ABSTRACT. We study “proper pairings” for finite simple graphs. These are combinatorial objects that Huang and Postnikov used to give a bijective proof of Pak and Postnikov’s reciprocity formula for the spanning forest polynomial $f_G$. We find that by introducing the “component graph” – a combinatorial object related to proper pairings – we are able to see new enumerative properties for these graph objects. As a result of our study, we give another combinatorial proof of the reciprocity theorem similar to Huang and Postnikov’s. Furthermore, we generalize $f_G$ to a polynomial $f_{G,H}$ that records the spanning trees of a graph that contain a fixed subgraph, and we show that these generalized polynomials exhibit a similar reciprocity property. As an application, we deduce a generalization of Cayley’s formula from our generalized reciprocity.

1. INTRODUCTION

A spanning tree of a graph is a connected acyclic subgraph whose vertex set is that of the graph. Counting the number of labeled spanning trees of a simple finite graph is a well-studied problem in classical combinatorics. A. Cayley’s formula states that there are $n^{n-2}$ spanning trees of the complete graph $K_n$ on $n$ vertices [3], and the Matrix-Tree Theorem expresses the number of spanning trees of any connected graph $G$ as the determinant of a matrix [2]. While these results can provide nice formulas for the number of spanning trees with labeled vertices of a graph, neither offers a way to list the spanning trees.

A famous algorithm for listing labeled spanning trees is due to H. Prüfer [6], whose coding provides a bijection between labeled spanning trees of $K_n$ and length-$(n-2)$ sequences of vertices from the vertex set $[n] = \{1, 2, \ldots, n\}$. The Prüfer sequence for a given tree $T$ contains each vertex $v$ with multiplicity $\deg_T(v) - 1$, and A. Rényi introduced a graph polynomial $t_G$ which records these multiplicities for each spanning tree of a given graph [7].

I. Pak and A. Postnikov modified Rényi’s polynomial to record the degree sequences of spanning rooted forests of a graph $G$, rather than those of spanning trees [5]. This graph polynomial $f_G$ exhibits a remarkable reciprocity property, which relates the spanning trees of a graph to those of its complement. Pak and Postnikov initially proved this reciprocity formula by an inductive algebraic argument.

S. Huang and A. Postnikov later gave a combinatorial proof of this reciprocity property by constructing a bijection which specializes to the Prüfer code [4]. This bijection allows each term in the reciprocity formula to be interpreted as a unique “proper pairing,” a combinatorial object that defines a new spanning tree – the “replacement graph” – by specifying edges to add and delete in an initial spanning tree. Inclusion and exclusion of these replacement graphs yields the correct spanning trees of the complement graph.

Date: July 31, 2019.
In this paper, we generalize the spanning forest polynomial $f_G$ to a polynomial $f_{G,H}$, which records the spanning forests of a graph $G$ that contain all edges in $H$. We then show that this polynomial exhibits a reciprocity property (Theorem 3.3) similar to the one that Pak and Postnikov proved for $f_G$. Our generalization of the spanning forest polynomial allows us to prove the generalized form of Cayley’s formula (Corollary 3.4). In our proof of the generalized reciprocity property, we introduce a directed multigraph called the “component graph” as an alternative way to interpret the relationship between proper pairings and terms in the spanning tree polynomial. More specifically, we establish a constant-to-one map from a subset of proper pairings to each component graph (Proposition 5.5) and then compute the number of component graphs with labeled vertices (Proposition 5.4) to show that the terms in the reciprocity formula are exactly the monomials of the “replacement graphs” obtained from the operation specified by the proper pairings. Our interpretation of the terms in the generalized spanning forest polynomial $f_{G,H}$ may provide a way to give further insight into the geometry of spanning tree polytopes.

The structure of this paper is as follows: in section 2, we review some definitions and the properties of the graph polynomial $f_G$ as discussed in [5]; in section 3, we give our generalized reciprocity formula; in section 4, we define and discuss proper pairings; in section 5, we define the component graph and use it to prove enumerative properties of proper pairings; in section 6, we give the combinatorial proof of the generalized reciprocity formula.

2. $f_G$ and the Reciprocity Formula

We use standard graph definitions and notation, most of which can be found in [2]. In brief, the degree of a vertex $v$ in a graph $G$ (i.e. the number of edges adjacent to $v$ in $G$) is denoted $\deg_G(v)$; the number of connected components of a graph $G$ will be denoted $k(G)$; the complete graph on $n$ vertices is denoted $K_n$ and the empty graph $E_n$; and the complement of a graph $G$ is denoted $\overline{G}$. Furthermore, a spanning tree of a graph $G$ is a connected acyclic subgraph whose vertex set is that of the graph, and a spanning tree on a set of $n$ vertices is a spanning tree of the complete graph on those vertices. From now on, we reserve the symbol $G$ for a finite simple graph, and $T$ for a spanning tree.

We now review the spanning forest polynomial $f_G$, its properties, and the reciprocity formula as presented in [5]. Consider a simple graph $G$ with vertex set $V(G) = [n]$ and edge set $E(G)$. We define the extended graph $\tilde{G}$ of $G$ to be the simple graph on vertices $V(\tilde{G}) = \{0\} \cup V(G)$ with edge set $E(\tilde{G}) = E(G) \cup \{0i : i \in V(G)\}$. We associate a variable $x_i$ to each vertex $i$ ($x$ is used in place of $x_0$) and assign each spanning tree $T$ of $\tilde{G}$ a monomial
\[ m(T) := \prod_{i \in V(T)} x_i^{\deg_{\tilde{G}}(i)-1}, \]
as is done in [5]. We now define the spanning forest polynomial $f_G$.

**Definition 2.1.** [5] For a graph $G$ on $[n]$, the spanning forest polynomial
\[ f_G(x; x_1, x_2, \ldots, x_n) := \sum_{T} m(T), \]
where the sum is over all spanning trees $T$ of the extended graph $\tilde{G}$.

**Remark 2.2.** Notice that, for a spanning tree $T$ on vertices $\{0, 1, \ldots, n\}$, we can construct a spanning rooted forest $F_T$ by deleting vertex 0 from $T$ and designating all vertices adjacent to 0 in $T$ as roots of $F_T$. From now on, we use the symbol $F_T$ for the spanning rooted forest constructed from $T$. Thus, $f_G$ is also a sum over all the spanning rooted forests of $G$.

This technique of constructing a rooted forest from a tree also appears in [2].
The graph polynomial \( f_G \) exhibits several useful properties. One relates the spanning forests of a disjoint union of two graphs to the spanning trees of each graph. Let \( G_1 \) and \( G_2 \) be two graphs such that \( V(G_1) \cap V(G_2) = \emptyset \). We associate the variables \( y_1, y_2, \ldots, y_{n_1} \) and \( z_1, z_2, \ldots, z_{n_2} \) to the vertices of \( G_1 \) and \( G_2 \) respectively, and we associate the variable \( x \) to vertex 0 in \( G_1, \ G_2, \) and \( G_1 \cup G_2 \). Then the following formula holds:

\[
 f_{G_1 \cup G_2} (x; y_1, \ldots, y_{n_1}, z_1, \ldots, z_{n_2}) = x \cdot f_{G_1} (x; y_1, \ldots, y_{n_1}) \cdot f_{G_2} (x; z_1, \ldots, z_{n_2}).
\]

The proof can be found in [5].

Another important property of \( f_G \) is the reciprocity property between a graph and its complement, which was discovered by S. D. Bedrosian [1] in the case \( x_1 = x_2 = \cdots = x_n = 1 \).

**Theorem 2.3** (Reciprocity). [5] Let \( G \) be a graph on \([n]\). Then

\[
 f_G (x; x_1, \ldots, x_n) = (-1)^{n-1} f_{\overline{G}} (-x - Y; x_1, \ldots, x_n),
\]

where \( Y = x_1 + \cdots + x_n \).

One can find an algebraic proof for Theorem 2.3 in [5]. Additionally, Huang and Postnikov gave a combinatorial proof of Theorem 2.3 in [1], which we discuss in Remark 4.3.

3. \( f_{G,H} \) and the Generalized Reciprocity Formula

We generalize the definition of \( f_G \) from [5] by introducing a polynomial \( f_{G,H} \), which records the spanning trees of \( G \) that contain \( H \) as a subgraph.

**Definition 3.1.** For graphs \( G, H \) on \([n]\) with \( E(H) \subseteq E(G) \), the generalized spanning forest polynomial

\[
 f_{G,H} (x; x_1, \ldots, x_n) := \sum_T m(T),
\]

where the sum is over all spanning trees \( T \) of \( \overline{G} \) with \( E(H) \subseteq E(T) \).

**Remark 3.2.** Notice that when \( H \) is not a forest, \( f_{G,H} \) is zero. In the case \( H = E_n \), \( f_{G,E_n} \) equals \( f_G \), the spanning rooted forest polynomial in [5].

As with \( f_G \), the generalized spanning forest polynomial \( f_{G,H} \) exhibits a disjoint union property. Let \( G_1 \) and \( G_2 \) be two graphs such that \( V(G_1) \cap V(G_2) = \emptyset \) and \( H_1, H_2 \) be graphs such that \( V(H_1) = V(G_1), \ E(H_1) \subseteq E(G_1) \), and \( V(H_2) = V(G_2), \ E(H_2) \subseteq E(G_2) \). We again associate the variables \( y_1, y_2, \ldots, y_{n_1} \) and \( z_1, z_2, \ldots, z_{n_2} \) to the vertices of \( G_1 \) and \( G_2 \) respectively, and we associate the variable \( x \) to vertex 0 in \( \overline{G_1}, \ \overline{G_2}, \) and \( \overline{G_1 \cup G_2} \). Then the following formula holds:

\[
 f_{G_1 \cup G_2, H_1 \cup H_2} (x; y_1, \ldots, y_{n_1}, z_1, \ldots, z_{n_2}) = x \cdot f_{G_1,H_1} (x; y_1, \ldots, y_{n_1}) \cdot f_{G_2,H_2} (x; z_1, \ldots, z_{n_2}).
\]

Notice that every spanning tree \( T \) such that \( E(H_1 \cup H_2) \subseteq E(T) \) in graph \( \overline{G_1 \cup G_2} \) splits into two spanning trees \( T_1 \) and \( T_2 \) in graphs \( \overline{G_1} \) and \( \overline{G_2} \) respectively. Since \( \deg_{T_1}(0) - 1 = (\deg_{T_2}(0) - 1) + (\deg_{T_2}(0) - 1) + 1 \), we need one additional \( x \) on the right hand side.

Furthermore, the polynomial \( f_{G,H} \) exhibits a reciprocity property similar to the reciprocity formula in [5]. The following is the main result of this paper.

**Theorem 3.3** (Generalized Reciprocity). Let \( G_1, G_2, H \) be graphs on \([n]\) with \( G_1 \cup G_2 = K_n \) and \( G_1 \cap G_2 = H \). Then

\[
 f_{G_2,H} (x; x_1, \ldots, x_n) = (-1)^{n-|E(H)|-1} f_{G_1,H} (-x - Y; x_1, \ldots, x_n),
\]

where \( Y = x_1 + \cdots + x_n \).
We prove Theorem 3.3 in section 5. In the specialization $G_1 = G$ and $G_2 = \mathcal{G}$ (so that $H = E_n$), this result becomes Theorem 2.3. Additionally, our proof gives another combinatorial proof of Theorem 2.3.

A mild generalization of Cayley’s formula [3] follows easily from Theorem 3.3.

**Corollary 3.4.** Let $F$ be an unrooted spanning forest on the vertex set $[n]$ with $k$ components $c_1, \ldots, c_k$, and suppose component $c_i$ contains $n_i$ vertices. Then the number of spanning trees $T$ of the complete graph $K_n$ on $[n]$ such that $E(F) \subseteq E(T)$ is

$$n^{k-2} \prod_{i=1}^{k} n_i.$$

**Proof of Corollary 3.4.** First, notice that

$$f_{G,H}(0; x_1, \ldots, x_n) = (x_1 + \cdots + x_n) \left( \sum_T m(T) \right),$$

where the sum is over all spanning trees $T$ of $G$ such that $E(H) \subseteq E(T)$. Consider a monomial of a spanning tree $T'$ of $\mathcal{G}$, and note that the degree of $x$ will be 0 in $m(T')$ if and only if the degree of vertex 0 is 1 in $T'$. Deleting this edge and vertex 0 will define a spanning tree $T$ of $G$. Since 0 can be adjacent to any of vertices $\{1, \ldots, n\}$, we have the $(x_1 + \cdots + x_n)$ on the right hand side.

Consider an unrooted forest $F$ on the vertex set $[n]$. Then the family of spanning trees $T$ of $\tilde{F}$ with $E(F) \subseteq E(T)$ is in bijective correspondence with the family of rooted forests $F_T$ with underlying unrooted forest $F$. Therefore,

$$f_{F,F}(x; x_1, \ldots, x_n) = \sum_{(r_1, \ldots, r_k)} x^{k-1} \prod_{i=1}^{k} x_{r_i},$$

where, for each $i$, $r_i$ is the root of component $c_i$. By Theorem 3.3 with $G_1 = F$ and $G_2 = K_n$, we see that

$$f_{K_n,F}(x; x_1, \ldots, x_n) = (-1)^{|E(F)|+n-1} f_{F,F}(-x - Y; x_1, \ldots, x_n)$$

$$= (x + Y)^{k-1} \sum_{(r_1, \ldots, r_k)} \prod_{i=1}^{k} x_{r_i},$$

where $Y = x_1 + \cdots + x_n$. Thus, we have

$$f_{K_n,F}(0; 1, \ldots, 1) = n^{k-1} \prod_{i=1}^{k} n_i.$$

Then the number of spanning trees $T$ of $K_n$ with $E(F) \subseteq E(T)$ is

$$n^{k-2} \prod_{i=1}^{k} n_i.$$

□

We note that Corollary 3.4 reduces to Cayley’s formula when $F$ is the empty graph on $n$ vertices.
4. Proper Pairings

In this section, we discuss a combinatorial object called a proper pairing. In later sections, we will classify proper pairings into families $\mathcal{P}(T,V)$, which will allow us to interpret the terms in $(-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)$ in Theorem 3.3.

We first define proper pairings. Consider a spanning tree $T$ on vertices $\{0, 1, \ldots, n\}$.

**Definition 4.1.** Let $n$ be a fixed positive integer.

1. A *pairing* is a triple $P = (T, \mathcal{V}, \mathcal{S})$, where $T$ is a spanning tree on vertices $\{0, 1, \ldots, n\}$, $\mathcal{V}$ is a multiset of $\rho := |\mathcal{V}|$ vertices from $[n]$, and $\mathcal{S} = \{(v_1, r_1), (v_2, r_2), \ldots, (v_\rho, r_\rho)\}$ is a set of ordered pairs of vertices such that $r_1, r_2, \ldots, r_\rho$ are distinct roots of $F_T$ and $\{v_1, v_2, \ldots, v_\rho\} = \mathcal{V}$ as a multiset.

2. Given a pairing $P = (T, \mathcal{V}, \mathcal{S})$, we call $T$ its *tree*, $\mathcal{V}$ its *multiset*, and $\mathcal{S}$ its *pair set*. We define $\rho := |\mathcal{V}| = |\mathcal{S}|$ to be the *size* of the pairing.

3. The *replacement graph* $R_P$ of a pairing $P = (T, \mathcal{V}, \mathcal{S})$ is a multi-graph obtained from $T$ by, for each $i \in [\rho]$, deleting the edge $0r_i$ and adding $v_ir_i$. A pairing $P$ is called a *proper pairing* if its replacement graph $R_P$ is a spanning tree on $\{0, 1, \ldots, n\}$.

The following are some intuitive remarks about Definition 4.1.

**Remark 4.2.**

1. Since the roots $r_1, \ldots, r_\rho$ are required to be distinct in the pair set $\{(v_1, r_1), \ldots, (v_\rho, r_\rho)\}$, the $\rho$ elements of the pair set will be distinct even if some $v_i = v_j$ for $i \neq j$.

2. The size $\rho$ of a pairing $P = (T, \mathcal{V}, \mathcal{S})$ cannot exceed the number of roots of $F_T$ because the roots $r_1, \ldots, r_\rho$ in $\mathcal{S}$ must be distinct.

3. The replacement graph $R_P$ will have the same number of edges, counted with multiplicity, as $T$.

Figure 4 gives an example of two pairings with the same tree $T$ and multiset $\mathcal{V}$.

We discuss the bijection that Huang and Postnikov constructed in [4] in the following remark.

**Remark 4.3.** In this remark, we will use our notation instead of that used in [4]. Huang and Postnikov use an algorithm similar to the Prüfer coding. Their algorithm essentially associates each proper pairing $P = (T, \mathcal{V}, \mathcal{S})$ to an ordered pair $(T, W)$. Here, $W$ is a permutation of the set $\mathcal{V} \cup \{0\}_{k(F_T) - 1 - \rho}$, where $\{0\}_{k(F_T) - 1 - \rho}$ denotes the multiset of $k(F_T) - 1 - \rho$ zeros. Using this bijection, Huang and Postnikov were able to interpret each monomial in $(-1)^{n-1}f_{G_1}(x - Y; x_1, \ldots, x_n)$ as a proper pairing. More details about the bijection are provided in Appendix A.

5. Classification of Proper Pairings by Component Graphs

In this section, we classify proper pairings by their trees and multisets, which will allow us to interpret terms in $(-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)$.

Let $\mathcal{P}(T,V)$ denote the family of all proper pairings with tree $T$ and multiset $\mathcal{V}$. Theorem 5.1 gives the key enumerative property needed for our interpretation of $(-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)$.

**Theorem 5.1.** Suppose $T$ is a spanning tree on $\{0, 1, \ldots, n\}$. Let $k := k(F_T)$, and let $\mathcal{V}$ be a multiset of $\rho$ vertices from $[n]$ so that, for each $i \in [n]$, vertex $i$ is listed with multiplicity $\rho_i$. Then

$$|\mathcal{P}(T,V)| = \binom{k-1}{\rho} \cdot \binom{\rho}{\rho_1, \rho_2, \ldots, \rho_n}.$$
A spanning tree $T$ on vertices $\{0, 1, \ldots, 4\}$ and a multiset $V = \{2, 3\}$.

Ex. 1. $P_1 = (T, V, \{(2, 1), (3, 4)\})$ has $R_{P_1}$

Ex. 2. $P_2 = (T, V, \{(2, 4), (3, 2)\})$ has $R_{P_2}$

Figure 1. For a spanning tree $T$ and a multiset $V = \{2, 3\}$, we give two examples of pairings with their replacement graphs. The dashed lines in the replacement graphs indicate deleted edges, and the red lines indicate added edges. The red vertices are the vertices in $V$. Note that $P_1$ is a proper pairing; $P_2$ is not.

This theorem follows from Propositions 5.4 and 5.5 below.

Consider a spanning tree $T$ on vertices $\{0, 1, \ldots, n\}$. The connected components of $F_T$ canonically induce a set partition of $[n]$, which we will denote by $B(F_T)$. Let the map $h : [n] \to B(F_T)$ send each vertex $v \in [n]$ to the component of $v$ in $F_T$. Instead of counting $|P(T,V)|$ directly, we introduce a new graph called the component graph for each pairing $P$.

**Definition 5.2.** Suppose $P = (T, V, \{(v_1, r_1), (v_2, r_2), \ldots, (v_\rho, r_\rho)\})$ is a pairing. Then the component graph $C_P$ of $P$ is the directed multigraph whose

1. vertex set $V(C_P) = B(F_T)$, and
2. directed edge set $E(C_P) = \{h(v_i)h(r_i) : i \in [\rho]\}$.

Figure 2 gives the component graphs for the proper pairing examples in Figure 1. Note that the red edges in the replacement graphs, which are the edges added due to the pair set, become directed edges in the component graphs. However, the black edges, which are the edges in $F_T$, are no longer present in the component graph.

Let $\mathcal{C}(T,V)$ denote the family of component graphs for all proper pairings in $P(T,V)$. We will count $|\mathcal{C}(T,V)|$ and present a numerical relationship between $|\mathcal{C}(T,V)|$ and $|P(T,V)|$ to prove Theorem 5.1.

We define a **cycle** in a (directed) multigraph to be any closed trail, regardless of edge direction, with exactly one repeated vertex (i.e. the start and finish vertex). In particular, a loop will be considered a cycle.

**Proposition 5.3.** Suppose $T$ is a spanning tree on $\{0, 1, \ldots, n\}$, and let $V$ be a multiset of $\rho$ vertices from $[n]$ so that, for each $i \in [n]$, vertex $i$ is listed with multiplicity $\rho_i$.

Consider a directed multigraph $H$ on vertex set $B(F_T)$. Then $H \in \mathcal{C}(T,V)$ if and only if all of the following conditions are satisfied:
Ex. 1. $P_1 = (T, V, \{(2, 1), (3, 4)\})$ has $R_{P_1}$ and $C_{P_1}$

Ex. 2. $P_2 = (T, V, \{(2, 4), (3, 2)\})$ has $R_{P_2}$ and $C_{P_2}$

Figure 2. We construct the component graphs for the pairings $P_1$ and $P_2$ shown in Figure 1. The vertices $c_1, c_2, c_3$ of the component graph are $\{1\}, \{2, 3\}, \{4\}$ respectively.

1. Exactly $\rho$ vertices in $H$ have indegree 1, and all other vertices have indegree 0.
2. For each $b \in B(F_T)$, the outdegree of vertex $b$ in $H$ is $\text{outdeg}_H(b) = \sum_{i \in b} \rho_i$.
3. $H$ contains no cycles.

Proof. Suppose $H \in \mathcal{C}_{(T, V)}$, i.e. $H$ is the component graph of some proper pairing of the form $P = (T, V, \{(v_1, r_1), \ldots, (v_\rho, r_\rho)\})$. Points (1) and (2) are straightforward. For (3), suppose to the contrary that $H$ contained a cycle $b_1b_2\cdots b_m$ of length $m \geq 1$, where $b_1, \ldots, b_m \in B(F_T)$ are distinct components of $F_T$. Note that the induced subgraph of $T$ on $b_i$ is a tree on $|b_i|$ vertices and therefore has $|b_i| - 1$ edges. The subgraph of $R_P$ induced on $\bigcup_{i=1}^m b_i$ has “original edges” from $T$ and “new edges” from $P$. We know that there are at least $m$ new edges: those which appear as edges in the cycle $b_1\cdots b_m$. Therefore, the subgraph has at least

$$\left(\sum_{i=1}^m |b_i| - 1\right) + m = \sum_{i=1}^m |b_i| = \left|\bigcup_{i=1}^m b_i\right|$$

edges, which contradicts the fact that $R$ is a tree.

Conversely, suppose that all three properties hold for some directed graph $H$. We construct a proper pairing $P = (T, V, S)$ with component graph $H$ by assigning an ordered pair $(v(e), r(e))$ of vertices in $[n]$ to each edge $e \in E(H)$ as follows:

1. $v(e)$: For each edge $e$ from the set of outdeg$_H(b)$ edges leaving $b \in V(H)$, we assign $v(e)$ to be a vertex in component $b$ so that, after some $v(e)$ has been assigned to each edge, the set $\{e' : v(e') = j\}$ contains $\rho_j$ elements.
2. $r(e)$: To each directed edge $e = h(v(e))h(r_i)$ in $H$, we assign the root $r_i$ as $r(e)$.

(We note that, although in general there are many ways to execute step 1, there is exactly one way to execute step 2.) If we let $S := \{(v(e), r(e)) : e \in E(H)\}$, then this assignment process defines a pairing $P = (T, V, S)$ whose component graph is $H$.

We now show that $P$ is a proper pairing, i.e. that its replacement graph $R_P$ is a tree. Because $R_P$ contains the same number of edges as $T$, it suffices to check that $R_P$ is acyclic. Suppose to the contrary that $R_P$ had a cycle. Since the rooted forest $F_T$ contained no cycles, the cycle must include at least one edge in $E(R_P - \{0\}) - E(F_T)$. When all components of $F_T$ are contracted within $R_P$ to form the underlying undirected graph of $H$, this cycle will become a closed trail with at least one edge, which contradicts condition (3). Hence, $P$ is a proper pairing, so $H \in \mathcal{C}_{(T, V)}$. \qed
As we see in Figure 2, \( C_{P_1} \) satisfies all conditions in Proposition 5.3 and \( C_{P_2} \) does not.

Using the characterization of \( \mathcal{C}(T,V) \) from Proposition 5.3, we now count \( |\mathcal{C}(T,V)| \) in the following proposition.

**Proposition 5.4.** Let \( T, V = \{v_1, \ldots, v_p\} \), \( \rho \), \( \rho_i \), and \( k \) be defined as in the previous proposition. Label the \( k \) components of \( F_T \) as \( c_1, \ldots, c_k \) (so that \( \{c_1, \ldots, c_k\} = B(F_T) \)). For each \( i \in [k] \), let \( q_i := \sum_{j \in \rho} \rho_j \).

Then \( |\mathcal{C}(T,V)| = \frac{(k-1)!}{(k-p-1)!} \cdot \frac{1}{q_1 \cdots q_k} \).

**Proof.** It suffices to count the number of directed graphs \( H \) on \( \{c_1, \ldots, c_k\} \) that satisfy the three requirements in Proposition 5.3. Let \( H_0 \) denote the empty graph on vertices \( \{c_1, \ldots, c_k\} \). For each vertex \( v_i \) in \( V \), we construct a new directed multigraph \( H_i \) by adding a directed edge \( h(v_i)F(v_j) \) to \( H_{i-1} \), where \( F(v_j) \) is some other vertex in \( \{c_1, \ldots, c_k\} \), so that \( H_i \) is acyclic and each vertex of \( H_i \) has indegree at most 1. For a given \( i \in [\rho] \), we call \( F(v_i) \) the “finish vertex,” and we will count the number of ways to assign distinct finish vertices \( F(v_i) \) for \( i \in [\rho] \) to create \( H_i \). (Figure 3 gives an example of this assignment process.)

We claim that, after \( F(v_1), \ldots, F(v_{i-1}) \) have been designated (so that \( H_{i-1} \) has been constructed), there are exactly \( k - i \) legal choices for \( F(v_i) \): all \( k \) vertices \( c_1, \ldots, c_k \) except

1. \( F(v_1), \ldots, F(v_{i-1}) \), which were already used as finish vertices, and
2. One additional vertex \( h(v_m) \), where \( \gamma_m \) is the last value on the list defined by the algorithm
(a) Start with \( \gamma = i \) (so that \( \gamma_0 = i \)). Set a counter \( m = 0 \).
(b) Loop:
   (i) Store the current value of \( \gamma \) as \( \gamma_m \).
   (ii) If \( h(v_m) \notin \{F(v_1), \ldots, F(v_{i-1})\} \), RETURN the list \( \gamma_0, \gamma_1, \ldots, \gamma_m \), and TERMINATE.
   (iii) Else, \( h(v_m) = F(v_{i'}) \) for some \( i' \in [i-1] \). Increment \( m \), reassign \( \gamma \) to be \( \gamma' \), and repeat the loop.

(Assuming there were no cycles formed from edges in \( \{h(v_1)F(v_1), \ldots, h(v_{i-1})F(v_{i-1})\} \), the algorithm will terminate.)

More intuitively, \( h(v_m) \) is the vertex “at the start” of the component of \( h(v_i) \) in \( H_{i-1} \). If \( h(v_m) \) were used as \( F(v_i) \), then the edge \( h(v_m)h(v_m) \) would close a cycle with other edges \( h(v_m)F(v_0), h(v_1)F(v_1), \ldots, h(v_m)F(v_{i-1}) \).
By construction, \( h(v_m) \) is not on the list in (1).

We claim that, for any of the \( k - i \) choices for \( F(v_i) \) described above, adding the edge \( h(v_i)F(v_i) \) to \( H_{i-1} \) will not create a cycle in \( H_i \). Assume that \( H_{i-1} \) was acyclic, and suppose to the contrary that adding edge \( h(v_i)F(v_i) \) created a cycle in \( H_i \). Then the cycle would contain the new edge \( h(v_i)F(v_i) \), so \( F(v_i) = h(v_j) \) for some edge \( h(v_j)F(v_j) \) \( (j \leq i) \) in the component of \( h(v_i) \) in \( H_{i-1} \), where \( h(v_j) \) was not previously used as a finish vertex. The only such vertex \( h(v_j) \) is \( h(v_{k_m}) \), which is forbidden by the above algorithm. We have shown that there are \( k - i \) legal choices for \( F(v_i) \), so there are \( \frac{(k-1)!}{(k-p-1)!} \) ways to assign finish vertices to the edges \( h(v_1)F(v_1), \ldots, h(v_p)F(v_p) \) in that order. To count \( \mathcal{C}(T,V) \), we need to divide by \( \prod_{i \in [\rho]} q_i \) since, for each \( i \), the \( q_i \) edges leaving vertex \( c_i \) are indistinguishable. Therefore, \( |\mathcal{C}(T,V)| = \frac{(k-1)!}{(k-p-1)!} \cdot \frac{1}{q_1 \cdots q_k} \) as desired. \( \square \)

Now that we have computed \( |\mathcal{C}(T,V)| \), we compute \( |\mathcal{P}(T,V)| \) by defining a map \( g : \mathcal{P}(T,V) \rightarrow \mathcal{C}(T,V) \) that sends each proper pairing \( P \mapsto C_P \).

**Proposition 5.5.** Let \( k, \rho_i \), and \( q_i \) be defined as in the previous proposition. The map \( g : \mathcal{P}(T,V) \rightarrow \mathcal{C}(T,V) \) is a surjective \( \frac{q_1!q_2! \cdots q_k!}{\rho_1!\rho_2! \cdots \rho_k!} \)-to-1 map.
As in the proof of Proposition 5.4, we construct the three component graphs $C_1$, $C_2$, $C_3$ for the $(T, V)$ shown in Figure 4 by assigning “finish vertices” to vertices $c_3, c_4$. Note that the two edges leaving $c_4$ are considered indistinguishable.

The components are
\{1, 2\}, \{3\}, \{4\}, \{5, 6, 7\}.

\begin{align*}
    P_1 &= \{(4, 2), (6, 3), (7, 4)\} \\
    P_2 &= \{(4, 2), (6, 4), (7, 3)\} \\
    P_3 &= \{(4, 3), (6, 2), (7, 4)\} \\
    P_4 &= \{(4, 3), (6, 4), (7, 2)\} \\
    P_5 &= \{(4, 5), (6, 2), (7, 3)\} \\
    P_6 &= \{(4, 5), (6, 3), (7, 2)\}
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{As in the proof of Proposition 5.4, we construct the three component graphs $C_1$, $C_2$, $C_3$ for the $(T, V)$ shown in Figure 4 by assigning “finish vertices” to vertices $c_3, c_4$. Note that the two edges leaving $c_4$ are considered indistinguishable.}
\end{figure}

The map $g$ is a surjective 2-to-1 map, and there are 3 component graphs in $C(T, V)$.

Proof. Consider a component graph $C_P \in C(T, V)$. We label the components of $F_T$ as $\{c_1, \ldots, c_k\}$ as before. We construct all $\prod_{\rho \in \rho_1, \rho_2, \ldots, \rho_m} \frac{\rho_1! \rho_2! \cdots \rho_m!}{\rho_1!}$ proper pairings $P = (T, V, S)$ in $g^{-1}(C_P)$ by assigning an ordered pair $(v_i, r_i)$ to each directed edge $c_jc_i$ in $C_P$ so that $h(v_i) = c_j$ and the multiset $\{v_1, \ldots, v_p\} = V$ (in
some order). We note that each ordered pair’s second element \( r_i \) is determined by the corresponding edge’s endpoint \( c_i \) (because there is only one root in each component of \( F_T \)), so it suffices to assign only the first vertex \( v_i \) to each edge.

For each \( j \in [k] \), the outdegree of \( c_j \) is \( q_j \). Since all directed edges in \( C_P \) have distinct finish vertices, there are

\[
q_j! \prod_{i \in c_j} \rho_i!
\]

ways to assign the \( q_j \) edges leaving the vertex \( c_j \in C_P \) to the \( q_j \) vertices in the component \( C_j \) of \( r_j \) in \( F_T \), noting that the \( \rho_i \) copies of the same vertex \( i \in V \) should not be distinguished during the assignment process. It follows that there are \( \frac{q_1!q_2!\cdots q_k!}{\rho_1!\cdots \rho_n!} \) such assignments possible, so there are \( \frac{q_1!q_2!\cdots q_k!}{\rho_1!\cdots \rho_n!} \) pairings with component graph \( C_P \). Because we began with a component graph \( C_P \), Proposition 5.3 shows that each of these pairings is proper.

□

For a graph \( G \) in \([n]\) with \( q \) vertices, in which each vertex \( j \) is not a forest, both \( G \) and \( \tilde{G} \) are graphs with components, \( \rho_1! \rho_2! \cdots \rho_n! \) such that \( 0 \leq r_1! \rho_1! + 0!0!1!2! = 2 \), we see that \( g \) is a 2-to-1 map. Furthermore, since \( \frac{(k-1)!}{(k-2)!} \cdot \frac{1!}{(k-1)!} = \frac{1}{k} \cdot \frac{1}{1} = 3 \), there are 3 component graphs in \( C(T, V) \). As Theorem 5.1 asserts, there are 6 proper pairings in \( P(T, V) \).

In the next section, we show how Theorem 5.1 allows us to interpret the terms in Theorem 3.3.

6. PROOF OF GENERALIZED RECIPROCITY

**Definition 6.1.** For a graph \( G \) on \([n]\) and a spanning subgraph \( H \) of \( G \),

1. Let \( S_i(G, H) \) denote the family of spanning trees \( T \) of \( G \) with \( H \subseteq T \) and \( \deg_T(0) = i + 1 \).

2. Define a degree-(\( n - i - 1 \)) polynomial in \( x_1, \ldots, x_n \) by

\[
a_i(G, H) = x^{-i} \sum_{T \in S_i(G, H)} m(T).
\]

Then \( f_{G,H}(x; x_1, \ldots, x_n) = a_{n-1}(G, H)x^{n-1} + a_{n-2}(G, H)x^{n-2} + \ldots + a_0(G, H) \).

3. Let \( Y := x_1 + \cdots + x_n \).

We note that if \( H \) is not a forest, both \( S_i(G, H) \) and \( a_i(G, H) \) are zero.

In the following proposition, we describe the monomials of the replacement graph of the proper pairings in \( P(T, V) \).

**Proposition 6.2.** Suppose \( i, j \) are integers such that \( 0 \leq j \leq i \leq n - 1 \).

1. For \( T \in S_i(G, H) \) and \( V \) such that \( |V| = i - j \),

\[
\sum_{P \in P(T, V)} m(R_P) = \binom{i}{j} \binom{i-j}{\rho_1, \ldots, \rho_n} x^{-(i-j)} m(T) \prod_{v \in [n]} x_v^{\rho_v},
\]

where \( \rho_1, \rho_2, \ldots, \rho_n \) are the multiplicities of vertices \( 1, 2, \ldots, n \) in \( V \).

2. Then

\[
x^j a_i(G, H) \binom{i}{j} x^{i-j} = \sum_{T \in S_i(G, H)} \sum_{\substack{V \subseteq [n]}} \sum_{P \in P(T, V)} m(R_P).
\]

**Proof.** Fix \( i, j \) such that \( 0 \leq j \leq i \leq n - 1 \), and let \( T \in S_i(G, H) \). First consider a fixed multiset \( V \) of \( i - j \) vertices, in which each vertex \( v \in [n] \) is listed with multiplicity \( \rho_v \). From the definition of the replacement graph, we observe the following:

1. The degree of \( x \) in \( \frac{m(R_P)}{m(T)} \) is \( -|V| = -(i - j) \).
(2) The degree of $x_i$ in $\frac{m(R_P)}{m(T)}$ is $\rho_i$, the multiplicity of vertex $i$ in $\mathcal{V}$.

Then, we see that the monomial of $R_P$ for any proper pairing $P \in \mathcal{P}(T, \mathcal{V})$ will be

$$m(R_P) = x^{-(i-j)}m(T) \prod_{v \in [n]} x_v^{\rho_v}.$$

By Theorem 5.1, there are $(i-j)(\rho_1, \ldots, \rho_n) = (i-j)(\rho_1, \ldots, \rho_n)$ proper pairings in $\mathcal{P}(T, \mathcal{V})$, and the replacement graph of each has the same monomial. This proves (1). Figure 5 is an example of (1).

For (2), we sum over all $T \in S_i(G, H)$ and $\mathcal{V}$ with $|\mathcal{V}| = i - j$:

$$\sum_{T \in S_i(G, H)} \sum_{\mathcal{V} \in \mathcal{P}(T, \mathcal{V})} m(R_P) = \sum_{T \in S_i(G, H)} \left[ \sum_{\mathcal{V} \in \mathcal{P}(T, \mathcal{V})} \left( \begin{array}{c} i-j \cr j \end{array} \right) \left( \frac{i-j}{\rho_1, \ldots, \rho_n} \right) x^{-(i-j)}m(T) \prod_{v \in [n]} x_v^{\rho_v} \right].$$

$$= x^{-(i-j)} \sum_{T \in S_i(G, H)} \left[ m(T) \sum_{\rho_1 + \cdots + \rho_n = i-j} \left( \begin{array}{c} i-j \cr \rho_1, \ldots, \rho_n \end{array} \right) \prod_{v \in [n]} x_v^{\rho_v} \right]$$

$$= x^j \left[ x^{-i} \sum_{T \in S_i(G, H)} m(T) \right] \left( \begin{array}{c} i \cr j \end{array} \right) \sum_{\rho_1 + \cdots + \rho_n = i-j} \left( \begin{array}{c} i-j \cr \rho_1, \ldots, \rho_n \end{array} \right) \prod_{v \in [n]} x_v^{\rho_v}$$

$$= x^j a_i(G, H) \left( \begin{array}{c} i \cr j \end{array} \right) \sum_{\rho_1 + \cdots + \rho_n = i-j} \left( \begin{array}{c} i-j \cr \rho_1, \ldots, \rho_n \end{array} \right) \prod_{v \in [n]} x_v^{\rho_v}$$

$$= x^j a_i(G, H) \left( \begin{array}{c} i \cr j \end{array} \right) Y^{i-j}. \quad \square$$

Proposition 6.2 provides an interpretation for the monomials of the replacement graphs. We now prove Theorem 3.3.

The proof of Theorem 3.3 relies on the following lemma.

**Lemma 6.3.** Given a spanning tree $T'$ on vertices $\{0, 1, \ldots, n\}$ and an unrooted forest $F$ in $G$ such that $E(F) \subseteq E(T')$, there is exactly one proper pairing $P = (T, \mathcal{V}, S)$ such that

1. $F$ is the underlying unrooted forest of the rooted forest $F_T$.
2. The replacement graph $R_P$ is $T'$.

**Proof.** Suppose $T'$ is a spanning tree on $\{0, 1, \ldots, n\}$ and $F$ an unrooted forest in $G$ such that $E(F) \subseteq E(T')$. Define the set $E(T', F) := E(F_{T'}) - E(F)$.

We prove by induction on $|E(T', F)|$ that, given $T'$ and $F$, exactly one proper pairing satisfies properties (1) and (2) in the lemma. If $E(T', F) = \emptyset$, then the only proper pairing satisfying (1) and (2) is $(T', \emptyset, \emptyset)$. Now suppose that the result holds whenever $|E(T', F)| < j$, and consider $|E(T', F)| = j$. We claim that there exists a component $\tilde{C}$ of $F$ that contains a vertex $x$ incident to an edge in $E(T', F)$ and a vertex $y$ adjacent to 0 in $T'$. If there were no such component $\tilde{C}$, then each component of $F$ would be one of two types:

1. No vertex in the component is adjacent to 0.
2. No vertex in the component is incident to any edge in $E(T', F)$.

Note that not all components are in type 1 because 0 cannot be an isolated point in $T'$, and not all components are in type 2 because $E(T', F) \neq \emptyset$. However, there are no edges in $F$ between any
We use this lemma to prove Theorem 3.3.
Proof of Theorem 3.3. We expand each \((x + Y)^i\) term in \((-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)\) and apply Proposition 6.2 to get

\[
(-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)
\]

\[
= (-1)^{-|E(H)|} \left[ a_{n-1}(G_1, H)(x + Y)^{n-1} - \ldots \pm a_0(G_1, H) \right]
\]

\[
= (-1)^{-|E(H)|} \sum_{j=0}^{n-1} x^j \sum_{i=j}^{n-1} (-1)^{n-i-1} a_i(G_1, H) \binom{i}{j} Y^{i-j}
\]

\[
= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{n-i-1-|E(H)|} \left[ \sum_{T \in S_i(G_1, H)} \sum_{|V|=i-j} \sum_{P \in P(T,V)} m(R_P) \right].
\]

We note that a spanning tree \(T \in S_i(G, H)\) has \(i + 1\) edges incident to vertex 0 and \(n - i - 1\) edges not incident to 0. Since \(|E(H)|\) of the edges not incident to 0 are edges of \(H\), we see that \(a_i(G_1, H) \neq 0\) iff \(n - i - 1 \geq |E(H)|\). In particular, \(a_i(G_1, H) = 0\) for all \(i > n - |E(H)| - 1\), so the previous expression can be rewritten as

\[
(*) \quad (-1)^{n-|E(H)|-1}f_{G_1,H}(-x - Y; x_1, \ldots, x_n)
\]

\[
= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{n-i-1-|E(H)|} \left[ \sum_{T \in S_i(G_1, H)} \sum_{|V|=i-j} \sum_{P \in P(T,V)} m(R_P) \right].
\]

For a given replacement graph \(R_P\), we show using Lemma 6.3 that the coefficient of \(m(R_P)\) in the above expression is 1 if every edge of \(R_P\) is in \(G_2\) (i.e. no edge of \(R_P\) is in \(G_1 - E(H)\)) and 0 otherwise.

By Lemma 6.3, the underlying unrooted forest \(F\) of \(F_T\) and a replacement graph \(T'\) together determine the proper pairing \(P\), so we can reindex the triple sum from equation (*) as

\[
\sum_{T \in S_i(G_1, H)} \sum_{|V|=i-j} \sum_{P \in P(T,V)} m(R_P) = \sum_F \sum_{T'} m(T'),
\]

where \(F\) is the underlying unrooted forest of \(F_T\) for \(T \in S_i(G_1, H)\) and \(T'\) is a spanning tree on \(\{0, 1, \ldots, n\}\) such that \(E(F) \subseteq E(T')\) with exactly \(j + 1\) edges incident to 0. Consider a particular spanning tree \(T'\) with \(j + 1\) edges incident to 0 and \(b(T')\) edges in \(G_1 - H\). Then \(F\) must have had \(n - i - 1 - |E(H)|\) edges in \(G_1 - H\), so there are \(\binom{b(T')}{n-i-1-|E(H)|}\) choices of \(F\) that give replacement tree \(T'\). It follows that

\[
(-1)^{n-i-1-|E(H)|} \sum_{T \in S_i(G_1, H)} \sum_{|V|=i-j} \sum_{P \in P(T,V)} m(R_P)
\]

\[
= \sum_{T'} (-1)^{n-i-1-|E(H)|} \binom{b(T')}{n-i-1-|E(H)|} m(T'),
\]

where the sum on the right hand side is over all spanning trees \(T'\) on \(\{0, 1, \ldots, n\}\) containing all the edges in \(H\) and exactly \(j + 1\) edges incident to 0.
From the above discussion, we see that \((-1)^{n-|E(H)|} f_{G_1,H}(-x - Y; x_1, \ldots, x_n)\) can be expressed as

\[
\sum_{j=0}^{n-1} \sum_{i=j}^{n-|E(H)|-1} \left( \sum_{T^\prime} b(T^\prime) m(T^\prime) \right) n - i - 1 - |E(H)|, \]

where \(T^\prime\) is again a spanning tree on \(\{0, 1, \ldots, n\}\) such that \(E(H) \subseteq E(T^\prime)\) with exactly \(j + 1\) edges incident to 0. Exchanging the first two summations and summing over all \(T^\prime\) such that \(E(H) \subseteq E(T^\prime)\) (note that the number of edges incident to 0 is no longer fixed to be \(j + 1\)), we see that

\[
(-1)^{n-|E(H)|} f_{G_1,H}(-x - Y; x_1, \ldots, x_n) = \sum_{i=0}^{n-1-|E(H)|} \left( \sum_{T^\prime} b(T^\prime) m(T^\prime) \right) n - i - 1 - |E(H)| \]

\[
= \sum_{T^\prime} \left[ \sum_{i=0}^{n-1-|E(H)|} (-1)^{n-i-1-|E(H)|} \left( n - i - 1 - |E(H)| \right) b(T^\prime) m(T^\prime) \right] \]

\[
= \sum_{T^\prime: \text{spanning tree of } G_2 \text{ s.t. } E(H) \subseteq E(T^\prime)} m(T^\prime). \]

Notice that the coefficient of \(m(T')\) for spanning trees \(T' \notin G_2\) becomes 0 due to cancellation. Thus, we are left with the monomials of the spanning trees of \(G_2\) that contains \(H\) as a subgraph, which is by definition \(f_{G_2,H}(x; x_1, \ldots, x_n)\). This finishes the proof of Theorem 3.3.

\[\square\]

7. Acknowledgments

We would like to thank the MIT Department of Mathematics for providing this research opportunity for us. Thank you to Professor Alexander Postnikov for suggesting this project. We would also like to thank Professor David Jerison and Professor Ankur Moitra for their extremely helpful feedback and comments throughout the course of this research.

We are very grateful to our mentor Pakawut Jiradilok for suggesting reading materials, reviewing the concepts, and creating problem sets to help us understand these concepts during the reading period. We appreciate that he spends much more than the expected amount of time to help us with this project. We also appreciate his invaluable suggestions and the “unrelated” topics that he occasionally brings up in meetings because it expands our understanding of mathematics. Thank you for the patience and guidance that you have provided us in the course of the research; without you this project would not have been possible.

References

Appendix A. Bijection by Huang and Postnikov

We present the bijection found by Huang and Postnikov in [4]. We will use our notation instead of that used in [4].

Let $A$ be the set of all proper pairings $P = (T, V, S)$, and let $B$ be the set of all $(T, W)$, where $W$ is a sequence of $k(F_T) - 1$ vertices chosen from $\{0, 1, \ldots, n\}$. The bijection between $A$ and $B$ works as follows:

I. $\varphi: A \rightarrow B$ (see Definition 4.1 from [4])

Consider some proper pairing $P = (T, V, S) \in A$. Start with a set $\mathcal{R} = \{r \in [n] : r$ is a root of $F_T\}$ and an empty sequence $W = ()$.

WHILE $|\mathcal{R}| > 1$:

1. Delete leaves from $V(R_P) - \mathcal{R}$ until all remaining leaves are in $\mathcal{R}$.
2. Let $\ell_{\text{max}}$ be the leaf of maximum index remaining in $R_P$.
   a. Append $\ell_{\text{max}}$'s neighbor to the end of $W$.
   b. Delete $\ell_{\text{max}}$ from both $R_P$ and $\mathcal{R}$.

RETURN $(T, W) \in B$.

We note that $\varphi$ specializes to the Prüfer code when $F_T = E_n$.

II. $\varphi^{-1}: B \rightarrow A$ (see Definition 4.3 from [4])

Consider some weight sequence $W \in B$. Again, let $\mathcal{R}$ be the set of roots of $F_T$. Start with a copy $T^*$ of the tree $T$ and an empty set $S = \{}$.

WHILE $|W| > 0$:

1. Let $v$ be the first element in the remaining sequence $W$, and let $r$ be the root of maximum index in $\mathcal{R}$ that is not in $v$’s component of $F_{T^*}$. (In the case $v = 0$, let $r$ be the root of maximum index, regardless of component.)
2. Delete $v$ from $W$ and $r$ from $\mathcal{R}$.
3. If $v \neq 0$:
   a. Add edge $vr$ and remove 0$r$ in $T^*$.
   b. Add $(v, r)$ to $S$.

RETURN $(T, V, S) \in A$. 

15