times F$ is Ricci flat. As in [Dai04], one can construct n-independent geodesic lines in $M \times F$ by choosing pairs of points p_i, q_i in the asymptotic end $(M \setminus K) \times F$ with distance comparable to the Euclidean distance. The Cheeger-Gromoll splitting theorem then implies that $M \times F$ is isometric to the Riemannian product $\mathbb{R}^n \times F$. In particular, M is isometric to $\mathbb{R}^n$ and $M \times F$ is a Riemannian product, so the warping function $v^2 = e^{-\frac{2f}{m}}$ is constant. This gives some geometric intuition for why f must be constant in the case of equality.

2.2 The conformal metric method

The warped product method only allows us to show that the Bakry-Émery mass is non-negative for $m \in \mathbb{N}$. In this section, we show that conformally changing the original metric g on our asymptotically Euclidean manifold M allows us to prove the Bakry-Emery positive mass theorem for $m \notin [1-n, 0]$. Comparing to the warped product method, this method is less geometrically intuitive, yet more intrinsic in the sense that we are only concerned with the original manifold. Our goal is to prove the following continuous (up to a set of finite measure) version of the Bakry-Émery positive mass theorem:

Theorem 2.9 (Conformal Bakry-Émery positive mass theorem)**.** *Suppose $(M^n, g, v^m d\mathrm{Vol}_g, m)$ is an asymptotically Euclidean smooth metric measure space of order $\tau > \frac{n-2}{2}$, where $3 \leq n \leq 7$ or M is spin. Assume $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$.*

(a) If $m \in S_n = \mathbb{R} \setminus [1-n, 0]$, $R^m_f \in L^1(M, g)$, and $R^m_f \geq 0$, then $\mathfrak{m}_{f,m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$ and f is identically 0.

(b) If $m \in [1-n, 0)$, $R^m_f \in L^1(M, g)$, $R^m_f \geq F^m_f$, and $R^{\varepsilon-n}_f \geq 0$ for some $\varepsilon \in [0, 1)$, then $\mathfrak{m}_{f,m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$ and f is identically 0.

Since $\mathbb{N} \subset S_n$, Theorem 2.9 is a strict improvement of Theorem 2.4. Additionally, since $(0, \infty) \subset S_n$, the condition $R^m_f \geq 0$ converges to $R_f \geq 0$ as $m \to \infty$, which is assumed in the weighted positive mass theorem proven in [BO22].

Besides conformally changing the metric, the proof also involves analyzing the following *Bakry-Émery barrier function*:

$$F^m_f := \frac{m+n-1}{m(1-n)} |\nabla f|^2. \tag{2.27}$$

Observe that $F^m_f \leq 0$ if $m \in S_n$.

In Section 2.2.3, we generalize Baldauf and Ozuch's positive mass theorem to allow for some negative weighted scalar curvature:

Corollary 2.10 (Generalized weighted positive mass theorem)**.** *Suppose (M^n, g, f), $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$, is a weighted asymptotically Euclidean spin manifold of order $\tau > \frac{n-2}{2}$, and assume $R_f \in L^1(M, g)$ and $R_f \geq -\frac{1}{n-\varepsilon}|\nabla f|^2$ for some $\varepsilon < 1$. Then $\mathfrak{m}_f(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$.*

To show that this is indeed a nontrivial extension of the weighted positive mass theorem, we proceed to construct the *Bakry-Émery logarithmic functions*, with respect to which the weighted mass is positive and the weighted scalar curvature is negative.

2.2.1 The Bakry-Émery conformal metric

As before, we let $(M^n, g, v^m d\mathrm{Vol}_g, m)$ be an asymptotically Euclidean smooth metric measure space of order $\tau > \frac{n-2}{2}$ with $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$. This time, however, we allow m to be any real number outside the interval $[1-n, 0]$ in lieu of restricting it to be a natural number.

Definition 2.11. *We define the* Bakry-Émery conformal metric $g_{f,m}$ *of $(M^n, g, v^m d\mathrm{Vol}_g, m)$ by*

$$g_{f,m} := e^{\frac{2f}{1-n}} g = v^{\frac{2m}{n-1}} g.$$

Since $v^{\frac{2m}{n-1}} = 1 + \mathcal{O}(r^{-\tau})$ and $g_{ij} = \delta_{ij} + \mathcal{O}(r^{-\tau})$ in asymptotic coordinates for g, it follows that $(g_{f,m})_{ij} = \delta_{ij} + \mathcal{O}(r^{-\tau})$ in the same coordinates. Analogous asymptotic conditions hold for the first and second derivatives of $g_{f,m}$ as well. Thus, $(M^n, g_{f,m})$ is also asymptotically Euclidean of order τ, and any asymptotic coordinate system for g is also an asymptotic coordinate system for $g_{f,m}$.

We thus fix a coordinate system for M_∞ that is simultaneously asymptotic for g and $g_{f,m}$. The mass-density vector field of $(M^n, g_{f,m})$ is then given by

$$\begin{aligned}
\mu_{f,m} &= [\partial_i (g_{f,m})_{ij} - \partial_j (g_{f,m})_{ii}]\, \partial_j \\
&= \left[\partial_i (v^{\frac{2m}{n-1}} g_{ij}) - \partial_j (v^{\frac{2m}{n-1}} g_{ii})\right] \partial_j \\
&= v^{\frac{2m}{n-1}} (\partial_i g_{ij} - \partial_j g_{ii})\partial_j + \frac{2m}{n-1} v^{\frac{2m-n+1}{n-1}} (v_i g_{ij} - v_j g_{ii})\partial_j \\
&= v^{\frac{2m}{n-1}} \mu + \frac{2m}{n-1} v^{\frac{2m-n+1}{n-1}} (v_i g_{ij} - v_j g_{ii})\partial_j,
\end{aligned} \tag{2.28}$$

where μ is the mass-density vector field of (M^n, g). Since $v = 1 + \mathcal{O}(r^{-\tau})$, $\nabla v = \mathcal{O}(r^{-\tau-1})$. Since (M^n, g) is asymptotically Euclidean, it follows that

$$\begin{aligned}
v_i g_{ij} &= v_i[\delta_{ij} + \mathcal{O}(r^{-\tau})] \\
&= v_i \delta_{ij} + \mathcal{O}(r^{-2\tau-1}).
\end{aligned}$$

It now follows from (2.28) and relabeling indices that

$$\begin{aligned}
\mu_{f,m} &= v^{\frac{2m}{n-1}} \mu + \frac{2m}{n-1} v^{\frac{2m-n+1}{n-1}} (v_i \delta_{ij} - v_j \delta_{ii})\partial_j + \mathcal{O}(r^{-2\tau-1}) \\
&= v^{\frac{2m}{n-1}} \mu + \frac{2m}{n-1} v^{\frac{2m-n+1}{n-1}} (v_j \delta_{ij} - v_i \delta_{jj})\partial_i + \mathcal{O}(r^{-2\tau-1}) \\
&= v^{\frac{2m}{n-1}} \mu + \frac{2m}{n-1} v^{\frac{2m-n+1}{n-1}} (1-n) \sum_{i=1}^{n} v_i \partial_i + \mathcal{O}(r^{-2\tau-1}) \\
&= v^{\frac{2m}{n-1}} \mu - 2m v^{\frac{2m-n+1}{n-1}} \sum_{i=1}^{n} v_i \partial_i + \mathcal{O}(r^{-2\tau-1}).
\end{aligned} \tag{2.29}$$

Additionally, the volume element with respect to $g_{f,m}$ is given by

$$\begin{aligned}
d\mathrm{Vol}_{g_{f,m}} &= \sqrt{\det(g_{f,m})}\; dx_1 \wedge \cdots \wedge dx_n \\
&= v^{\frac{2mn}{2(n-1)}} \sqrt{\det(g)}\; dx_1 \wedge \cdots \wedge dx_n \\
&= v^{\frac{mn}{n-1}} d\mathrm{Vol}_g.
\end{aligned} \tag{2.30}$$

Ultimately, we want to show that the Bakry-Émery mass is related to the ordinary mass of $(M^n, g_{f,m})$, which is non-negative provided $R_{g_{f,m}} \geq 0$ by the ordinary positive mass theorem. The first step is to use (2.29) and (2.30) to compute the mass with respect to $g_{f,m}$:

$$\begin{aligned}
\mathfrak{m}(g_{f,m}) &= \lim_{\rho\to\infty} \int_{S_\rho} \mu_{f,m} \lrcorner d\mathrm{Vol}_{g_{f,m}} \\
&= \underbrace{\lim_{\rho\to\infty} \int_{S_\rho} v^{\frac{2m}{n-1}} \mu \lrcorner v^{\frac{mn}{n-1}} d\mathrm{Vol}_g}_{=I_1(m)} + \underbrace{\lim_{\rho\to\infty} \int_{S_\rho} -2m v^{\frac{2m-n+1}{n-1}} v_i \partial_i \lrcorner v^{\frac{mn}{n-1}} d\mathrm{Vol}_g}_{=I_2(m)}.
\end{aligned} \tag{2.31}$$

We now prove the following surprising relationship between the mass of $(M^n, g_{f,m})$ and the Bakry-Émery mass of $(M^n, g, v^m d\mathrm{Vol}_g, m)$:

Lemma 2.12. *For any $m \in S_n$,*

$$\mathfrak{m}_{f,m}(g) = \mathfrak{m}(g_{f,m}).$$

Proof. Since $v^k = 1 + \mathcal{O}(r^{-\tau})$ for any $k \in \mathbb{R}$ and $\mu = \mathcal{O}(r^{-\tau-1})$, we can write $I_1(m)$ as

$$\begin{aligned} I_1(m) &= \lim_{\rho\to\infty} \int_{S_\rho} v^{\frac{3m}{n-1}} \mu \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} [1 + \mathcal{O}(r^{-\tau})] \mu \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} \mu \lrcorner v^m d\mathrm{Vol}_g + \lim_{\rho\to\infty} \int_{S_\rho} \mathcal{O}(r^{-2\tau-1}) \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} \mu \lrcorner v^m d\mathrm{Vol}_g. \end{aligned} \tag{2.32}$$

Similarly, since $v_i = \mathcal{O}(r^{-\tau-1})$, $v^k \cdot v_i = \mathcal{O}(r^{-\tau-1})$. Then we can write $I_2(m)$ as

$$\begin{aligned} I_2(m) &= \lim_{\rho\to\infty} \int_{S_\rho} v^{\frac{m(n+2)-(n-1)(m+1)}{n-1}} \cdot (-2mv \cdot v_i \partial_i) \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} [1 + \mathcal{O}(r^{-\tau})](-2mv \cdot v_i) \partial_i \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} (-2mv \cdot v_i \partial_i) \lrcorner v^m d\mathrm{Vol}_g + \lim_{\rho\to\infty} \int_{S_\rho} \mathcal{O}(r^{-2\tau-1}) \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} (-2mv \cdot v_i \partial_i) \lrcorner v^m d\mathrm{Vol}_g. \end{aligned} \tag{2.33}$$

Substituting (2.32) and (2.33) into (2.31), it follows that

$$\begin{aligned} \mathfrak{m}(g_{f,m}) &= I_1(m) + I_2(m) \\ &= \lim_{\rho\to\infty} \int_{S_\rho} (\mu - 2mv \cdot v_i \partial_i) \lrcorner v^m d\mathrm{Vol}_g \\ &= \lim_{\rho\to\infty} \int_{S_\rho} \overline{\mu} \lrcorner v^m d\mathrm{Vol}_g \\ &= \mathfrak{m}_{f,m}(g), \end{aligned}$$

where $\overline{\mu}$ is as in (2.2). ☐

As an immediate consequence of Lemma 2.12 and the ordinary positive mass theorem, we have:

Corollary 2.13. *Suppose $(M^n, g, v^m d\mathrm{Vol}_g, m)$, $m \in S_n$, is an asymptotically Euclidean smooth metric measure space of order $\tau > \frac{n-2}{2}$, where $3 \le n \le 7$ or M is spin. Assume $v^m = e^{-f}$ for some $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$, $R_{g_{f,m}} \in L^1(M, g)$, and $R_{g_{f,m}} \ge 0$. Then $\mathfrak{m}_{f,m}(g) \ge 0$, with equality if and only if $(M^n, g_{f,m})$ is isometric to $(\mathbb{R}^n, \delta_{ij})$.*

2.2.2 Proof of the Conformal Bakry-Émery Positive Mass Theorem

Using the Bakry-Émery conformal metric, we now prove that the Bakry-Émery mass of $(M^n, g, v^m d\mathrm{Vol}_g, m)$ is non-negative assuming R_f^m is bounded below by (2.27).

Lemma 2.14. *The Bakry-Émery scalar curvature R_f^m of (M^n, g) satisfies $R_f^m \ge F_f^m$ if and only if $R_{g_{f,m}} \ge 0$, with equality if and only if $R_{g_{f,m}} = 0$.*

Before proving this result, it is worthwhile to provide some nontrivial examples of functions satisfying $R_f^m \ge F_f^m$. Observe that this bound may be written as

$$R_g \ge -2\Delta f + \frac{n-2}{n-1}|\nabla f|^2. \tag{2.34}$$

Let $k \in \mathbb{R}$. By the second theorem on page 16 of [CSCB79], $\Delta : \mathcal{C}^{2,\alpha}_{-\tau}(M) \to \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$ is an isomorphism. Assume $h_k \in \mathcal{C}^{2,\alpha}_{-\tau-2}(M)$ satisfies (2.34) on M_{cpct} and $h_k = k\Delta^{-1}r^{-\tau-2}$ on $\partial M_\infty \cong B_R$. Define the function $f_k \in \mathcal{C}^{2,\alpha}_{-\tau-2}(M)$ by

$$f_k(x) = \begin{cases} \Delta^{-1}\varphi_k(x) & x \in M_\infty \\ h_k(x) & x \in M_{\text{cpct}} \end{cases},$$

where $\varphi_k = kr^{-\tau-2}$ on $M \backslash M_{\text{cpct}}$. Observe that f is well-defined since $\Delta^{-1}\varphi$ agrees with h on ∂M_∞. Since $R_g \in \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$, there is a constant $C_1 > 0$ such that $R_g \geq -C_1|x|^{-\tau-2}$ on M_∞. Since $\nabla f_k \in \mathcal{C}^{1,\alpha}_{-\tau-1}(M)$, $|\nabla f_k|^2 = \mathcal{O}(r^{-2\tau-2})$, so $|\nabla f_k|^2 = \mathcal{O}(r^{-\tau-2})$. Then there is a constant $C_2(k) > 0$ such that $|\nabla f_k|^2 \leq C_2(k)|x|^{-\tau-2}$. Additionally, $\Delta f_k = kr^{-\tau-2}$ on M_∞.

Proposition 2.15. *If $k \geq \frac{1}{2}C_1 + \frac{n-2}{2(n-1)}C_2(k)$, then $R^m_{f_k} \geq F^m_{f_k}$ on all of M.*

Proof. We have

$$\begin{aligned} \left(-C_1 - \frac{n-2}{(n-1)}C_2(k)\right)|x|^{-\tau-2} &= -2\left(\frac{1}{2}C_1 + \frac{n-2}{2(n-1)}C_2(k)\right)|x|^{-\tau-2} \\ &\geq -2k|x|^{-\tau-2} \\ &= -2\Delta f_k \end{aligned}$$

on M_∞. Then

$$\begin{aligned} R_g &\geq -C_1|x|^{-\tau-2} \\ &\geq -2\Delta f_k + \frac{n-2}{n-1}C_2(k)|x|^{-\tau-2} \\ &\geq -2\Delta f_k + \frac{n-2}{n-1}|\nabla f_k|^2 \end{aligned}$$

on M_∞. By definition, $R^m_{f_k} \geq F^m_{f_k}$ on M_{cpct} as well, so the proposition follows. $\square$

Informally speaking, functions whose Laplacians fit within certain 'asymptotic slices' satisfy our lower bound for R^m_f, where the lower bound of the slice is determined by R_g and ∇f.

Proof of Lemma 2.14. We first recall that if $g_{f,m} = \varphi^{\frac{4}{n-2}}g$, then

$$R_{g_{f,m}} = \varphi^{-\frac{n+2}{n-2}}\left(\frac{-4(n-1)}{n-2}\Delta\varphi + R_g\varphi\right).$$

This formula can be found in [Bes07, Corollary 1.161(a)], albeit the opposite convention for the Laplacian is used there. In our case, $\varphi^{\frac{4}{n-2}} = v^{\frac{2m}{n-1}}$, so $\varphi = v^{\frac{m(n-2)}{2(n-1)}}$. Then

$$\begin{aligned} R_{g_{f,m}} &= \left(v^{\frac{m(n-2)}{2(n-1)}}\right)^{-\frac{n+2}{n-2}}\left(\frac{-4(n-1)}{n-2}\Delta v^{\frac{m(n-2)}{2(n-1)}} + R_g v^{\frac{m(n-2)}{2(n-1)}}\right) \\ &= v^{-\frac{m(n+2)}{2(n-1)}}\left(\frac{-4(n-1)}{n-2}\Delta(e^{-\frac{f}{m}})^{\frac{m(n-2)}{2(n-1)}} + R_g v^{\frac{m(n-2)}{2(n-1)}}\right) \\ &= -\frac{4(n-1)}{n-2}v^{-\frac{m(n+2)}{2(n-1)}}\Delta e^{-\frac{(n-2)f}{2(n-1)}} + v^{-\frac{2m}{n-1}}R_g \\ &= -\frac{4(n-1)}{n-2}e^{\frac{(n+2)f}{2(n-1)}}\Delta e^{-\frac{(n-2)f}{2(n-1)}} + e^{\frac{2f}{n-1}}R_g. \end{aligned}$$

Letting $k = -\frac{n-2}{2(n-1)}$, we compute

$$\begin{aligned}\Delta e^{kf} &= \operatorname{div}(\nabla e^{kf}) \\ &= k \operatorname{div}((\nabla f)e^{kf}) \\ &= ke^{kf}\Delta f + k\langle \nabla f, \nabla(e^{kf})\rangle \\ &= ke^{kf}\Delta f + k^2 e^{kf}|\nabla f|^2 \\ &= e^{kf}(k\Delta f + k^2|\nabla f|^2).\end{aligned}$$

Then

$$\begin{aligned}R_{g_{f,m}} &= -\frac{4(n-1)}{n-2} e^{\frac{(n+2)f}{2(n-1)}} e^{-\frac{(n-2)f}{2(n-1)}} \left[-\frac{n-2}{2(n-1)}\Delta f + \left(\frac{n-2}{2(n-1)}\right)^2 |\nabla f|^2\right] + e^{\frac{2f}{n-1}} R_g \\ &= e^{\frac{2f}{n-1}} \left[\frac{4(n-1)}{n-2} \cdot \frac{n-2}{2(n-1)}\Delta f - \frac{4(n-1)}{n-2} \cdot \frac{(n-2)^2}{4(n-1)^2}|\nabla f|^2 + R_g\right] \\ &= e^{\frac{2f}{n-1}} \left(2\Delta f - \frac{n-2}{n-1}|\nabla f|^2 + R_g\right) \\ &= e^{\frac{2f}{n-1}} \left(2\Delta f - \frac{n-2}{n-1}|\nabla f|^2 + R_f^m - 2\Delta f + \frac{m+1}{m}|\nabla f|^2\right) \\ &= e^{\frac{2f}{n-1}} \left[R_f^m + \left(\frac{m+1}{m} - \frac{n-2}{n-1}\right)|\nabla f|^2\right] \\ &= e^{\frac{2f}{n-1}} \left(R_f^m + \frac{m+n-1}{m(n-1)}|\nabla f|^2\right) \\ &= e^{\frac{2f}{n-1}} \left(R_f^m - F_f^m\right). \end{aligned} \tag{2.35}$$

Since $e^{\frac{2f}{n-1}}$ is strictly positive, it follows from (2.35) that $R_{g_{f,m}} \geq 0$ if and only if $R_f^m \geq F_f^m$, with equality if and only if $R_f^m = F_f^m$. ☐

Proof of Theorem 2.9.

Part (a) We first note that $F_f^m \in L^1(M, g)$. Indeed, $|\nabla f|^2 = \mathcal{O}(r^{-2\tau-2})$ and $-2\tau - 2 < -n$. Moreover, recall that $R_f^m \in \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$. Then (2.35) and the assumption that $R_f^m \in L^1(M, g)$ implies

$$\begin{aligned}R_{g_{f,m}} &= e^{\frac{2f}{n-1}}(R_f^m - F_f^m) \\ &= [1 + \mathcal{O}(r^{-\tau})](R_f^m - F_f^m) \\ &= (R_f^m - F_f^m) + \mathcal{O}(r^{-\tau})\mathcal{O}(r^{-\tau-2}) \\ &= (R_f^m - F_f^m) + \mathcal{O}(r^{-2\tau-2}) \in L^1(M, g).\end{aligned} \tag{2.36}$$

Since $F_f^m \leq 0$ for $m \in S_n$, it follows that $R_f^m \geq 0 \geq F_f^m$. Then Lemma 2.14 implies $R_{g_{f,m}} \geq 0$, hence $\mathfrak{m}_{f,m}(g) \geq 0$ by Corollary 2.13.

It remains to prove the sharpness condition. By Lemma 2.12, $\mathfrak{m}_{f,m}(g) = 0$ if and only if $\mathfrak{m}(g_{f,m}) = 0$, which is equivalent to $R_{g_{f,m}} = 0$ since $(M^n, g_{f,m})$ is isometric to $(\mathbb{R}^n, \delta_{ij})$ by Corollary 2.13. By (2.35), this is equivalent to

$$0 \leq R_f^m = \frac{m+n-1}{m(1-n)}|\nabla f|^2 = F_f^m \leq 0,$$

which holds if and only if

$$\frac{m+n-1}{m(1-n)}|\nabla f|^2 = 0.$$

Since $m \neq 1 - n$, this is equivalent to $|\nabla f|^2 = 0$, i.e. f is constant. Since $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$, f is constant if and only if $f = 0$, or equivalently, $g_{f,m} = g$. Therefore, Corollary 2.13 implies $\mathfrak{m}_{f,m}(g) = 0$ if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$, as desired.

Part (b) The non-negativity of $\mathfrak{m}_{f,m}(g)$ follows from (2.36) and Lemma 2.14. If $\mathfrak{m}_{f,m}(g) = 0$, choose $k > 0$ such that $km \in [\varepsilon - n, 1-n)$. Since $R_{g_{f,m}} = 0$, it follows from (2.35) that $R_f^m = F_f^m$. Consequently,

$$\begin{aligned} R_f^{km} &= R_f^m + \left(\frac{m+1}{m} - \frac{km+1}{km}\right)|\nabla f|^2 \\ &= \left(\frac{m+n-1}{m(1-n)} + \frac{m+1}{m} - \frac{km+1}{km}\right)|\nabla f|^2 \\ &= \frac{km+n-1}{km(1-n)}|\nabla f|^2 \\ &\leq 0, \end{aligned}$$

for $km < 1-n$. Since $R_f^{\varepsilon-n} \geq 0$, it follows that

$$0 \leq R_f^{\varepsilon-n} \leq R_f^{km} = \frac{km+n-1}{km(1-n)}|\nabla f|^2 \leq 0.$$

Since $km \neq 1-n$, it follows that $|\nabla f|^2 = 0$, hence $f = 0$. Then as before, Corollary 2.13 implies $\mathfrak{m}_{f,m}(g) = 0$ if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$, as desired. ☐

Proof of Corollary 2.10. If $R_f \geq -\frac{1}{n-\varepsilon}|\nabla f|^2$, then $R_f^{\varepsilon-n} = R_f + \frac{1}{n-\varepsilon}|\nabla f|^2 \geq 0$. Moreover, since $|\nabla f|^2 = \mathcal{O}(r^{-2\tau-2})$ and $-2\tau - 2 < -n$, it follows that $\frac{1}{n-\varepsilon}|\nabla f|^2 \in L^1(M, g)$. This along with the assumption that $R_f \in L^1(M, g)$ implies $R_f^{\varepsilon-n} \in L^1(M, g)$. Since $\varepsilon - n < 1 - n$, Theorem 2.9(a) with $m = \varepsilon - n$ now implies $\mathfrak{m}_f(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$. ☐

Remark 2.16. Theorem 2.9(a) may be modified to extend to all $m \in \mathbb{R}\backslash\{0\}$ at the expense of the sharpness condition. In particular, $R_f^m \geq F_f^m$ implies $\mathfrak{m}_{f,m}(g) \geq 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, e^{\frac{2f}{n-1}}\delta_{ij})$. In general, if $f = 0$, then $\mathfrak{m}_{f,m}(g) = \mathfrak{m}(g)$, in which case the Bakry-Émery positive mass theorems reduce to the ordinary positive mass theorem.

Remark 2.17. In the case when M is spin, Theorem 2.9 provides a different Witten's formula for the Bakry-Émery mass than that in Theorem 2.4. From the standard Witten's formula for $(M, g_{f,m})$, we obtain the following Witten's formula for the Bakry-Émery mass when $R_f^m \geq 0$. Here $\overline{\psi}$ is a section of the spinor bundle $\Sigma\overline{M}$ on M associated to the metric $g_{f,m}$. It may be identified with a weighted Witten spinor as introduced in [BO22]:

$$\mathfrak{m}_{f,m}(g) = 4\int_M \left[|\nabla_{g_{f,m}}\overline{\psi}|^2 + \frac{1}{4}R_{g_{f,m}}|\overline{\psi}|^2\right] d\mathrm{Vol}_{g_{f,m}}.$$

Recall the bundle isometry $\mathbf{G}_u : \Sigma M \to \Sigma\overline{M}$ induced by the isometry $(TM, g) \to (TM, g_{f,m}), X \mapsto e^{\frac{f}{n-1}}X$. Setting $\overline{\psi} = \mathbf{G}_u(\psi)$, for ψ a section of the spinor bundle on M associated to the metric g, the formulas for the spin connection of a conformal change [BHM+15, Proposition 2.33] yield the following in an orthonormal frame:

$$(\nabla_{g_{f,m}})_{e_i}\overline{\psi} = \overline{(\nabla_g)_{e_i}\psi} + \frac{1}{2(n-1)}\overline{e_i \cdot \nabla f \cdot \psi} + \frac{1}{2(n-1)}e_i(f)\overline{\psi}.$$

Using that $\mathbf{G}_u$ is a bundle isometry, the expression for $R_{g_{f,m}}$, and the identity $d\mathrm{Vol}_{g_{f,m}} = e^{-\frac{nf}{n-1}}d\mathrm{Vol}_g$, the Witten's formula for $\mathfrak{m}_{f,m}(g)$ can be written entirely in terms of f, m, and a weighted Witten spinor on M.

2.2.3 The Bakry-Émery logarithmic functions

We now present some interesting consequences of Corollary 2.10. In particular, we prove the following surprising result:

Theorem 2.18. *Suppose (M^n, g) is asymptotically Euclidean of order $\tau \in \left(\frac{n-2}{2}, n-2\right)$. Assume the scalar curvature R of (M, g) is non-negative, positive somewhere, and belongs to $\mathcal{C}^{0,\alpha}_{-\tau-2}(M)$. There is a one-parameter family $\{f_m\}_{m\in(-\infty,1-n)}$ of non-constant functions in $\mathcal{C}^{2,\alpha}_{-\tau}(M) \cap \mathcal{C}^\infty(M)$ satisfying $R_{f_m} < 0$ and $\mathfrak{m}_{f_m}(g) > 0$. In particular, there exist non-constant functions $f \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$ for which $R_f < 0$ and $\mathfrak{m}_f(g) > 0$.*

Informally speaking, positive mass is a property that is not limited to positively (weighted) curved manifolds, and Bakry-Émery theory allows us to find continuous families of functions that exemplify this phenomenon. We call the members of the one-parameter family $\{f_m\}$ in Theorem 2.18 the *Bakry-Émery logarithmic functions.* These functions also serve as counterexamples to the converse of [BO22, Theorem 2.13]. To prove Theorem 2.18, we first solve the equation $R_f^m = 0$ for any arbitrary $m \in \mathbb{R} \setminus [-1, 0]$:

Proposition 2.19. *Suppose (M^n, g) satisfies the hypotheses of Theorem 2.18. Then for each $m \in \mathbb{R} \setminus [-1, 0]$, there exists nonzero $f \in \mathcal{C}^{2;\alpha}_{-\tau}(M)$ such that $R_f^m = 0$.*

Proof. We want to solve the equation

$$R + 2\Delta f - \frac{m+1}{m}|\nabla f|^2 = 0, \quad f \in \mathcal{C}^{2;\alpha}_{-\tau}(M). \tag{2.37}$$

Let us first transform this into a linear problem. In asymptotic coordinates on M, set

$$K = \sqrt{\frac{m+1}{m}}, \quad \lambda = \frac{2}{K},$$
$$f_\lambda(x) = \frac{1}{\lambda} f(\lambda x), \quad R_\lambda(x) = R(\lambda x),$$
$$w(x) = e^{-K f_\lambda(x)} - 1.$$

With these substitutions, (2.37) becomes

$$R_\lambda + K\Delta f_\lambda - K^2 |\nabla f_\lambda|^2 = 0. \tag{2.38}$$

We also compute

$$-\Delta w = e^{-K f_\lambda}(K\Delta f_\lambda - K^2 |\nabla f_\lambda|^2).$$

Combining this with (2.38) yields

$$R_\lambda e^{-K f_\lambda} - \Delta w = 0,$$

that is

$$-\Delta w + R_\lambda w = -R_\lambda. \tag{2.39}$$

This is a linear elliptic equation. Keeping in mind the definition of w, solving (2.37) is therefore equivalent to finding a function w such that

$$\begin{cases} -\Delta w + R_\lambda w = -R_\lambda, \\ w > -1, \\ f(x) = -\dfrac{\lambda}{K} \log\left(1 + w\left(\dfrac{x}{\lambda}\right)\right) \in \mathcal{C}^{2,\alpha}_{-\tau}(M). \end{cases} \tag{2.40}$$

We argue that the operator

$$-\Delta + R_\lambda : \mathcal{C}^{2,\alpha}_{-\tau}(M) \to \mathcal{C}^{0,\alpha}_{-\tau-2}(M) \tag{2.41}$$

is injective. Suppose $-\Delta u + R_\lambda u = 0$ for some $u \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$. Since $u(x) \to 0$ as $|x| \to \infty$ and $R_\lambda \geq 0$, the maximum principle applied to $-\Delta + R_\lambda$ and progressively larger balls in M implies that u is identically zero. So (2.41) is indeed injective; it is then surjective by [LP87, Theorem 9.2(d)] and the fact that $\tau \in \left(\frac{n-2}{2}, n-2\right)$. This, together with the hypothesis $R_\lambda \in \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$, guarantees the existence of a unique $w \in \mathcal{C}^{2;\alpha}_{-\tau}(M)$ satisfying (2.39). Since R_λ is not identically zero, neither is w. By (2.39), the function $\tilde{w} = w+1$ satisfies $(-\Delta + R_\lambda)\tilde{w} = 0$, is nonconstant and is asymptotically 1. By the maximum principle, $\tilde{w}$ attains a strictly positive minimum, so $w \geq -1 + \epsilon$ on M for some $\epsilon > 0$.

It remains to verify the last property in (2.40). Since $w(x/\lambda) = \mathcal{O}(|x|^{-\tau})$ as $|x| \to \infty$, it eventually falls into the radius of convergence of the Taylor expansion of $\log(1 + \bullet)$. Thus for large $|x|$ we have

$$f(x) = -\frac{\lambda}{K} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[w\left(\frac{x}{\lambda}\right)\right]^k = \mathcal{O}(|x|^{-\tau}).$$

Differentiating f in the ith direction, we get

$$\partial_i f(x) = -\frac{1}{K(1+w(x/\lambda))}\partial_i w\left(\frac{x}{\lambda}\right).$$

Since $1+w(x/\lambda) \geq \epsilon > 0$ and $\partial_i w = \mathcal{O}(|x|^{-\tau-1})$, it follows that

$$|\nabla f(x)| = \mathcal{O}(|x|^{-\tau-1}).$$

Similarly, one computes and estimates

$$\begin{aligned}\partial_i\partial_j f(x) &= \frac{1}{\lambda K(1+w(x/\lambda))^2}\partial_i w\left(\frac{x}{\lambda}\right)\partial_j w\left(\frac{x}{\lambda}\right) - \frac{1}{\lambda K(1+w(x/\lambda))}\partial_i\partial_j w\left(\frac{x}{\lambda}\right) \\ &\leq \mathcal{O}(|x|^{-2\tau-2}) + \mathcal{O}(|x|^{-\tau-2}).\end{aligned} \tag{2.42}$$

where the decay $\partial_i\partial_j w(x) = \mathcal{O}(|x|^{-\tau-2})$ is used. Thus

$$|\nabla^2 f(x)| = \mathcal{O}(|x|^{-\tau-2}).$$

Finally, we will show that

$$[\nabla^2 f]_{\mathcal{C}^\alpha(B_{|x|/2}(x))} = \mathcal{O}(|x|^{-\tau-2-\alpha}). \tag{2.43}$$

Let $x \in M$ be arbitrary and let $y, z \in B_{|x|/2}(x)$. By (2.42), we have

$$\begin{aligned}|\partial_i\partial_j f(y) - \partial_i\partial_j f(z)| \leq &\left|\frac{C}{(1+w(y/\lambda))^2}\partial_i w\left(\frac{y}{\lambda}\right)\partial_j w\left(\frac{y}{\lambda}\right) - \frac{C}{(1+w(z/\lambda))^2}\partial_i w\left(\frac{z}{\lambda}\right)\partial_j w\left(\frac{z}{\lambda}\right)\right| \\ &+ \left|\frac{C}{(1+w(y/\lambda))}\partial_i\partial_j w\left(\frac{y}{\lambda}\right) - \frac{C}{(1+w(z/\lambda))}\partial_i\partial_j w\left(\frac{z}{\lambda}\right)\right| \\ =:& A + B.\end{aligned} \tag{2.44}$$

Using that $w \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$ and $1+w \geq \epsilon > 0$, we can bound

$$\begin{aligned}B &\leq \left|\frac{C}{(1+w(y/\lambda))}\left(\partial_i\partial_j w\left(\frac{y}{\lambda}\right) - \partial_i\partial_j w\left(\frac{z}{\lambda}\right)\right)\right| + \left|\left(\frac{C}{1+w(y/\lambda)} - \frac{C}{1+w(z/\lambda)}\right)\partial_i\partial_j w\left(\frac{z}{\lambda}\right)\right| \\ &\leq C[\partial_i\partial_j w]_{C^\alpha(B_{|x|/2\lambda}(\frac{x}{\lambda}))}\left|\frac{y}{\lambda} - \frac{z}{\lambda}\right|^\alpha + C\left|w\left(\frac{z}{\lambda}\right) - w\left(\frac{y}{\lambda}\right)\right|\left|\frac{z}{\lambda}\right|^{-\tau-2} \\ &\leq C\left(|x|^{-\tau-2-\alpha}|y-z|^\alpha + \left|w\left(\frac{z}{\lambda}\right) - w\left(\frac{y}{\lambda}\right)\right||x|^{-\tau-2}\right).\end{aligned} \tag{2.45}$$

Since y and z are connected by a path in $B_{|x|/2}(x) \subset B_{2|x|}(0)$, one has

$$\begin{aligned}|w(y) - w(z)| &\leq |y-z| \sup_{B_{2|x|}(0)} |\nabla w| \leq C|y-z||x|^{-\tau-1} \leq C|y-z|^\alpha|x|^{-\tau-\alpha} \\ &\leq C|y-z|^\alpha|x|^{-\alpha},\end{aligned}$$

and hence

$$[w]_{\mathcal{C}^\alpha(B_{|x|/2}(x))} \leq C|x|^{-\alpha}.$$

Substituting a rescaled version of this into (2.45) yields the bound

$$B \leq C|y-z|^\alpha|x|^{-\tau-2-\alpha}.$$

The same bound for A (given in (2.44)) can be obtained by entirely analogous means. Thus (2.43) follows, and the proposition is proved. ☐

Proof of Theorem 2.18. By Proposition 2.19, for each $m \in (-\infty, 1-n)$, there exists a non-constant function $f_m \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$ satisfying $R^m_{f_m} = 0$. Then

$$R_{f_m} = R^m_{f_m} + \frac{1}{m}|\nabla f_m|^2 = \frac{1}{m}|\nabla f_m|^2 < 0.$$

Since $m < 1-n$, $R_{f_m} \geq -\frac{1}{n-\varepsilon}|\nabla f|^2$ with $\varepsilon = m+n < 1$, so Corollary 2.10 implies $\mathfrak{m}_{f_m}(g) \geq 0$. Moreover, since $R^m_{f_m} \geq 0$ and f_m is not identically zero, Theorem 2.9 applied to m implies $\mathfrak{m}_{f_m}(g) > 0$. ☐

2.2.4 A Bochner-type theorem for spin manifolds

In the case when M is spin, it is of interest to note that the Bakry-Émery conformal metric $g_{f,m} = e^{\frac{2f}{1-n}} g$ corresponds to precisely the conformal change that recovers the weighted Dirac operator in the sense described in Remark 1.4 (in particular (1.6)). From this observation, we obtain the following consequence of Lemma 2.14, which is a generalization of Corollary 1.7(d).

Corollary 2.20. *Let M be a closed spin manifold and $f \in C^\infty(M)$.*

(a) If $m \in \mathbb{R}$, $R_f^m \geq F_f^m$, and $R_f^m > F_f^m$ at some point, then M admits no nontrivial harmonic spinors.

(b) If $m \in \mathbb{R} \setminus [1-n, 0]$, $R_f^m \geq 0$, and $R_f^m > 0$ at some point, then M admits no nontrivial harmonic spinors.

Proof. By Lemma 2.14, the condition that $R_f^m \geq F_f^m$ and $R_f^m > F_f^m$ at some point implies that $g_{f,m}$ has scalar curvature $R_{g_{f,m}} \geq 0$, yet $R_{g_{f,m}} > 0$ at some point. By integrating the Lichnerowicz formula on the spinor bundle determined by $g_{f,m}$ as in [LM89, Corollary 8.9], one sees that there is no nontrivial harmonic spinor in that spinor bundle. Then (1.6) shows there is also no nontrivial harmonic spinor in the spinor bundle determined by the metric g, with respect to the *weighted* Dirac operator D_f. Now (a) follows from the fact that D and D_f are unitarily equivalent, hence have the same eigenvalues [BO22, Proposition 1.20]. Assertion (b) follows immediately from (a), because $F_f^m \leq 0$ when $m \in \mathbb{R} \setminus [1-n, 0]$. □

When m is negative it holds that $R_f^m \geq R_f$, so this corollary does *not* follow from the weighted Friedrich inequality [BO22, Theorem 1.23]. Corollary 2.20 also raises interest in finding functions f so that R_f^m is constant on closed manifolds. To study this, we first note the following fact about the Bakry-Émery scalar curvatures, based on Theorem 2.1 in [DLD87].

Lemma 2.21. *Let (M, g) be a Riemannian manifold, with $f \in C^\infty(M)$. Let $u = e^{-\frac{f(m+1)}{2m}}$. We have*

$$-\frac{4m}{m+1}\Delta_g u + Ru = R_f^m u.$$

Proof. Set $w = e^{-\frac{f}{m}}$, then $u = w^{\frac{m+1}{2}} = e^{-\frac{f(m+1)}{2m}}$. We omit the g subscript in the Laplacian. We have

$$\begin{aligned}
\Delta u &= \frac{(m+1)}{2} w^{\frac{m-1}{2}} \Delta w + \frac{(m+1)}{2}\frac{(m-1)}{2} w^{\frac{m-3}{2}} |\nabla w|^2 \\
\Delta w &= -\frac{1}{m} e^{-\frac{f}{m}} \Delta f + \frac{1}{m^2} |\nabla f|^2 e^{-\frac{f}{m}} \\
|\nabla w|^2 &= \frac{1}{m^2} e^{-\frac{2f}{m}} |\nabla f|^2.
\end{aligned}$$

These imply that

$$\begin{aligned}
-\frac{4m}{m+1}\Delta u + Ru &= -\frac{4m}{m+1} w^{\frac{m-1}{2}} \Delta w + \frac{(m+1)}{2}\frac{(m-1)}{2} w^{\frac{m-3}{2}} |\nabla w|^2 + Ru \\
&= -\frac{4m}{m+1}\left(-\frac{(m+1)}{2m} e^{-\frac{f(m+1)}{2m}} \Delta f + \frac{(m+1)}{2m^2} e^{-\frac{f(m+1)}{2m}} |\nabla f|^2\right) \\
&\quad - \frac{4m}{m+1}\left(\frac{(m+1)(m-1)}{4m^2} e^{-\frac{f(m+1)}{2m}} |\nabla f|^2\right) \\
&= 2e^{-\frac{f(m+1)}{2m}} \nabla f - \frac{2}{m} e^{-\frac{f(m+1)}{2m}} - \frac{(m-1)}{m} e^{-\frac{f(m+1)}{2m}} |\nabla f|^2 \\
&= R_f^m u.
\end{aligned}$$

□

Corollary 2.22. *Let (M, g) be a closed Riemannian manifold. For each $m \in \mathbb{R} \setminus \{0\}$, there is a unique constant $\lambda_m \in \mathbb{R}$ and a smooth function $f \in C^\infty(M)$ unique up to an additive constant, such that $R_f^m = \lambda_m$. If additionally the scalar curvature R of g is non-negative, then $\lambda_m \geq 0$.*

Proof. Assume $m \in \mathbb{R} \setminus [-1, 0]$. By Lemma 2.21, solving $R_f^m = \lambda_m$ is equivalent to finding a positive solution u to the eigenvalue problem

$$Lu := \left(-\frac{4m}{m+1}\Delta_g + R\right) u = \lambda_m u,$$

where u is related to f by $u = e^{-\frac{(m+1)f}{2m}}$. Standard elliptic theory tells us that only the principal eigenfunctions of L do not change sign; thus, λ_m must equal the principal eigenvalue, which has multiplicity one and is given by the Rayleigh quotient

$$\lambda_m = \inf_{v \in H^1(M), \|v\|_{L^2(M)}=1} \left[\int_M \left(\frac{4m}{m+1} |\nabla v|^2 + R v^2 \right) d\mathrm{Vol}_g \right].$$

This is clearly non-negative when $R \geq 0$. Finally, since the eigenspace of λ_m is one-dimensional, u is unique up to scaling, so f is unique up to translation. □

Note that, as $m \to \infty$, the eigenvalue λ_m approaches the first eigenvalue $\lambda_P(g)$ of the operator $-4\Delta + R$, while as $m \to 1 - n$ from the left, the eigenvalue λ_m approaches the first eigenvalue $\mu_1(g)$ of the conformal laplacian $-4\frac{n-1}{n-2}\Delta + R$. The weighted Friedrich Inequality implies that any eigenvalue λ of the Dirac operator on M satisfies $\lambda^2 \geq \frac{n}{4(n-1)}\lambda_P(g)$, while the Hijazi inequality implies that any eigenvalue satisfies $\lambda^2 \geq \frac{n}{4(n-1)}\mu_1(g)$. In particular, the inequality $\lambda^2 \geq \frac{n}{4(n-1)}\lambda_P(g)$ is equivalent to the weaker inequalities $\lambda^2 \geq \frac{n}{4(n-1)}\lambda_m(g)$ holding for all $m > 0$, while the inequality $\lambda^2 \geq \frac{n}{4(n-1)}\mu_1(g)$ is equivalent to the weaker inequalities $\lambda^2 \geq \frac{n}{4(n-1)}\lambda_m(g)$ holding for all $m < 1 - n$.

A Appendix

A.1 Proof of Lemma 1.3

Recall we work on $M^n \times F^m$ with the metric $\overline{g} = g \oplus v^2 h$, where (F,h) is a Riemannian manifold admitting parallel spinors. We work near a point (m_0, f_0), and choose coordinates $\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m$ for $M \times F$ so that $\xi_1, \ldots, \xi_n$ are geodesic normal coordinates at a point $m_0 \in M$ and $\zeta_1 = e^{\frac{f}{m}}\partial_1, \ldots, \zeta_m = e^{\frac{f}{m}}\partial_m$ are orthonormal at $f_0 \in F$, with ∂_i geodesic normal coordinates at $f_0 \in F$. We write ξ_α, ζ_a, and interchangeably treat the index a as running from 1 to m or $n+1$ to $n+m$. We will also interchangeably write $\xi_\alpha = \partial_\alpha$. Note the metric on $M \times F$ in the coordinate frame ξ_α, ∂_a has the following form (up to first order) at (m_0, f_0)

$$\bar{g} = \begin{bmatrix} I_n & 0 \\ 0 & e^{-\frac{2f}{m}} I_m \end{bmatrix}.$$

We use the standard formulas for the Christoffel symbols of the Levi-Civita connection in the coordinate frame ξ_α, ζ_a. We have

$$\begin{aligned}
\nabla_{\xi_\alpha}\xi_\beta &= \frac{1}{2} g^{ip}(g_{p\alpha,\beta} + g_{p\beta,\alpha} - g_{\alpha\beta,p})\partial_i \\
&= 0
\end{aligned}$$

since $g_{p\alpha}, g_{p\beta}, g_{\alpha\beta}$ are equal to 1 or 0 up to first order,

$$\begin{aligned}
\nabla_{\xi_\alpha}\zeta_a &= \xi_\alpha(e^{f/m})\partial_a + \frac{e^{f/m}}{2} g^{ip}(g_{p\alpha,a} + g_{pa,\alpha} - g_{\alpha a,p})\partial_i \\
&= \frac{1}{m}\xi_\alpha(f)e^{f/m}\partial_a + \frac{e^{f/m}}{2} g^{ii}(g_{i\alpha,a} + g_{ia,\alpha} - g_{\alpha a,i})\partial_i \\
&= \frac{1}{m}\xi_\alpha(f)e^{f/m}\partial_a + \frac{e^{f/m}}{2} g^{aa} g_{aa,\alpha}\partial_a \\
&= \frac{1}{m}\xi_\alpha(f)e^{f/m}\partial_a + \frac{e^{f/m}}{2} e^{2f/m}\xi_\alpha(e^{-2f/m})\partial_a \\
&= 0
\end{aligned}$$

since g is diagonal, $g_{\alpha a} = 0$,$g_{i\alpha} = 1$ or 0 up to first order,

$$\begin{aligned}
\nabla_{\zeta_a}\xi_\alpha &= \frac{e^{f/m}}{2} g^{ip}(g_{pa,\alpha} + g_{p\alpha,a} - g_{a\alpha,p})\partial_i \\
&= \frac{e^{f/m}}{2} g^{ii}(g_{ia,\alpha} + g_{i\alpha,a} - g_{a\alpha,i})\partial_i \\
&= \frac{e^{f/m}}{2} g^{aa}(g_{aa,\alpha})\partial_a \\
&= \frac{e^{f/m}}{2} e^{2f/m}\xi_\alpha(e^{-2f/m})\partial_a \\
&= -\frac{1}{m}\xi_\alpha(f)\zeta_a
\end{aligned}$$

since g is diagonal, $g_{a\alpha} = 0$, $g_{i\alpha} = 1$ or 0 up to first order,

$$
\begin{aligned}
\nabla_{\zeta_a}\zeta_b &= \frac{e^{2f/m}}{2} g^{ip}(g_{pa,b} + g_{pb,a} - g_{ab,p})\partial_i \\
&= \frac{e^{2f/m}}{2} g^{ii}(g_{ia,b} + g_{ib,a} - g_{ab,i})\partial_i \\
&= -\frac{e^{2f/m}}{2} g^{\alpha\alpha}(g_{ab,\alpha})\xi_\alpha \\
&= -\delta_{ab}\frac{e^{2f/m}}{2}\xi_\alpha(e^{-2f/m})\xi_\alpha \\
&= \frac{1}{m}\delta_{ab}\nabla f
\end{aligned}
$$

where we use that $\partial_a(f) = 0, g$ is diagonal, and all metric components are constant in the F directions. This completes the calculation of the Levi-Civita connection on $M \times F$. We use 1.4 to compute the spin connection. Recall the following connections and tensors

$$
\begin{aligned}
T(X,Y) &:= (\nabla_{X^V} Y^V)^H + (\nabla_{X^V} Y^H)^V \\
A(X,Y) &:= (\nabla_{X^H} Y^V)^H + (\nabla_{X^H} Y^H)^V \\
\nabla^Z_X Y &:= (\nabla_{X^V} Y^V)^V \\
\nabla^V_X Y &:= (\nabla_{X^H} Y^V)^V.
\end{aligned}
$$

By the above calculations, the connections ∇^V, ∇^Z on the vertical distributions are trivial, and $A(\xi_\alpha, \xi_\beta) = 0, A(\xi_\alpha, \zeta_a) = 0, T(\zeta_a, \zeta_b) = \frac{1}{m}\delta_{ab}\nabla f$. It follows from 1.4 that for a section $\psi \in \Gamma(\Sigma(M \times F))$ we have

$$
\begin{aligned}
\nabla_{\xi_\alpha}\psi &= \xi_\alpha(\psi) \\
\nabla_{\zeta_a}\psi &= \zeta_a(\psi) + \frac{1}{2m}\zeta_a \cdot \nabla f \cdot \psi
\end{aligned}
$$

A.2 Proof of (2.14)

All the computations below are in a geodesic frame based at a fixed point $p \in M$. By Cartan's magic formula, we have

$$
\begin{aligned}
d(e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) &= \mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) - e_i \lrcorner (d(e_j \lrcorner d\mathrm{Vol}_{\overline{g}})) \\
&= \mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) - e_i \lrcorner (\mathcal{L}_{e_j} d\mathrm{Vol}_{\overline{g}} - e_j \lrcorner dd\mathrm{Vol}_{\overline{g}}) \\
&= \mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) - e_i \lrcorner \mathrm{div}(e_j) d\mathrm{Vol}_{\overline{g}} \\
&= \mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}).
\end{aligned} \tag{A.1}
$$

Observe that $\mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}})$ is an $(n-1)$-form, determined by its values on $(n-1)$-tuples $(e_{\sigma(1)}, ..., e_{\sigma(n-1)})$, where σ is strictly increasing. If $\sigma(k) = j$ for some j, it is easily checked that $(\mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}))(e_{\sigma(1)}, ..., e_{\sigma(n-1)}) = 0$. Meanwhile, by the Leibniz formula for the Lie derivative, we have

$$
\begin{aligned}
(\mathcal{L}_{e_i}(e_j \lrcorner d\mathrm{Vol}_{\overline{g}}))(e_1, ..., \hat{e_j}, ..., e_n) &= e_i[d\mathrm{Vol}_{\overline{g}}(e_j, e_1, ..., \hat{e_j}, ..., e_n)] - (e_j \lrcorner d\mathrm{Vol}_{\overline{g}})([e_i, e_1], ..., \hat{e_j}, ..., e_n) \\
&\quad - \cdots - (e_j \lrcorner d\mathrm{Vol}_{\overline{g}})(e_1, ..., \hat{e_j}, ..., e_{n-1}, [e_i, e_n]) \\
&= 0,
\end{aligned}
$$

where we used the fact that the Lie brackets vanish at p. Then by (A.1), $d(e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) = 0$. Then

$$
\begin{aligned}
d\alpha &= d\langle [e^i, e^j]\psi_0, \xi\rangle_{\overline{g}} \wedge (e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) \\
&= [\langle [e^i, e^j]\nabla_{e_k}\psi_0, \xi\rangle_{\overline{g}} + \langle [e^i, e^j]\psi_0, \nabla_{e_k}\xi\rangle_{\overline{g}}]]\varepsilon^k \wedge (e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}).
\end{aligned} \tag{A.2}
$$

Since interior contraction satisfies a graded Leibniz rule with respect to the wedge product of forms, it follows that

$$\begin{aligned}\varepsilon^k \wedge (e_i \lrcorner e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) &= e_i \lrcorner (\varepsilon^k \wedge (e_j \lrcorner d\mathrm{Vol}_{\overline{g}})) + (e_i \lrcorner \varepsilon^k) \wedge (e_j \lrcorner d\mathrm{Vol}_{\overline{g}}) \\ &= -e_i \lrcorner (-e_j \lrcorner (\varepsilon^k \wedge d\mathrm{Vol}_{\overline{g}}) + (e_j \lrcorner \varepsilon^k) d\mathrm{Vol}_{\overline{g}}) + \delta_i^k e_j \lrcorner d\mathrm{Vol}_{\overline{g}} \\ &= \sum_{k \le n+m} (-\delta_j^k e_i \lrcorner d\mathrm{Vol}_{\overline{g}} + \delta_i^k e_j \lrcorner d\mathrm{Vol}_{\overline{g}}). \end{aligned} \tag{A.3}$$

Plugging (A.3) into (A.2), we obtain

$$\begin{aligned} d\alpha = &-\langle [e^i, e^j] \nabla_{e_j} \psi_0, \xi \rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}} - \langle [e^i, e^j] \psi_0, \nabla_{e_j} \xi \rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}} + \langle [e^i, e^j] \nabla_{e_i} \psi_0, \xi \rangle_{\overline{g}} e_j \lrcorner d\mathrm{Vol}_{\overline{g}} \\ &+ \langle [e^i, e^j] \psi_0, \nabla_{e_i} \xi \rangle_{\overline{g}} e_j \lrcorner d\mathrm{Vol}_{\overline{g}} \\ &= -2\langle [e^i, e^j] \nabla_{e_j} \psi_0, \xi \rangle e_i \lrcorner d\mathrm{Vol}_{\overline{g}} - 2\langle [e^i, e^j] \psi_0, \nabla_{e_j} \xi \rangle_{\overline{g}} e_i \lrcorner d\mathrm{Vol}_{\overline{g}}, \end{aligned}$$

as claimed. □

A.3 Weighted function spaces

Here, we provide a brief discussion on weighted Lebesgue, $\mathcal{C}^k$ and Hölder spaces, which serve as the foundation of the asymptotic analysis employed throughout the paper.

Let (M^n, g) be an asymptotically flat manifold with asymptotic coordinates $\{x^i\}$ on M_∞, and let $r(x) = |x|$ on M_∞.

Definition A.1 (Weighted Lebesgue space). *Let $q \ge 1$ and $\beta \in \mathbb{R}$. The* weighted Lebesgue space *$L^q_{0,\beta}(M)$ is the set of all $u \in L^1_{loc}(M)$ for which the norm*

$$||u||_{q,0,\beta} := \left(\int_M |r^{-\beta} u|^q \rho^{-n} d\mathrm{Vol}_g \right)^{1/q}$$

is finite.

Definition A.2 (Weighted Sobolev space). *For $k \in \mathbb{N}_0$, the* weighted Sobolev space *$L^q_{k,\beta}(M)$ is the set of functions u for which $|\nabla^i u| \in L^q_{0,\beta-i}(M)$ for all $0 \le i \le k$ equipped with the norm*

$$\begin{aligned} ||u||_{q,k,\beta} :&= \sum_{0 \le i \le k} ||\nabla^i u||_{q,0,\beta-i} \\ &= \sum_{0 \le i \le k} \left(\int_N |r^{i-\beta} u|^q r^{-n} d\mathrm{Vol}_g \right)^{1/q}. \end{aligned}$$

We note that if $u \in L^q_{k,\beta}(M)$, then $\nabla^j u \in L^q_{k-j,\beta-j}(M)$ for any $0 \le j \le k$ since

$$\begin{aligned} ||\nabla^j u||_{q,k-j,\beta-j} &= \sum_{0 \le i \le k-j} ||\nabla^{i+j} u||_{q,0,\beta-i-j} \\ &= ||u||_{q,k,\beta} - \sum_{k-j < i \le k} ||\nabla^i u||_{q,0,\beta-i} \\ &\le ||u||_{q,k,\beta} < \infty. \end{aligned}$$

Definition A.3 (Weighted $\mathcal{C}^k$ space). *For $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$ the* weighted $\mathcal{C}^k$ space *$\mathcal{C}^k_\beta(M)$ is the set of $u \in \mathcal{C}^k(M)$ for which the norm*

$$||u||_{\mathcal{C}^k_\beta(M)} := \sum_{0 \le i \le k} \sup_{x \in M} \left[r(x)^{i-\beta} |\nabla^i u(x)| \right]$$

is finite.

We note that if $u \in \mathcal{C}^k_\beta(M)$, then $\nabla^j u \in \mathcal{C}^{k-j}_{\beta-j}(M)$ for $0 \leq j \leq k$ since

$$
\begin{aligned}
||\nabla^j u||_{\mathcal{C}^{k-j}_{\beta-j}(N)} &= \sum_{0\leq i\leq k-j} \sup_{x\in N} \left[r(x)^{i+j-\beta}|\nabla^{i+j}u(x)|\right] \\
&= ||u||_{\mathcal{C}^k_\beta(M)} - \sum_{k-j<i\leq k} \sup_{x\in N} \left[r(x)^{i-\beta}|\nabla^i u(x)|\right] \\
&\leq ||u||_{\mathcal{C}^k_\beta(M)} < \infty.
\end{aligned}
$$

Definition A.4 (Weighted Hölder space)**.** *For $0 < \alpha < 1$, $k \in \mathbb{N}_0$, and $\beta \in \mathbb{R}$, the* weighted Hölder space *$\mathcal{C}^{k,\alpha}_\beta(M)$ is the set of $u \in \mathcal{C}^k_\beta(M)$ for which the norm*

$$
||u||_{\mathcal{C}^{k,\alpha}_\beta(M)} := ||u||_{\mathcal{C}^k_\beta(M)} + \sup_{x\in M} r(x)^{k+\alpha-\beta}[\nabla^k u]_{\mathcal{C}^\alpha(B_{r(x)/2}(x))}
$$

is finite, where $B_{r(x)/2}(x)$ is the metric ball of radius $r(x)/2$ centered at x and

$$
[\nabla^k u]_{\mathcal{C}^\alpha(B_{r(x)/2}(x))} = \sup_{y,z\in B_{r(x)/2}(x)} \frac{|\nabla^k u(y) - \nabla^k(z)|}{|y-z|^\alpha}.
$$

We note that if $u \in \mathcal{C}^{k,\alpha}_\beta(M)$, then $\nabla^j u \in \mathcal{C}^{k-j,\alpha}_{\beta-j}(M)$ since $\nabla^j u \in \mathcal{C}^{k-j}_{\beta-j}(M)$ and

$$
\begin{aligned}
||\nabla^j u||_{\mathcal{C}^{k-j,\alpha}_{\beta-j}(M)} &= ||\nabla^j u||_{\mathcal{C}^{k-j}_{\beta-j}(M)} + \sup_{x\in M} r(x)^{k-j+\alpha-\beta+j}[\nabla^{k-j}\nabla^j u]_{\mathcal{C}^\alpha(B_{r(x)/2}(x))} \\
&= ||\nabla^j u||_{\mathcal{C}^{k-j}_{\beta-j}(M)} + (||u||_{\mathcal{C}^{k,\alpha}_\beta(M)} - ||u||_{\mathcal{C}^k_\beta(M)}) < \infty.
\end{aligned}
$$

An important property of weighted spaces is that the index β reflects the order of growth. For instance, any function in $\mathcal{C}^{k,\alpha}_\beta(M)$ grows at most like r^β. To see this, if $u \in \mathcal{C}^{k,\alpha}_\beta(M)$, there is a constant $C > 0$ such that

$$
\sup_{x\in N} r^{-\beta}(x)|u(x)| \leq \sum_{0\leq i\leq k} \sup_{x\in N} \left[r(x)^{i-\beta}|\nabla^i u(x)|\right] \leq C.
$$

Another important property of weighted Holder spaces is the inclusion

$$
\mathcal{C}^{k,\alpha}_\beta(M) \subseteq \mathcal{C}^{k,\alpha}_{\beta'}(M) \tag{A.4}
$$

for any $\beta' > \beta$. The proof uses the following Sobolev embedding lemma:

Lemma A.5 (Weighted Sobolev lemma)**.** *Suppose $q > 1$, $k \in \mathbb{N}_0$, and $\ell \in \mathbb{N}_0$ satisfy*

$$
\ell - k - \alpha > \frac{n}{q}.
$$

Then for each $\varepsilon > 0$, there are continuous embeddings

$$
\mathcal{C}^{\ell,\alpha}_{\beta-\varepsilon}(M) \subset L^q_{\ell,\beta}(M) \subset \mathcal{C}^{k,\alpha}_\beta(M).
$$

Proof of (A.4). Choose $q > 1$ so that $1 - \alpha = (3-2) - \alpha > \frac{n}{q}$ (such a q exists since $0 < \alpha < 1$). Then by the weighted Sobolev lemma, $\mathcal{C}^{3,\alpha}_\beta(M) \subseteq \mathcal{C}^{2,\alpha}_{\beta+(\beta'-\beta)}(M) = \mathcal{C}^{2,\alpha}_{\beta'}(M)$, and since $\mathcal{C}^{2,\alpha}_\beta(M) \subseteq \mathcal{C}^{3,\alpha}_\beta(M)$, it follows that $\mathcal{C}^{2,\alpha}_\beta(M) \subseteq \mathcal{C}^{2,\alpha}_{\beta'}(M)$. ☐

Lemma A.6. *The square of the Dirac operator $D^2 : \mathcal{C}^{2,\alpha}_{-\tau}(M) \to \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$ is an isomorphism.*

Proof. If $2 - n < -\tau < 0$, then Theorem 9.2(d) and the appendix of [LP87] imply $D^2 = -\Delta + \frac{1}{4}R_g$ is an isomorphism. If instead $-\tau \leq 2 - n$, then $\mathcal{C}^{2,\alpha}_{-\tau}(M) \subseteq \mathcal{C}^{2,\alpha}_\beta(M)$ for any $2 - n < \beta < 0$ by the inclusion property (A.4). If $\varphi \in \mathcal{C}^{2,\alpha}_{-\tau}(M)$ satisfies $D^2\varphi = 0$, then $\varphi = 0$ since $\varphi \in \mathcal{C}^{2,\alpha}_\beta(M)$ and $D^2 : \mathcal{C}^{2,\alpha}_\beta(M) \to \mathcal{C}^{0,\alpha}_{\beta-2}(M)$ is an isomorphism. Then $D^2 : \mathcal{C}^{2,\alpha}_{-\tau}(M) \to \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$ is injective, so Theorem 9.2(d) of [LP87] now implies $D^2 : \mathcal{C}^{2,\alpha}_{-\tau}(M) \to \mathcal{C}^{0,\alpha}_{-\tau-2}(M)$ is an isomorphism. ☐

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