ON KAKEYA-TYPE PROBLEMS FOR HYPERPLANES IN $\mathbb{R}^d$
SPUR FINAL PAPER, SUMMER 2017

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Abstract. It is well known that in $\mathbb{R}^2$, there exist so-called Kakeya sets, which are compact measure zero sets containing a unit line segment in every direction. It is also well known that for $d \geq 3$, a compact set with a unit $d-1$ disk in every direction cannot have measure zero. For $E \subset \mathbb{R}^d$, $d \geq 3$ and $A \in \mathbb{R}$, we first consider bounds on the measure of sets $\Gamma_A(E)$ of directions on $\mathbb{S}^{d-1}$ for which there is a hyperplane slice of E with $d-1$ dimensional measure larger than A. In particular, we study certain families of ellipsoids in $\mathbb{R}^3$ to compare the size of their $\Gamma_A(E)$ sets to previously known bounds. Then, in an attempt to develop tools to analyze $\Gamma_A(E)$, we use a maximal operator related to the Kakeya maximal function and derive estimates that elucidate some properties of Kakeya-like sets containing a unit $d-1$ disk normal to every direction.

Contents

Date: January 2, 2018.

1. Background

Our problem is motivated by the exposition in [2] on Kakeya sets, which we summarize below. We begin with a regularity property enjoyed by all subsets of $\mathbb{R}^d$ of finite measure, when $d \geq 3$. We let μ_d denote the Lebesgue measure on $\mathbb{R}^d$, dropping the subscript when there is no ambiguity. In addition, for $e \in \mathbb{S}^{d-1}, t \in \mathbb{R}$, we let $P_{e,t}$ denote the hyperplane with normal e and a signed distance t from the origin. That is,

$$P_{e,t} := \{x \in \mathbb{R}^d \mid \langle x, e \rangle = t\}.$$

Furthermore, for $E \subset \mathbb{R}^d$ we let

$$E_{e,t} = E \cap P_{e,t}$$

Finally, we let σ denote the uniform measure on $\mathbb{S}^{d-1}$, defined as

$$\sigma(F) = d\mu_d \left(\bigcup_{\lambda \in [0,1]} \lambda F \right)$$

where $F \subset \mathbb{S}^{d-1}$. With these definitions in mind, we present the following theorem, found in [2]:

Theorem 1.1. *Suppose E is a set of finite measure in $\mathbb{R}^d$, with $d \geq 3$. Then for almost every $e \in \mathbb{S}^{d-1}$:*

(1) $E_{e,t}$ is measurable for all $t \in \mathbb{R}$
(2) $\mu_{d-1}(E_{e,t})$ is continuous in $t \in \mathbb{R}$.

This theorem has an important corollary regarding measure zero sets in $\mathbb{R}^{d \geq 3}$:

Corollary 1.2. *Suppose E is a set of measure zero in $\mathbb{R}^d$ with $d \geq 3$. Then, for almost every $e \in S^{d-1}$, the slice $E_{e,t}$ has measure zero for all t.*

When $d = 2$, however, this is remarkably not true, due to the existence of a *Kakeya (Besicovitch) set* in $\mathbb{R}^2$, satisfying the following:

Theorem 1.3. *There exists a set $K \subset \mathbb{R}^2$ such that*

(1) K is compact
(2) $\mu(K) = 0$
(3) Contains a unit line segment in every direction.

In general, we can define the notion of a Kakeya set in $\mathbb{R}^d$ as follows:

Definition 1.4. A **Kakeya set** or **Besicovitch set** K in $\mathbb{R}^d$ is a measure zero compact set containing a translate of a unit line segment in each direction ; that is, for every $e \in \mathbb{S}^{d-1}$, there is some $v \in \mathbb{R}^d$ such that for $L = \{\lambda e \mid \lambda \in [0,1]\}$, $K \cap (L+v) = L+v$.
Analogously, a ***(d, n)*–Kakeya set** K is a measure zero compact set containing a translate of every $n < d$ dimensional unit disk.

Notice that Corollary 1.2 implies there exist no $(d, d-1)$-Kakeya sets. Whether other (d, n)-Kakeya sets exist remains an open question. Nonetheless, there certainly are compact sets containing a unit $d-1$ disk in every direction; Corollary 1.2 simply prohibits them from having zero measure. A closed unit ball for instance trivially fulfills this criteria. In this paper, we will be interested in this class of sets, which we call **Kakeya-like sets**.

1.1. **The Radon Transform and Theorem 1.1.** The *Radon transform*, denoted by $\mathcal{R}$, for an appropriate function f, is defined by

$$\mathcal{R}f(e,t) := \int_{P_{e,t}} f \, d\mu_{d-1}.$$

The radon transform is relevant to this discussion because, for measurable sets E, we have

$$\mathcal{R}(\mathbb{1}_{E_{e,t}})(e,t) = \mu_{d-1}(E_{e,t}).$$

Notice that if f is continuous and has compact support, the radon transform behaves nicely: it exists and is continuous for all $(e,t) \in S^{d-1} \times \mathbb{R}$ and has compact support in t. Thus, we can prove all our key results for functions of compact support, and use standard density arguments to extent them to $L^1 \cap L^2$ functions.

Theorem 1.1 is proven using estimates involving the following "maximal" radon transform:

$$\mathcal{R}^* f(e) := \sup_t |\mathcal{R}f(e,t)|$$

In particular, the next theorem provides the principal tool for proving Theorem 1.1:

Theorem 1.5. *Suppose f is continuous and has compact support in $\mathbb{R}^d$, $d \geq 3$. Then*

$$\|\mathcal{R}^* f(e)\|_{L^1(\mathbb{S}^{d-1})} \leq c_d(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)})$$

for some constant $c > 0$ depending only on d.

Theorem 1.5 is important to us because it provides us with a bound on measure of $\Gamma_A(E)$. We describe this in greater detail in section 2.

1.2. **The Kakeya Maximal Function.** Suppose $K \subset \mathbb{R}^d$ is a Kakeya set, and let K_δ be its δ-neighborhood. If we let $\ell(e) \subset K$ be the unit line segment in the $e \in \mathbb{S}^{d-1}$ direction contained within K, then its δ neighborhood $\ell_\delta(e)$ is essentially a tube in the e direction with width 2δ in all orthogonal directions. As such, K_δ can be regarded as a union of these tubular objects. Integrating the characteristic function $\mathbb{1}_{K_\delta}$ gives us a notion of the size of this neighborhood, and letting $\delta \to 0$, we can hope to draw conclusions about the "fullness" of the Kakeya set itself. The main tool in making this precise is the Kakeya maximal function, which we now define.

Definition 1.6. Let $f \in L^1_{loc}(\mathbb{R}^d)$, and define the **Kakeya maximal function** $f^*_\delta(e) : \mathbb{S}^{d-1} \to \mathbb{R}$ as

$$f^*_\delta(e) := \sup_T \frac{1}{\mu(T)} \int_T |f(x)| dx,$$

where the supremum is taken over all $1 \times \delta \times ... \times \delta$ tubes T in the e direction.

The Kakeya maximal function is the subject of this area's most important conjecture, the Kakeya Maximal Function Conjecture:

Conjecture 1.7 (Kakeya Maximal Function Conjecture)**.** *For any $\epsilon > 0$, there exists some constant C_ϵ so that*

$$\|f^*_\delta\|_{L^d(\mathbb{S}^{d-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^d(\mathbb{R}^d)}$$

This conjecture has wide-ranging consequences, the most famous of which is the Kakeya Conjecture :

Conjecture 1.8. *Let $K \subset \mathbb{R}^d$ be a Kakeya set. Then it has full Hausdorff dimension.*

Both remain unsolved for $d \geq 3$. The case $d = 2$ is settled by the following estimate, due to Córdoba:

Theorem 1.9 (Córdoba)**.** *In $\mathbb{R}^2$, we have*

$$\|f^*_\delta\|_{L^2(\mathbb{S}^1)} \lesssim \sqrt{\log(\delta^{-1})} \|f\|_{L^2(\mathbb{R}^2)}. \tag{1}$$

In addition to settling the Kakeya Maximal Function conjecture in $\mathbb{R}^2$, this estimate allows us to derive the following lower bound the measure of K_δ in terms of δ, by setting $f = \mathbb{1}_{K_\delta}$:

Corollary 1.10. *For a Kakeya set $K \subset \mathbb{R}^2$ with $\delta-$neighborhood K_δ,*

$$\frac{1}{\log(\delta^{-1})} \lesssim \mu(K_\delta). \tag{2}$$

Inspired by this result, we wished to derive a bound of this form for a related class of sets by seeking an estimate like Córdoba's in (1) with a related maximal function. In the next section, we describe the problem in greater detail.

2. Problem Statement and Summary of Results

Let $\mathfrak{M}_d$ be the collection of measurable subsets of $\mathbb{R}^{d\geq 3}$ with unit measure. For $E \in \mathfrak{M}_d$ consider the following set:

$$\Gamma_A(E) := \{e \in \mathbb{S}^{d-1} \mid \sup_t \mu(E_{e,t}) > A\}$$

and notice

$$\Gamma_A(E) = \{e \in \mathbb{S}^{d-1} \mid \mathcal{R}^*(\mathbb{1}_E)(e) > A\}.$$

Then define for a subcollection $\mathscr{F} \subset \mathfrak{M}_d$

$$\gamma(\mathscr{F}, A) = \sup \sigma(\Gamma_A(E))$$

Theorem 1.5 and Chebychev's inequality give us the following bound on the measure of $\Gamma_A(E)$:

$$\sigma(\Gamma_A(E)) \leq \frac{c_d(\mu(E) + \mu(E)^{1/2})}{A}$$

Therefore, we have the bound

$$\gamma(\mathfrak{M}_d, A) \leq \frac{c_d}{A} \tag{3}$$

The principal goal of this project was to analyze how good of a bound this is.

2.1. **Experiments with a family of Ellipsoids.**

To simplify the discussion, we first analyze the family $\mathscr{E}$ of ellipsoids in $\mathbb{R}^3$ of unit volume and with equation $\frac{x^2}{L^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$, where $L > r > 0$ and $4/3\pi L r^2 = m$. Letting E_L denote such an ellipsoid, we derive the following bound:

$$\sigma(\Gamma_A(E_L)) \lesssim \frac{1}{LA}$$

For any given A, if we seek to restrict L so that $\sigma(\Gamma_A(E)) > 0$, we observe that A cannot be larger than the measure of the slice through the origin along the ellipsoid's major axis; this tells us we must have $A^2 \lesssim L$, and so we obtain the upper bound

$$\gamma(\mathscr{E}, A) \lesssim \frac{1}{A^3}$$

We detail these computations in section 3.

Our next approach to understand $\Gamma_A(E)$ lies with estimates of a maximal function related to the Kakeya maximal function and to the Radon transform.

2.2. **Estimates with a Kakeya-like maximal function.** Before being introduced to the Kakeya Maximal function, we encountered a direct proof of Corollary 1.10, as presented by Larry Guth, that does not reference the Kakeya Maximal function. He suggested we try to adapt this proof to Kakeya-like sets in $\mathbb{R}^d$ in the hopes of finding a positive lower bound for the measure of such sets. We sketch his proof:

Proof Sketch of Corollary 1.10. For each unit line segment in K, its δ-neighborhood is essentially a tube with unit length and with width δ. We make a δ-net on $\mathbb{S}^1$ by taking δ^{-1} of these tubes, call them $T_1, T_2, \ldots$ such that T_i and T_j are at an angle of $|i-j|\delta$ to each other. We see then by Cauchy-Schwartz that

$$\begin{aligned} 1 &= \left(\int \left(\sum_{i}^{\delta^{-1}} \mathbb{1}_{T_i} \right) \mathbb{1}_{K_\delta} \, d\mu \right)^2 \\ &\leq \int \left(\sum_{i}^{\delta^{-1}} \mathbb{1}_{T_i} \right)^2 d\mu \int \mathbb{1}_{K_\delta}^2 \, d\mu = \mu(K_\delta) \sum_{i,j}^{\delta^{-1}} \mu(T_i \cap T_j), \end{aligned}$$

thus giving

$$\frac{1}{\sum_{i,j} \mu(T_i \cap T_j)} \leq \mu(K_\delta). \tag{4}$$

We now proceed to estimate the denominator in (4). Because the angle between T_i and T_j is δ, the parallelogram $T_i \cap T_j$ has height δ and base $\lesssim \frac{\delta}{|i-j|\delta}$. Thus $\mu(T_i \cap T_j) \lesssim \frac{\delta}{|i-j|}$. Hence,

$$\sum_{i,j}^{\delta^{-1}} \mu(T_i \cap T_j) \lesssim \sum_{i,j}^{\delta^{-1}} \frac{\delta}{|i-j|} \lesssim \delta^{-1} \sum_{z=1}^{\delta^{-1}} \frac{\delta}{z} \lesssim \log(\delta^{-1}).$$

This proves the claim $\square$

Letting K now denote a Kakeya-like set, we wished to derive a sharp bound of the form

$$L_\delta \lesssim \mu(K_\delta) \tag{5}$$

where $L_\delta \to l > 0$, thus allowing us to nontrivially bound the size of $\mu(K)$ from below. Such an estimate would offer a direct, quantified proof of Corollary 1.2, which we hoped could be used in order to derive a better bound for $\Gamma_A(E)$.

We hoped to do this via a similar calculation to that presented in Guth's direct proof of (2). After a few attempts in $\mathbb{R}^3$, we decided to approach the problem more theoretically, via the use of maximal functions. Specifically, we define a maximal function $M_\delta f(e)$ which uses thin cylinders with normals in each $e \in \mathbb{S}^{d-1}$ instead of using tubes as in the definition of f^*_δ. We then sought to derive an L^2 estimate of the form

$$\|M_\delta f\|_{L^2(\mathbb{S}^{d-1})} \lesssim A_\delta \|f\|_{L^2(\mathbb{R}^d)} \tag{6}$$

that could give (5) as a corollary. In Section 5, however, we prove that if we have an estimate as in (6) then, though we still arrive at an estimate like (5) as a corollary, we can only do so with $L_\delta \to 0$ as $\delta \to 0$, which means this approach cannot lead to a nontrivial, and much less a sharp, lower bound on the measure of Kakeya-like sets. Nevertheless, we prove that (6) holds with $A_\delta = \sqrt{\log(\delta^{-1})}$, and interpret this result in terms of the fullness of what we call weak Kakeya-like sets, which we define in section 5.

3. Ellipsoid Computations

Let $E_L \in \mathscr{E}$ be a skinny ellipsoid with equation $\frac{x^2}{L^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$, where $L > r$ and $\mu(E_L) = 1$. Letting $p = (1, 0, 0)$, which is direction of the principal axis, the largest possible slice (with measure $\pi r L$) occurs at $E_{e,0}$ for any e orthogonal to p. That is, $\Gamma_{\pi r L}(E)$ is the 1-sphere $\mathbb{S}^2_{p,0} := \mathbb{S}^2 \cap P_{p,0}$. Given the symmetry present in E, we know that for $A < \pi r L$, $\Gamma_A(E)$ forms a band with uniform width on $\mathbb{S}^2$ around $\mathbb{S}^2_{p,0}$. As such, letting ρ be the natural metric on $\mathbb{S}^2$, there is a constant θ

$$\Gamma_A(E_L) := \{x \in \mathbb{S}^2 \mid \rho(x, \mathbb{S}^2_{p,0}) \le \theta\}.$$

Now fix $z = (0, 0, 1)$. Observe that the intersection of $\Gamma_A(E)$ with the plane $\mathrm{span}(p, z)$ will form an arc of length 2θ, with center z. Hence θ is the angle between z and the rightmost point on the arc. To find $\sigma(\Gamma_A(E))$, we must carry out the following computation:

$$\int_{-\theta}^{\theta} 2\pi\cos(x)\,dx \sim \sin\theta$$

We make use of the following formula for the area of the intersection between a general ellipsoid and a plane through the origin, found in [5]:

Fact 1. Suppose $e = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$, and let $F \subset \mathbb{R}^3$ be an ellipsoid with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Then the area of the slice $F_{e,0}$ is given by

$$\mu(F_{e,0}) = \frac{\pi abc}{\sqrt{a^2\xi_1^2 + b^2\xi_2^2 + c^2\xi_2^2}}$$

Suppose $e = (\xi_1, 0, \xi_2)$ is the rightmost point mentioned above. Using this formula, we get that

$$A = \frac{\pi r^2 L}{\sqrt{L^2\xi_1^2 + r^2\xi_2^2}}$$

But $\xi_1 = \sin(\theta)$ and $\xi_2 = \cos(\theta)$, and so, recalling that $r^2L \sim m$, the above becomes

$$L^2\sin^2(\theta) + \frac{1}{L}(1 - \sin^2(\theta)) \sim \frac{1}{A^2}$$
$$\Rightarrow \sin^2(\theta) \sim \frac{\frac{1}{A^2} - \frac{1}{L}}{L^2 - \frac{1}{L}} \lesssim \frac{1}{A^2L^2}$$

and we obtain

$$\Gamma_A(E_L) \lesssim \frac{1}{AL}$$

As we noted in the introduction, in order for $\Gamma_A(E)$ to have positive measure, we must have $A^2 \lesssim L$, and thus we arrive at the bound

$$\gamma(\mathscr{E}, A) \lesssim \frac{1}{A^3}.$$

The Chebychev bound on the other hand gives

$$\gamma(\mathscr{E}, A) \lesssim \frac{1}{A},$$

and so clearly the Chebychev bound is a poor one for $\gamma_3(\mathscr{E}, \cdot)$. Unfortunately, this calculation offers little insight as to how to produce better bounds for γ_3 or γ_d in general, and so we turn now to our next topic.

4. L^2 Estimates for a Kakeya-like maximal function

The maximal operator we consider is related to the radon transform and to the Kakeya maximal operator. First, for each $e \in \mathbb{S}^{d-1}$ and $v \in \mathbb{R}^d$, we define the cylinder

$$C_e^\delta(v) := \{x \in \mathbb{R}^d \mid -\delta \leq \langle x - v, e\rangle \leq \delta \text{ and } \operatorname{proj}_{e^\perp} x \leq 1\}$$

These cylinders take the place of tubes in our maximal function, which we define as follows:

$$M_\delta f(e) := \sup_v \frac{1}{\mu(C_e^\delta(v))} \int_{C_e^\delta(v)} |f(x)|\, dx$$

Our main result is the following:

Theorem 4.1. *For $f \in L^2(\mathbb{R}^d)$, we have the estimate*

$$\|M_\delta f\|_{L^2(\mathbb{S}^{d-1})} \lesssim \sqrt{\log(\delta^{-1})}\|f\|_2. \tag{7}$$

Observe that in our subsequent work, we may assume f is nonnegative, for $\|M_\delta(|f|)\|_p = \|M_\delta f\|_p$ and $\|(|f|)\|_p = \|f\|_p$ for any $1 \leq p \leq \infty$. We also note the following basic fact:

Proposition 4.2. *To prove (7) holds, it suffices to show it holds for $M_\delta f$ restricted to the set of directions that lie within $1/10$ of the vertical.*

We omit the proof; it is identical to the proof of Fact 2.1.4 in [4], except we substitute $M_\delta f$ in for f_δ^*. For the rest of this section, we work with the restriction of $M_\delta f$ to the directions within $1/10$ of the vertical without mention. This allows us to avoid certain technicalities in the geometric estimates we use arising from the interaction between antipodal points, and allows us to use the estimate $\sin(\theta) \sim \theta$ comfortably.

Moving on, we present our two central lemmas, following the treatment of f_δ^* found in [3].

Lemma 4.3. *For any pair of directions e_1, e_2 within $1/10$ of the vertical, and any pair $v, w \in \mathbb{R}^d$, we have*

$$\mu(C_{e_1}^\delta(v) \cap C_{e_2}^\delta(w)) \lesssim \frac{\delta^2}{|e_1 - e_2|}. \tag{8}$$

Proof. The case where $d = 2$ is a well known result by Córdoba, and can be found in [4]. We proceed to prove the general case by induction. Suppose n is a number such that the Lemma holds. First, observe that

$$\mu(C_{e_1}^\delta(v) \cap C_{e_2}^\delta(w)) \leq \mu(C_{e_1}^\delta(v) \cap C_{e_2}^\delta(v)) = \mu(C_{e_1}^\delta(0) \cap C_{e_2}^\delta(0)),$$

and so we assume without loss of generality that our cylinders are centered at the origin, and we denote them simply by C_1 and C_2. Now consider the following subsets of the boundaries of our C_i:

$$G_i := \{x \in C_i \mid \operatorname{proj}_{e_i^\perp} x = 1\}.$$

The intersection $G_1 \cap G_2$ consists of two antipodal regions. Observe that C_i can be circumscribed by a box S_i that is a rotated copy of $[-\delta, \delta] \times [0,1]^{d-1}$. We circumscribe the C_i so that the centers of each antipodal region in $G_1 \cap G_2$ lie on opposite faces of S_i (faces with same dimensions as $[-\delta, \delta] \times [0,1]^{d-2}$). Choose one of these antipodal regions, let F_1 be the face of S_1 that contains the center of this region, and let F_2 be the corresponding face for S_2. Then F_1 and F_2 intersect at an angle $|e_1 - e_2|$ to each other. Since having (8) for $\delta \times 1 \times \cdots \times 1-$cylinders implies the same result for $\delta \times 1 \times \cdots \times 1-$boxes and vice versa, the induction hypothesis gives us

$$\mu(F_1 \cap F_2) \lesssim \frac{\delta^2}{|e_1 - e_2|}.$$

However, $S_1 \cap S_2$ is the parallelepiped with base $F_1 \cap F_2$ and unit height. Therefore, given that $\mu(C_1 \cap C_2) \leq \mu(S_1 \cap S_2)$, this gives us (8). □

Central to our analysis is the notion of a maximal $\delta-$separated set, which we now define.

Definition 4.4. Let (X, ρ) be a metric space. A set $\Omega \subset X$ is a $\delta-$**separated set** if for each $x, y \in X$, $\rho(x, y) \geq \delta$. It is a **maximal** $\delta-$**separated set** if it is $\delta-$separated and is not properly contained in any other δ-separated sets.

Fact 2. Let (X, ρ) be a metric space and let $\Omega \subset X$ be a δ-separated set. Define $D_\delta(x)$ to be the δ-neighborhood around $x \in X$, and set

$$\mathcal{D} := \bigcup_{\omega \in \Omega} D_\delta(\omega).$$

Then $\mathcal{D} = X$ if and only if Ω is maximally δ-separated.

Proof. Suppose $\mathcal{D} = X$. Then for every point $x \in X$, there is some $w \in \Omega$ so that $\rho(x, \omega) < \delta$, and so $\Omega \cup \{x\}$ is no longer δ-separated, proving maximality of Ω. On the other hand, if $x \in X \setminus \mathcal{D} \neq \emptyset$, then $\Omega \cup \{x\}$ is δ-separated, proving Ω cannot be maximal. □

Fact 3. If Ω is a maximal $\delta-$separated subset of $\mathbb{S}^{d-1}$, then Ω has on the order of $\delta^{-(d-1)}$ elements.

Proof. This follows from the fact that the sphere has measure ~ 1 and a disk $D_\delta(x)$ has measure $\sim \delta^{d-1}$. □

Lemma 4.5. *Let $\{e_k\} \subset \mathbb{S}^{d-1}$ be a maximal δ-separated set and $\{v_k\}$ be any collection of points in $\mathbb{R}^d$. In addition, let $1 < p < \infty$ and write $1/q = 1 - 1/p$. If we have*

$$\| \sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)} \|_q \lesssim A$$

for some sequence $\{\alpha_k\}$ with the property $\delta^{d-1} \sum_k |\alpha_k|^q = 1$, then for any $f \in L^p(\mathbb{R}^d)$ we have the bound

$$\| \ M_\delta f \|_{L^p(\mathbb{S}^{d-1})} \lesssim \delta^{\frac{d-2}{p}} A \|f\|_p$$

Proof. We discretize the domain of $M_\delta f$ via the following argument. Let $D_\delta(e_k) := \{x \in \mathbb{S}^{d-1} \mid |e_k - x| < \delta\}$ denote the δ-neighborhood in $\mathbb{S}^{d-1}$ around e_k. Then notice that because $\{e_k\}$ is a maximal $\delta-$separated set, $\mathbb{S}^{d-1} = \bigcup_k D_\delta(e_k)$. Furthermore, we for $|e - e_k| < \delta$ can cover $C^\delta_e(v)$ with some bounded number of cylinders $C_{e_k}(v_i)$, implying that there is a constant λ for which $M_\delta f(e) \leq \lambda M_\delta f(e_k)$ when $e \in D_\delta(e_k)$. Ergo, since $\int_{D_\delta(e_k)} d\mu \sim \delta^{d-1}$,

$$\|M_\delta f\|_p \leq \left(\sum_k \int_{D_\delta(e_k)} M_\delta f(e)^p \, d\mu \right)^{1/p} \lesssim \left(\delta^{d-1} \sum_k M_\delta f(e_k)^p \right)^{1/p}. \tag{9}$$

We now exploit the duality between ℓ^p and ℓ^q, where $1/p + 1/q = 1$. Let $F : \ell^q \to \mathbb{R}$ be the functional on ℓ^q so that for $\alpha = \{\alpha_k\}$,

$$F(\alpha) := \sum_k \beta_k \alpha_k$$

where $\beta = \{\beta_k\}$ is defined by $\beta_k := \delta^{\frac{d-1}{p}} M_\delta f(e_k)$ when k is smaller than the number of points in the maximally separated δ set, and zero otherwise. Let α be the sequence such that

$$\delta^{\frac{d-1}{q}} \alpha_k := \left(\frac{\beta_k}{\|\beta\|_{\ell^p}} \right)^{q-1}.$$

Then, since $1/q - 1 = -1/p$ and $\delta^{d-1}\|\alpha\|_{\ell^q} = 1$, we obtain

$$\delta^{d-1} \sum_k \alpha_k M_\delta f(e_k) \sim F(\delta^{\frac{d-1}{q}} \alpha) = \|\beta\|_{\ell^p} = \left(\delta^{d-1} \sum_k M_\delta f(e_k)^p \right)^{1/p}.$$

In turn, using the definition of $M_\delta f$ and observing $|C_e^\delta(v)| \sim \delta$ for any $e \in \mathbb{S}^{d-1}, v \in \mathbb{R}^d$,

$$\begin{aligned}\delta^{d-1}\sum_k \alpha_k M_\delta f(e_k) &\lesssim \delta^{d-2}\sum_k \alpha_k \int_{C^\delta_{e_k}(v_k)} f(x)\\ &= \delta^{d-2}\int_{\mathbb{R}_d}\left(\sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)}(x)\right) f(x)\,d\mu\end{aligned}$$

As such, by Holder's inequality, (9) becomes

$$\|M_\delta f\|_p \lesssim \delta^{d-2}\int_{\mathbb{R}_d}\left(\sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)}(x)\right) f(x)\,d\mu \leq \delta^{\frac{d-2}{p}} A\|f\|_p \tag{10}$$

□

We can now proceed to the proof of Theorem 4.1. We mimic the proof of Theorem 1.9 found in [4].

Proof of Theorem 4.1. By the previous lemma, it suffices to show that for any sequence $\alpha \in \ell^2$ with $\delta^{d-1}\sum_k \alpha_k^2 = 1$ and any maximal δ-separated $\{e_k\} \subset \mathbb{S}^{d-1}$,

$$\|\sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)}\|_2 \lesssim \delta^{-(d-2)}\sqrt{\log(\delta^{-1})}.$$

Using the geometric lemma, the left hand side reduces to

$$\|\sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)}\|_2 = \sum_{k,j}\alpha_k\alpha_j \mu(C^\delta_{e_k}(v_k)\cap C^\delta_{e_j}(v_j)) \lesssim \sum_{k,j}\alpha_k\alpha_j\frac{\delta^2}{|e_k - e_j|}.$$

Writing $\alpha_k\alpha_j\frac{\delta^2}{|e_k-e_j|} = \delta^{1/2}\alpha_k\delta^{1/2}\alpha_j\frac{\delta}{|e_k-e_j|}$ and using Cauchy-Schwartz we get

$$\|\sum_k \alpha_k \mathbb{1}_{C^\delta_{e_k}(v_k)}\|_2^2 \leq \sum_k \delta\alpha_k^2\sum_j\frac{\delta}{|e_k-e_j|} \lesssim \log(\delta^{-1})\delta\sum_k \alpha_k^2 = \delta^{-(d-2)}\log(\delta^{-1}).$$

This proves the theorem. □

5. Weak Kakeya-like Sets

As we shall see, any estimate as in Theorem 4.1 that bounds $\|M_\delta f\|_p$ fails to yield a good bound on $\mu(K_\delta)$ for Kakeya-like sets K. However, the bound is better for a "weakened" class of sets, which we briefly explore in this section.

Definition 5.1. In $\mathbb{R}^d$, a **weak Kakeya-like set**, or a weak set for brevity, is a measure zero compact set containing a unit $d-1$ disk with normals in a dense subset Γ of $\mathbb{S}^{d-1}$. We call Γ the **set of characteristic directions** for W.

We would like to prove things about $M_\delta \mathbb{1}_{W_\delta}$ for weak sets W in parallel to proofs about $(\mathbb{1}_{K_\delta})^*_\delta$. As before, we will make extensive use of the notion of a δ-separted set. This time, however, we will use a maximal δ-separated set on the subspace of characteristic directions Γ of a weak set, and not on the whole sphere. A few facts about such sets:

Fact 4. Let X be a metric space, and let $\Gamma \subset X$ be a dense subset. If $\Omega \subset \Gamma$ is maximally δ-separated in the subspace Γ (and not necessarily in all of X), then the closed disks $\{\overline{D}_\delta(\omega) \mid \omega \in \Omega\}$ cover X. In particular, if $X = \mathbb{S}^{d-1}$, then $|\Omega| \sim \delta^{-(d-1)}$

Proof. The first claim follows from the fact that $\{D_\delta(\omega) \cap \Gamma \mid x \in \Omega\}$ covers Γ and $\bar{\Gamma} = X$. The second is true by the same reasoning as Fact 3. $\square$

Proposition 5.2. *Let $W \subset \mathbb{R}^d$ a weak set, and let W_δ be its δ-neighborhood. Then,*

$$\frac{1}{\log(\delta^{-1})} \lesssim \mu(W_\delta) \tag{11}$$

Of course, the bound above also holds for Kakeya-like sets; it is a terrible bound however for $d \geq 3$, since for such a set K, $\mu(K_\delta) \to \mu(K) > 0$ as $\delta \to 0$ (see Remark). Weak Kakeya-like sets on the other hand have measure zero, and so the bound is much tighter. Hence, weak Kakeya-like sets behave much more like Kakeya sets in $\mathbb{R}^2$ in a measure-theoretic sense. Furthermore, the existence of weak Kakeya-like sets means that any attempt at finding a nontrivial lower bound for the measure of a Kakeya-like set by using this maximal function cannot work, as encapsulated in the next proposition, the proof of which implies Proposition 5.2.

Theorem 5.3. *Let $1 \leq p, q < \infty$, and suppose we have an estimate*

$$\|M_\delta f\|_q \lesssim A_\delta \|f\|_p.$$

Then

$$A_\delta^{-1} \lesssim \mu(W_\delta)^{1/p} \text{ and } A_\delta^{-1} \to 0 \text{ as } \delta \to 0$$

Proof. Let W be a weak set, and let W_δ be its δ neighborhood. Then for any cylinder C_e^δ normal to e,

$$M_\delta \mathbb{1}_{W_\delta}(e) \gtrsim \frac{1}{\mu(C_e^\delta)} \int_{C_e^\delta} \mathbb{1}_{W_\delta} \, d\mu \sim \delta^{-1} \mu(W_\delta \cap C_e^\delta). \tag{12}$$

Let Γ be the set of characteristic directions for W, and let $\{e_k\} \subset \Gamma$ be a maximal $\delta-$separated set on Γ. Choose $e \in D_{\frac{\delta}{2}}(e_k)$ and any set of points $\{v_k\} \subset \mathbb{R}^d$. Clearly $\frac{1}{2}C_e^\delta(v_k) \subset C_{e_k}^\delta\left(\frac{1}{2}v_k\right)$, meaning

$$\mu(W_\delta \cap C_e^\delta(v_k)) \geq \mu(C_{e_k}^\delta(v_k) \cap C_e^\delta(v_k)) \geq \mu(1/2C_e^\delta(v_k)) \sim \delta.$$

Hence, for $e \in D_{\frac{\delta}{2}}(e_k)$, (12) becomes

$$M_\delta \mathbb{1}_{W_\delta}(e) \gtrsim 1.$$

However, the sets $D_{\frac{\delta}{2}}(e_k)$ are disjoint from one another and

$$\mathcal{D} := \bigcup_k D_{\frac{\delta}{2}}(e_k) \subset \mathbb{S}^{d-1}. \tag{13}$$

In addition, because $\{e_k\}$ has $\sim \delta^{-(d-1)}$ terms,

$$\sum_k \mu(D_{\delta/2}(e_k)) \sim 1.$$

Therefore,

$$1 \lesssim \sum_k \int_{D_{\frac{\delta}{2}}(e_k)} (M_\delta \mathbb{1}_{W_\delta})^q = \int_{\mathcal{D}} (M_\delta \mathbb{1}_{W_\delta})^q \leq \|M_\delta \mathbb{1}_{W_\delta}\|_q^q \lesssim A_\delta^q \mu(W_\delta)^{q/p}. \tag{14}$$

Because $\mu(W_\delta) \to 0$, we must have $A_\delta^{-1} \to 0$ □

Proof of Proposition 5.2. Setting $p = q = 2$ and $A_\delta = \sqrt{\log(\delta^{-1})}$, the claim follows directly from (14). □

Remark. Letting E be an arbitrary set and E_δ its δ-neighborhood, it is of course not in general true that $\mu(E_\delta) \to \mu(E)$ as $\delta \to 0$, because it is in general not true that the E_δ converges to E in the sense that

$$\bigcap_{\delta \in (0,\infty)} E_\delta = E.$$

For instance, if $E = \mathbb{Q}$, we have that $E_\delta = \mathbb{R}$ for all $\delta > 0$. However, if E is closed, then the above equality does hold, and we do in fact have that $\mu(E_\delta) \to \mu(E)$ as $\delta \to 0$. Hence, since Kakeya-like sets and weak sets are compact, we are justified in claiming, as we do above, that $\mu(K_\delta) \to \mu(K)$ and $\mu(W_\delta) \to 0$ for a Kakeya-like set K and a weak set W.

6. Future Directions

We conclude with a list of questions we would like to pursue in the future.

Question 1. We did not shed much light on $\gamma(\mathfrak{M}_d, \cdot)$. We would like to explore this problem further.

Question 2. For $E \in \mathfrak{M}_d$, what can we say about the Hausdorff dimension of

$$\Gamma_\infty(E) = \bigcap_{n\geq 1} \Gamma_n(E)?$$

Question 3. For Kakeya-like set K, can we find a meaningful bound

$$A_\delta \leq \mu(K_\delta)?$$

Or can we find bounds on the relative size of K to K_δ, for instance bounds such as

$$A_\delta \lesssim \mu(K_\delta \setminus K)$$

or

$$A_\delta \lesssim \frac{\mu(K_\delta)}{\mu(K)}?$$

Question 4. What are the possible Hausdorff dimensions of weak sets?

Question 5. Let $\mathscr{W}$ be the class of weak sets in $\mathbb{R}^d$, and let Γ_W be the characteristic set of directions for the weak set W. What is $\sup_{W\in\mathscr{W}} \dim(\Gamma_W)$? Does there exist a weak set W with $\dim(\Gamma_W) = d-1$?

Question 6. Suppose $\Gamma \subset \mathbb{S}^d$ is a dense subset with the property that Γ intersects every great circle at least once. For $d \geq 2$, does such a Γ exist so that it is the characteristic set of directions for some weak set W?

This is interesting because if there is some weak set W whose set of directions Γ intersects every great circle at least once, then W contains a line segment in every direction; such a W is also a Kakeya set. To see this, suppose such a W exists, choose any e on the sphere, and consider the set $e^\perp$ of directions on the sphere orthogonal to e. This set is a great circle and intersects Γ at some point, say p. Hence, there is a unit disk in W orthogonal to p, and in this unit disk there is a unit line segment parallel to e. Thus W is a Kakeya set. As such, bounding the Hausdorff dimension of Weak sets can give us some information on the Hausdorff dimension of Kakeya sets—if for some extraordinary and unexpected reason weak sets are never full-dimensional, this would disprove the Kakeya conjecture, and if weak sets whose set of directions intersect every great circle at least once are full dimensional, then we would have an example of Kakeya sets with full dimension.

Question 7. Can we improve the bound (11) on $\mu(W_\delta)$, or is it optimal?

Question 8. In the operator M_δ, we integrate over cylinders that are rotated copies of $\mathbb{B}^{d-1} \times [-\delta, \delta]$. If we define maximal operators $M_{\delta,n}$ that use rotated copies of $\mathbb{B}^n \times [-\delta, \delta]$ instead, we could use this operator to investigate $(d, n)-$Kakeya sets and $(d, n)-$Kakeya-like sets. What estimates can we derive for such operators $M_{\delta,n}$? Do they show any interesting behavior as we vary n? Do they shed any light on the $(d, n)-$Kakeya conjecture?

7. Acknowledgements

First and foremost, I would like to thank my mentor Paxton Turner for his guidance and patience, as well as and for the joyous levity he brought to our meetings. Next, I would like to thank Professor Guth for suggesting this project and for his numerous insights. Our weekly meetings with Professors Jerison and Moitra proved to be invaluable as well, and so I thank them together with Professor Gerovitch and the rest of the SPUR faculty and staff for all the work they put into the SPUR program. Finally, I want to thank Max Engelstein, friend and mentor in life and in math, for his support during the last six weeks.

References

[1] L. Guth. *On a Kakeya-type problem for hyperplanes in Rd.*. Private communication, May 2017.

[2] E. Stein and R. Shakarchi, *Real Analysis: Measure theory, Integration, and Hilbert Spaces*. Volume III of *Princeton Lectures in Analysis*. Princeton University Press, 2005.

[3] T. Wolff, *Lectures on Harmonic Analysis*. Volume 29 of *University Lecture Series*. American Mathematical Society, 2003

[4] E. Kroc, *The Kakeya Problem*. The University of British Columbia, August 2010. `http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf`

[5] P. Klein, *On the Ellipsoid and Plane Intersection Equation*. Computing Center, University of Technology Clausthal, 2012. `https://file.scirp.org/pdf/AM20121100009_89014420.pdf`

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