ON A CONNECTION BETWEEN AFFINE SPRINGER FIBERS AND L–U DECOMPOSABLE MATRICES

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PANAGIOTIS DIMAKIS
MENTOR: GUANGYI YUE
PROJECT SUGGESTED BY ROMAN BEZRUKAVNIKOV

Abstract. In this paper we study the spaces X defined as $\mathbb{O} \cap B_{-}B_{+}$ where $\mathbb{O}$ is a regular semi-simple orbit of a semi-simple group G over $\mathbb{C}$ and B_{-}, B_{+} are opposite Borel subgroups in G in the special case where $G = SL_n$. We describe a conjectural correspondence between affine Springer fibers over SL_n and the above spaces and verify it when $G = SL_2$. Finally we give a conjecture about the shape of X when $G = SL_3$ based on the above correspondence.

Contents

1. Introduction

The study of affine Springer fibers was initiated by Kazhdan and Lusztig and aims to generalize the theory of Springer fibers. Some of the reasons that motivate the study of affine Springer fibers are the connections between these fibers and the representation theory of p-adic groups as well as their applications to geometry. Understanding the classical definition and properties of affine Springer fibers requires a strong algebraic background. However, one can also describe them combinatorially as spaces that classify chains of lattices inside an $\mathbb{C}((t))$-vector space which are invariant under some $\mathbb{C}((t))$-linear transformation.

In this paper we prove a conjectured homotopy between the quotient of a specific type of affine Springer fiber by a group that naturally acts on it, and the space X already defined in the abstract for the special case $G = SL_2$. More specifically, let X_γ be the affine Springer

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fiber corresponding to the regular semisimple $\gamma = diag(m_1, ..., m_n)t \in \mathfrak{sl}_n(\mathbb{C}((t)))$. There exists a natural free action of the group $\Lambda_\gamma = \mathbb{Z}^{n-1}$ on X_γ. It is conjectured that the quotient X_γ/Λ_γ is homotopic to to X for $G = SL_n$. We prove this conjecture for $n = 2$.

The aim of trying to construct such a homotopy is twofold. On the one hand, it relates a space which is hard to understand and to use to test conjectured properties, to a space that, at least when the dimension is low, can be explicitly understood in terms of equations, and where some conjectured properties can be easily tested either by hand or with the help of a computer. On the other hand, the space X as defined in the abstract, is strongly connected to classical Lie theory. Thus if the conjectured homotopy is true in genera,l then it implies that there is a subtle but important connection between the theory of affine Springer fibers and classical Lie theory.

The structure of the paper is organized in the following way:

Section 2 describes most of the prerequisites someone needs in order to be able to understand section 3. In the fisrt subsection we give the nessesary definitions in order to be able to give a rigorous and at the same time easy to understand description of the space X when $G = SL_n$. We also present a lemma that will be useful in section 3, but which we considered appropriate to include here since it is true for any n and thus might prove useful to someone who wants to study the case where $n > 2$. In the second subsection we give a quick introduction to affine Springer fibers. We have tried to make this subsection as elementary as possible, giving mostly combinatorial descriptions of the objects in question. For the shake of completion, we also include a number of statements that we not prove. In the last subsection we state a number of theorems from algebraic topology and classical homotopy theory that play an essential role in the study of the topology of X in section 3.

Section 3 contains the original results of the present paper. In this section we use the tools described in section 2 to describe the spaces X and X_γ/Λ_γ and show that they are both homotopic to $S^2 \vee S^2 \vee S^1$. The analysis of X is based on topological methods. We first obtain a concrete description of X as a subspace of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and use classical results from algebraic topology found in [Hat01] to find its homology. Then, using classical homotopy theory results we show that X can be one of 4 possible spaces up to homotopy. Using the homology we have already calculated, we obtain the desired result. Next we treat the space X_γ/Λ_γ. We first describe the lattices $\Lambda \in F^2$ which are stable under $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. Using this description we derive the explicit form of X_γ from which it is straightforward to find X_γ/Λ_γ. We conclude the section with a conjecture on the shape of X when $G = SL_3$.

2. Preliminary definitions and results

2.1. The space X.

The aim of this subsection is to give a concrete description of the space X when the semisimple group considered is $SL_n(\mathbb{C})$. Any space appearing below carries the induced topology as a subspace of SL_n.

Definition 2.1. A semisimple regular orbit $\mathbb{O} \subset SL_n$ is defined as the congugacy class of a diagonal matrix in SL_n, all of whose diagonal entries are different from each other.

Although we could give a general definition of Borel subgroups of a semisimple group G, since our study restricts only to the special case where G is the special linear group, we will define these only for this group.

Definition 2.2. For SL_n we may assume that B_+ is the subgroup of upper triangular matrices and B_- the subgroup of lower triangular matrices.

Definition 2.3. The set of matrices in SL_n that can be expressed as a product of a lower triangular matrix L and an upper triangular matrix U are called $L-U$ decomposable matrices and will be denoted by $\mathbb{G}_0$.

Definition 2.4. The space X for $G = SL_n$ is defined as the intersection of a semisimple orbit $\mathbb{O} \subset SL_n$ and the set $\mathbb{G}_0$, topologized as a subspace of SL_n.

Lemma 2.5. *A matrix $A \in GL_n$ is $L-U$ decomposable if and only if all of its principal minors are nonsingular i.e. $det((a_{i,j})_{1\leq i,j\leq k}) \neq 0$ for all $1 \leq k \leq n$.*

Proof. First we show that if $A = LU$ then $det((a_{i,j})_{1\leq i,j\leq k}) \neq 0$ for all $1 \leq k \leq n$. It is easy to see that $(a_{i,j})_{1\leq i,j\leq k} = (l_{i,j})_{1\leq i,j\leq k} \times (u_{i,j})_{1\leq i,j\leq k}$ and thus $det((a_{i,j})_{1\leq i,j\leq k}) = det((l_{i,j})_{1\leq i,j\leq k})det((u_{i,j})_{1\leq i,j\leq k}) = \prod_{i=1}^{k} l_{i,i}u_{i,i} \neq 0$ since L and U belong to SL_n and thus $l_{i,i}, u_{i,i} \neq 0$ for all $1 \leq i \leq n$. So in this case all principal minors are nonsingular.

Now assume that all principal minors are nonsingular. We will prove that A is $L-U$ decomposable by induction on the dimension of A. The case $n = 1$ is trivial. Write A$=\begin{pmatrix} \hat{A} & \vec{a} \\ \vec{b}^t & a_{n,n} \end{pmatrix}$. By induction, we can write $\hat{A} = \hat{L}\hat{U}$ with $\hat{L}, \hat{U} \in GL_{n-1}$. We now show that we can express $A = \begin{pmatrix} \hat{L} & 0 \\ \vec{l}^t & l_{n,n} \end{pmatrix}\begin{pmatrix} \hat{U} & \vec{u} \\ 0 & u_{n,n} \end{pmatrix}$ with $l_{n,n}u_{n,n} \neq 0$. Note that this implies the result by the induciton hypothesis. The above equality implies that $\vec{l}^t = (a_{n,1}, ..., a_{n,n-1})\hat{U}^{-1}$ and $\vec{u} = \hat{L}^{-1}(a_{1,n}, ..., a_{n-1,n})^t$. Now the condition $l_{n,n}u_{n,n} \neq 0$ is equivalent to requiring $\vec{l}^t \cdot \vec{u} \neq a_{n,n}$. Assume that $\vec{l}^t \cdot \vec{u}$ were equal to $a_{n,n}$ or equivalently $(a_{n,1}, ..., a_{n,n-1})\hat{U}^{-1}\hat{L}^{-1}(a_{1,n}, ..., a_{n-1,n})^t = (a_{n,1}, ..., a_{n,n-1})\hat{A}^{-1}(a_{1,n}, ..., a_{n-1,n})^t = a_{n,n}$. But this implies that $(a_{n,1}, ..., a_{n,n-1})\hat{A}^{-1}[\hat{A}, (a_{1,n}, ..., a_{n-1,n})^t] = (a_{n,1}, ..., a_{n,n})$. This implies that the last row of A is a linear combination of the first $n-1$ rows so $det(A) = 0$ which is absurd. Therefore we are done. □

Remark 2.6. In case we restrict to SL_n, it can be seen from the above solution that we can choose $l_{n,n}$ and $u_{n,n}$ so that $det(L) = det(U) = 1$.

2.2. **Connection with affine Springer fibers.**

We have tried to make the presentation of the material on this subsection as elementary as possible, emphasizing the combinatorial nature of the subject rather than the algebro-geometric. We begin by giving a number of definitions.

Definition 2.7. Let $F = \mathbb{C}((t))$ be the field of Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ the formal power series over the complex numbers.

Definition 2.8. $\Lambda^s = \mathcal{O}^n$ will be called the standard lattice viewed as a submodule of F^n. A lattice $\Lambda \subset F^n$ is an $\mathcal{O}^n$-submodule such that:

(1) There exists $N > 0$ such that $t^N \Lambda^s \subset \Lambda \subset t^{-N}\Lambda^s$.
(2) $t^{-N}\Lambda^s/\Lambda$ is (locally) free of finite rank over $\mathbb{C}$.

More concretely, a full rank lattice Λ is the $\mathcal{O}$ span of n vectors $\{v_1, ..., v_n\}$ in F^n.

Unfortunately, there is no well defined notion of absolute dimension of a lattice in this case. However there is a notion of relative dimension between two lattices:

Definition 2.9. Let $\Lambda_1, \Lambda_2 \subset F^n$. The relative dimension between the two is given by:

$$[\Lambda_1 : \Lambda_2] := dim_{\mathbb{C}}(\Lambda_1/\Lambda_1 \cap \Lambda_2) - dim_{\mathbb{C}}(\Lambda_1/\Lambda_1 \cap \Lambda_2)$$

Above, considering the dimension of the quotients $(\Lambda_i/\Lambda_1 \cap \Lambda_2)$ is well defined since by definition this is a finitely generated vector space over $\mathbb{C}$.

Definition 2.10. We define the affine flag variety $\mathfrak{Fl}_n$ as the space parametrizing chains of lattices $...\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1...$ in F^n such that:

(1) $[\Lambda_i : \mathcal{O}^n] = i$ for all $i \in \mathbb{Z}$
(2) $\Lambda_i = t\Lambda_{i+n}$ for all $i \in \mathbb{Z}$

Denote by $\mathfrak{sl}_n(F)$ to be the special linear Lie algebra with matrix entries in F instead of $\mathbb{C}$. Explicitly, $\mathfrak{sl}_n(F)$ is the set of matrices M with entries in F such that $tr(M) = 0$. Also, note that since any lattice is of the form $\mathcal{O}v_1 \oplus ... \oplus \mathcal{O}v_n$ for some n-tuple of vectors in F^n, there is a well defined action of an element $\gamma \in \mathfrak{sl}_n(F)$ on the lattice, given by the action of γ on the v_i. Finally we are able to describe the affine Springer fibers over the affine flag variety.

Definition 2.11. For $\gamma \in \mathfrak{sl}_n(F)$ we define the affine Springer fiber X_γ to be the space parametrizing chains of lattices inside $\mathfrak{Fl}_n$ with the additional restriction that $\gamma\Lambda_i \subset \Lambda_i$ for any $i \in \mathbb{Z}$.

From here on we will restrict only to regular semisimple $\gamma \in \mathfrak{sl}_n(F)$ or more explicitly to diagonal matrices with distinct entries in $\mathfrak{sl}_n(F)$ with trace zero. Also denote by $\mathbb{G}_m$ to be the the multiplicative group $\mathbb{C}^*$, and by G_γ the centralizer of γ inside $SL_n(F)$.

Definition 2.12. Under the above assumptions, let $\Lambda_\gamma := Hom_F(\mathbb{G}_m, G_\gamma)$.

Theorem 2.13 (Kazhdan-Lusztig)**.** *The action of Λ_γ on X_γ is free and the quotient is proper over $\mathbb{C}$.*

From here on, we further specialize the form of γ. We only study the fibers with $\gamma = diag(m_1, ..., m_n)t \in \mathfrak{sl}_n(F)$. Now we present the main conjecture whose proof in the case $n = 2$ occupies most of section 3 below.

Conjecture 2.14. There exists a homeomorphism $X_\gamma/\Lambda_\gamma \simeq X$, where X is the space defined in the previous subsection.

Finally we give a lemma characterising the lattice Λ_γ.

Lemma 2.15. *Assuming the above restrictions for γ, $\Lambda_\gamma \cong \mathbb{Z}^{n-1}$, with generators given by the $n-1$ matrices of the form $diag(1, ..., 1, t, t^{-1}, 1, ..., 1)$ and their inverses.*

Proof. The proof of this claim follows easily from the more general case where $G = GL_n$, treated in [Yu16] (Ex. 2.3.4). $\square$

2.3. **Topology background.**

This subsection is only included for completeness. However, we encourage the interested reader to look at the proofs of the following theorems for two reasons. The first one is that in section 3 we sometimes use some idea appearing in the proof of these theorems. The second one is that some of the proofs are elegant in their own right.

Theorem 2.16 ([Hat01]). *If K is a compact, locally contractible subspace of a closed orientable n manifold M , then $H_i(M, M - K;\mathbb{Z}) \cong H^{n-i}(K;\mathbb{Z})$ for all i.*

Theorem 2.17 ([Fr13]). *The pullback of a homotopy equivalence along a fibration is again a homotopy equivalence.*

Theorem 2.18 ([St44]). *The k-sphere bundles over S^1 are of two types: the product bundle, and the generalized Klein bottle.*

Theorem 2.19 (Kunneth Theorem). *There exists the following short exact sequence for topological spaces X and Y where the cohomology is assumed to be integral:*

$$0 \to \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y) \to H^k(X \times Y) \to \bigoplus_{i+j=k-1} Tor_1(H^i(X), H^j(Y)) \to 0$$

3. Main results and conjectures

In this section we give an explicit description of the spaces X and X_γ/Λ_γ, where $G = SL(2, \mathbb{C})$, showing that they are both homotopic to $S^2 \vee S^2 \vee S^1$. In the end of the section we present a conjecture on the shape of the above spaces for $G = SL_3$.

First we analyze X.

Lemma 3.1. *The space X is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^1 - \mathbb{C}P^1$, such that if the coordinates of the first $\mathbb{C}P^1$ in the product are given by $[a : b]$ with $a, b \in \mathbb{C}$ and of the second by $[c : d]$ with $c, d \in \mathbb{C}$ then the first $\mathbb{C}P^1$ we remove has coordinates $([a : b], [a : b])$ and the second $([a : b], [a, bt^2])$.*

Proof. X is by definition the intersection of the orbit $\mathbb{O}_t$ of the matrix $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in $SL(2,\mathbb{C})$, where $t \neq 0, \pm 1$ so that the orbit is regular semisimple, with the set $\mathbb{G}_0 \subset SL(2,\mathbb{C})$ of the Gauss decomposable matrices. Thus,
$X = \left\{ \begin{pmatrix} d & -b \\ c & a \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad-bc = 1, adt^2 - bc \neq 0 \right\}$.The last constraint comes from the fact that the matrix $\begin{pmatrix} d & -b \\ c & a \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$ $\begin{pmatrix} adt - bct^{-1} & bdt - bdt^{-1} \\ act^{-1} - act & adt^{-1} - bct \end{pmatrix}$ lies in $\mathbb{G}_0$ if and only if the upper left entry is different from zero.

First, for any diagonalizable matrix B, writing $B = A^{-1}DA$ where D is diagonal with fixed eigenvalues, is unique up to rescaling of the rows of A. Therefore, since we can ignore the scaling of the rows of A, the following map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow ([a:b],[c:d])$ is injective and thus points of X correspond to points $([a:b],[c:d]) \in \mathbb{C}P^1 \times \mathbb{C}P^1$.

. Now we need to remove from $\mathbb{C}P^1 \times \mathbb{C}P^1$ the points that do not come from points of X. Since scaling is irrelevant, the conditions that must be satisfied by points of $\mathbb{C}P^1 \times \mathbb{C}P^1$ to come from X_t are $ad - bc \neq 0$ and $adt^2 - bc \neq 0$. It is straightforward that the points in $\mathbb{C}P^1 \times \mathbb{C}P^1$ satisfying $ad - bc = 0$ are precisely those of the form $([a:b],[a:b])$ and those satisfying $adt^2 - bc$ are precisely those of the form $([a:b],[a,bt^2])$. Therefore X has the required description as wanted.

$\square$

Lemma 3.2. *The homology of X is given by:*

$$H_i(X) = \begin{cases} \mathbb{Z} \textit{ for } n = 0,1 \\ \mathbb{Z} \oplus \mathbb{Z} \textit{ for } n = 2 \\ 0 \textit{ otherwise} \end{cases}$$

Proof. First we consider the space $Y = \mathbb{C}P^1 \times \mathbb{C}P^1 - X$. The above lemma implies that this space is the union of two copies of $\mathbb{C}P^1$ given by $\{([a:b],[a:b]) \mid [a:b] \in \mathbb{C}P^1\}$ and $\{([a:b],[a:bt^2]) \mid [a:b] \in \mathbb{C}P^1\}$ respectively. From this parametrization, it can be seen that the two $\mathbb{C}P^1$ intersect in exactly two points, namely $([1:0],[1:0])$ and$([0:1],[0:1])$. Since $\mathbb{C}P^1 \cong S^2$, Y can be seen to be homotopic to $S^2 \vee S^2 \vee S^1$. Therefore,

$$H^i(Y) = \begin{cases} \mathbb{Z} \text{ for } n = 0,1 \\ \mathbb{Z} \oplus \mathbb{Z} \text{ for } n = 2 \\ 0 \text{ otherwise} \end{cases}$$

Next, we consider the long exact sequence in homology of the pair $(\mathbb{C}P^1 \times \mathbb{C}P^1, X)$. Since $\mathbb{C}P^1 \times \mathbb{C}P^1$ is an orientable closed 4-manifold as the product of two closed orientable 2-manifolds and Y is a compact locally contractible subspace, we can apply theorem 2.16 to get the following commutative diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_i(X) & \longrightarrow & H_i(\mathbb{C}P^1 \times \mathbb{C}P^1) & \longrightarrow & H_i(\mathbb{C}P^1 \times \mathbb{C}P^1, X) & \longrightarrow & H_{i-1}(X) & \longrightarrow & \dots \\
 & & & & \uparrow \cong & & \uparrow \cong & & & & \\
 & & & & H^{4-i}(\mathbb{C}P^1 \times \mathbb{C}P^1) & \xrightarrow{i^*} & H^{4-i}(\mathbb{C}P^1 \times \mathbb{C}P^1 - X) & \cong & H^{4-i}(Y) & &
\end{array}$$

Above i^* is induced by the inclusion $i : Y \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ as can be seen from the proof of the theorem. Since $\mathbb{C}P^1 \cong S^2$, the Kunneth theorem and the diagram above imply

$$H_i(\mathbb{C}P^1 \times \mathbb{C}P^1) = \begin{cases} \mathbb{Z} \text{ for } n = 0, 4 \\ \mathbb{Z} \oplus \mathbb{Z} \text{ for } n = 2 \\ 0 \text{ otherwise} \end{cases}, H_i(\mathbb{C}P^1 \times \mathbb{C}P^1, X) = \begin{cases} \mathbb{Z} \text{ for } n = 3, 4 \\ \mathbb{Z} \oplus \mathbb{Z} \text{ for } n = 2 \\ 0 \text{ otherwise} \end{cases}$$

This immediately implies that $H_i(X) = 0$ for $i > 4$ from the above long exact sequence. For $i = 4$ the map i^* is an isomorphism since it is induced by the inclusion of a connected subspace into a connected space. Therefore, from the exact sequence

$$0 \;=\; H_5(\mathbb{C}P^1 \times \mathbb{C}P^1, X) \;\longrightarrow\; H_4(X) \;\longrightarrow\; \mathbb{Z} \;\xrightarrow{\cong}\; \mathbb{Z} \;\longrightarrow\; H_3(X) \;\longrightarrow\; 0$$

we obtain that $H_4(X) = H_3(X) = 0$. Similarly, the exact sequence

$$0 \;=\; H_1(\mathbb{C}P^1 \times \mathbb{C}P^1, X) \;\longrightarrow\; H_0(X) \;\longrightarrow\; \mathbb{Z} \;\longrightarrow\; H_0(\mathbb{C}P^1 \times \mathbb{C}P^1, X) \;=\; 0$$

implies that $H_0(X) = \mathbb{Z}$.

Finally, we find the first and second homology of X through deriving an explicit form of the map i^* in the following exact sequence:

$$0 \;\longrightarrow\; \mathbb{Z} \;\longrightarrow\; H_2(X) \;\longrightarrow\; \mathbb{Z} \oplus \mathbb{Z} \;\xrightarrow{i^*}\; \mathbb{Z} \oplus \mathbb{Z} \;\longrightarrow\; H_1(X) \;\longrightarrow\; 0$$

In order to describe i^* we need to take a closer look at how each of the two $\mathbb{C}P^1 \simeq S^2$ in Y lie inside $\mathbb{C}P^1 \times \mathbb{C}P^1$. By lemma 3.1, both copies can be considered to be placed diagonally in $\mathbb{C}P^1 \times \mathbb{C}P^1$. This, together with the fact that $H^2(Y) \cong H^2(\mathbb{C}P^1) \oplus H^2(\mathbb{C}P^1)$, where the two $\mathbb{C}P^1$ are as above, implies that $i^* = \Delta_1^* \oplus \Delta_2^* : H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \to H^2(\mathbb{C}P^1) \oplus H^2(\mathbb{C}P^1)$, induced by the "diagonal" inclusions Δ_i of the two copies of $\mathbb{C}P^1$ inside the product. Therefore it is enough to see where Δ_i^* sends the generators of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$ inside $H^2(\mathbb{C}P^1)$.

To do this consider the following diagram on the left and apply the cohomology functor to obtain the diagram on the right.

$$\begin{array}{ccccc}
\mathbb{C}P^1 & \xleftarrow{id} & \mathbb{C}P^1 & \xrightarrow{id} & \mathbb{C}P^1 \\
\downarrow \cong & & \downarrow \Delta & & \downarrow \cong \\
\mathbb{C}P^1 & \xleftarrow{pr_1} & \mathbb{C}P^1 \times \mathbb{C}P^1 & \xrightarrow{pr_2} & \mathbb{C}P^1
\end{array}
\;\to\;
\begin{array}{ccccc}
H^*(\mathbb{C}P^1) & \xrightarrow{\cong} & H^*(\mathbb{C}P^1) & \xleftarrow{\cong} & H^*(\mathbb{C}P^1) \\
\uparrow \cong & & \uparrow \Delta^* & & \uparrow \cong \\
H^*(\mathbb{C}P^1) & \xrightarrow{pr_1^*} & H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) & \xleftarrow{pr_2^*} & H^*(\mathbb{C}P^1)
\end{array}$$

Since both generators of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$ come from the image of the generator of $H^2(\mathbb{C}P^1)$ under pr_1^* and pr_2^* respectively as can be seen by looking at a proof of the Kunneth theorem, the commutativity of the right diagram implies that both generators of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$ map to the generator of $H^2(\mathbb{C}P^1)$.Therefore, i^* is given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. From this it is easy to see that $H_1(X) = coker(i^*) \cong \mathbb{Z}$. Also, $ker(i^*) \cong \mathbb{Z}$ so we have the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Since $\mathbb{Z}$ is free and thus projective as a module over $\mathbb{Z}$, this implies that the sequence is split and therefore $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ as wanted.

$\square$

Next, consider the following projection:

$$\begin{array}{ccc} ([a:b],[c:d]) & \in & X \\ \downarrow & & \downarrow \pi \\ [a:b] & \in & \mathbb{C}P^1 \end{array}$$

From lemma 3.1, the fibers above any point $[a:b] \in \mathbb{C}P^1$ different from $[1:0]$ and $[0:1]$ are of the form $\mathbb{C}P^1$ minus two distinct points and thus homotopic to S^1 and above the two points $[1:0]$ and $[0:1]$ they are of the form $\mathbb{C}P^1$ minus one point and therefore homotopic to D^2.

It is easy t see that $\pi^{-1}(\mathbb{C}P^1 - [1:0] - [0:1]) \to \mathbb{C}P^1 - [1:0] - [0:1]$ is a fiber bundle with fibers homotopic to S^1. Since $S^1 \hookrightarrow (\mathbb{C}P^1 - [1:0] - [0:1]) = S^1 \times I$, where $I = (0,1)$, is a homotopy equivalence, theorem 2.18 implies that $\pi^{-1}(\mathbb{C}P^1 - [1:0] - [0:1])$ is homotopic to a circle bundle over S^1. Applying theorem 2.17, this shows that $\pi^{-1}(\mathbb{C}P^1 - [1:0] - [0:1])$ can only be of two forms, namely either $T \times I$ or $K \times I$ where T is the usual torus, K the usual Klein bottle.

Deforming the two D^2 fibers to points, we obtain a space homotopic to X. Using the above description of $\pi^{-1}(\mathbb{C}P^1 - [1:0] - [0:1])$ and due to symmetry of the space X around the points $[1:0]$ and $[0:1]$, , this deformation can have exactly one of the following effects. The one is to lead the neighboring fibers of the two disks to converge to the resulting points of the deformation, which implies X is homotopic to one of the suspensions ΣT or ΣK. The other is to not affect the topology of the neighboring fibers. In this case it can be seen that X will be homotopic one of the following two spaces. Either a torus attached around a sphere, namely $T^2 \cup_{S^1} S^2$ where S^1 can be considered as an equator of both the sphere and the torus, or as $K \cup_{S^1} S^2$ where S^1 can be considered as an equator of both tha sphere and the Klein bottle.

Therefore there are 4 different possibilities for what X looks like. We now use lemma 3.2 in order to obtain the correct answer. Since the homology of a suspension of a space shifts by one, or more concretely $H_{i+1}(\Sigma X) \cong H_i(X)$, using the fact that $H_2(T) \cong H_2(K) \cong \mathbb{Z}$ we rule out the first two possibilities since X has zero third homology . It is easily seen that $T^2 \cup_{S^1} S^2 \simeq S^2 \vee S^2 \vee S^1$ and $K^2 \cup_{S^1} S^2 \simeq S^2 \vee \mathbb{R}P^2$. However, the first homology of the latter space is $H_1(S^2 \vee \mathbb{R}P^2) \cong H_1(S^2) \oplus H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$. Therefore we have shown that $X \simeq S^2 \vee S^2 \vee S^1$ as wanted.

Next, we analyze the space X_γ/Λ_γ. Notice that in the $n = 2$ case, $\gamma = \begin{pmatrix} ct & 0 \\ 0 & -ct \end{pmatrix}$, where $c \in \mathbb{C}^*$. Thus c can be cosidered to be equal to 1 since it is invertible in F.

Lemma 3.3 (2-dimensional lattices)**.** *All lattices $\Lambda \subset F^2$ that are invariant under the action of $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, are generated by one of the following pairs of vectors:*

(1) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^a + \begin{pmatrix} 0 \\ c \end{pmatrix} t^{b-1}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^b$ *for $a < b$.*

(2) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^b$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^b$.

(3) $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^a + \begin{pmatrix} c \\ 0 \end{pmatrix} t^{b-1}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^b$ *for $a + 1 < b$.*

(4) $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^{b-1}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^b$ *for $a < b$*

Proof. Assume that Λ is generated by $\{v_1, v_2\}$. Then it is easy to see that we can we can write $v_i = \hat{e}_j t^a +$ higher order terms, where $i, j \in \{1, 2\}$ and $\{\hat{e}_1, \hat{e}_2\}$ the standard basis. After we perform row reduction on the two vectors, it can be seen that we obtain one of the following two possibilities for $\{v_1, v_2\}$:

- $v_1 = \begin{pmatrix} 1 \\ c_a \end{pmatrix} t^a + \begin{pmatrix} 0 \\ c_{a+1} \end{pmatrix} t^{a+1} + \ldots + \begin{pmatrix} 0 \\ c_{b-1} \end{pmatrix} t^{b-1}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^b$ for $a \leq b$.
- $v_1 = \begin{pmatrix} c_a \\ 1 \end{pmatrix} t^a + \begin{pmatrix} c_{a+1} \\ 0 \end{pmatrix} t^{a+1} + \ldots + \begin{pmatrix} c_{b-1} \\ 0 \end{pmatrix} t^{b-1}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^b$ for $a < b$.

Next, we consider the condition $\gamma\Lambda \subset \Lambda$ for $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. In the first case this implies that $\gamma v_1 = \begin{pmatrix} 1 \\ -c_a \end{pmatrix} t^{a+1} + \begin{pmatrix} 0 \\ -c_{a+1} \end{pmatrix} t^{a+2} + \ldots + \begin{pmatrix} 0 \\ -c_{b-1} \end{pmatrix} t^b$ is a linear combination of $\{v_1, v_2\}$ over $\mathcal{O}$, or that $(\gamma - t)v_1 = \begin{pmatrix} 0 \\ -2c_a \end{pmatrix} t^{a+1} + \begin{pmatrix} 0 \\ -2c_{a+1} \end{pmatrix} t^{a+2} + \ldots + \begin{pmatrix} 0 \\ -2c_{b-1} \end{pmatrix} t^b \in \mathcal{O}v_2$. This forces $c_a = \ldots = c_{b-2} = 0$. Therefore in this case we obtain first form of lattices from the theorem unless $a = b$, in which case $c_{b-1} = 0$ as well, and thus we obtain the second form. the second case is equivalent to the first so we omit it.

□

From definition 2.9, it is straightforward that if Λ has one of the above forms, then $[\Lambda : \mathcal{O}^2] = -(a + b)$.

Lemma 3.4. *The space X_γ is an infinite chain of $\mathbb{C}P^1$'s indexed by $\mathbb{Z}$. If a $\mathbb{C}P^1$ is indexed by $i \in \mathbb{Z}$, then it intersects only with the $\mathbb{C}P^1$'s indexed by $\{i-1, i+1\}$, in exactly one point with each.*

Proof. In the special case we study, from definition 2.11, $X_\gamma := \{\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 = t^{-1}\Lambda_0 \mid \gamma\Lambda_i \subset \Lambda_i$ and $[\Lambda_i : \mathcal{O}^2] = i\}$. A crucial observation is that for any Λ_0 not generated by

$\{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-n}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^n\}$ where $n \in \mathbb{Z}$, Λ_1 is uniquely determined and is generated by $\{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-n-1}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^n\}$ for some $n \in \mathbb{Z}$. Similarly, if Λ_1 is not generated by $\{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-n-1}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^n\}$ for some $n \in \mathbb{Z}$, then Λ_0 is uniquely determined and is generated by $\{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-n}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^n\}$ where $n \in \mathbb{Z}$. The proof of this observation can be easily deduced from lemma 3.3 and is left as an exersice to the interested reader.

Now, since in this special case, $\Lambda_\gamma \cong \mathbb{Z}$ is generated by $\begin{pmatrix} t^{\pm 1} & 0 \\ 0 & t^{\mp 1} \end{pmatrix}$ and acting with it on $\Lambda_0 = \mathcal{O} \oplus \mathcal{O}$, we obtain any Λ_0 of the form $\Lambda_0 = \mathcal{O}t^n \oplus \mathcal{O}t^{-n}$. Using the same action on $\mathcal{O}t^{-1} \oplus \mathcal{O}$ we obtain any $\mathcal{O}t^{-n-1} \oplus \mathcal{O}t^n$ for all $n \in \mathbb{Z}$. Therefore we only need to analyze the case where we fix $\Lambda_0 = \mathcal{O} \oplus \mathcal{O}$ and the case where we fix $\Lambda_1 = \mathcal{O}t^{-1} \oplus \mathcal{O}$.

When $\Lambda_0 = \mathcal{O} \oplus \mathcal{O}$, it can be seen that Λ_1 can be any lattice generated by $\{v_1 = \begin{pmatrix} 1 \\ c \end{pmatrix} t^{-1},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ with $c \in \mathbb{C}$ or $\{v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^{-1},\ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. These lattices are parametrized by the slopes a line can have in $\mathbb{C}^2$ or more explicitly by the "vector" coefficient in front of the t^{-1} on the generator v_1. But this is precisely the space parametrizing lines in $\mathbb{C}^2$ and thus isomorphic to $\mathbb{C}P^1$. A very similar arguement shows that fixing $\Lambda_1 = \mathcal{O}t^{-1} \oplus \mathcal{O}$, the space parametrizing the possible Λ_0 is a $\mathbb{C}P^1$.

For $n \neq m$, two copies of $\mathbb{C}P^1$ corresponding to $\mathcal{O}t^n \oplus \mathcal{O}t^{-n}$ and $\mathcal{O}t^m \oplus \mathcal{O}t^{-m}$ clearly cannot intersect and the same holds for two copies of $\mathbb{C}P^1$ corresponding to $\mathcal{O}t^n \oplus \mathcal{O}t^{-n-1}$ and $\mathcal{O}t^m \oplus \mathcal{O}t^{-m-1}$. Finally, two copies of $\mathbb{C}P^1$ corresponding to $\mathcal{O}t^n \oplus \mathcal{O}t^{-n}$ and $\mathcal{O}t^m \oplus \mathcal{O}t^{-m-1}$ can intersect if and only if $n = \{m, m+1\}$ and the points of intersection are precisely:

- $\mathcal{O}t^n \oplus \mathcal{O}t^{-n} \subset \mathcal{O}t^n \oplus \mathcal{O}t^{-n-1} \subset \mathcal{O}t^{n-1} \oplus \mathcal{O}t^{-n-1}$.
- $\mathcal{O}t^n \oplus \mathcal{O}t^{-n} \subset \mathcal{O}t^{n-1} \oplus \mathcal{O}t^{-n} \subset \mathcal{O}t^{n-1} \oplus \mathcal{O}t^{-n-1}$.

Therefore we are done. □

Finally, taking the quotient X_γ/Λ_γ, it is easy to see by the above description that all $\mathbb{C}P^1$ corresponding to Λ_0 of the form $\mathcal{O}t^n \oplus \mathcal{O}t^{-n}$ go to the same $\mathbb{C}P^1$, all $\mathbb{C}P^1$ corresponding to Λ_1 of the form $\mathcal{O}t^n \oplus \mathcal{O}t^{-n-1}$ go to the same $\mathbb{C}P^1$ and the two intersect at precisely two points which is easily seen to be homotopic to $S^2 \vee S^2 \vee S^1$. Therefore $X_\gamma/\Lambda_\gamma \simeq S^2 \vee S^2 \vee S^1$ as wanted.

We close this section by giving a conjectural description of the space X in the SL_3 case.

Conjecture 3.5. When $G = SL_3$, the space X has six components. Three of them are isomorphic to the flag variety $FL_3(\mathbb{C})$. The other three are isomorphic to a $\mathbb{C}P^1$-bundle over the subspace of $\mathbb{C}P^2 \times \mathbb{C}P^2$, cut out by the equations $x_0y_0 = x_1y_1 = x_2y_2$. The latter space is also known as the blow-up of $\mathbb{C}P^2$ at three points.

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References

[St44] N.E Steenrod, *The classification of sphere bundles*, The Annals of Mathematics, 2nd Ser., Vol. 45, No. 2 (Apr., 1944), pp. 294–311

[Hat01] A. Hatcher, *Algebraic Topology*, online (2001), available at `https://www.math.cornell.edu/~hatcher/AT/AT.pdf`

[Fr13] M. Frankland, *Math 527 - Homotopy Theory, Homotopy pullbacks*, online (2013), available at `http://www.home.uni-osnabrueck.de/mfrankland/Math527/Math527_0308_20170508.pdf`

[Yu16] Z. Yun, *Lectures on Springer theories and orbital integrals*, online (2016), available at `https://arxiv.org/pdf/1602.01451.pdf`

Massachusetts Institute of Technology, Department of Mathematics, Cambridge MA 02139

E-mail address: `pdimakis@mit.edu`