An Investigation on an Extension of Mullineux Involution

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Abstract

In this paper, the action of Mullineux Involution on the b-core and b-quotient of a partition is studied. We then offer a new perspective on a conjecture regarding Mullineux Involution by considering the implicit extension of the involution present in the conjecture, which is then approached via its action on the b-quotient. When the b-quotient has rank at most 2, we give a description of this action and a conjecture on the distribution of lengths of cycles it creates among partitions with size n. The original conjecture is then revisited, where future approaches for work on the conjecture and the extension of Mullineux Involution are given.

1 Introduction

The purpose of this paper is to investigate Mullineux involution. In representation theory, Mullineux involution is strongly tied to the representation theory of the symmetric group S_n when working in characteristic p. As the irreducible representations are in bijection with p-regular partitions of n, Mullineux involution induces an involution on the p-regular partitions. In [**?**], Mullineux gives a purely combinatorial description of this algorithm, and moreover one that works for parameters b that are not necessarily prime. We are also motivated by an unpublished conjecture of Bezrukavnikov, in which there is an extension of the involution to non b-regular partitions. We rephrase the conjecture in terms of the functions C_b^* and investigate them through the use of b-cores and b-quotients. We then conjecture that for the cases when the b-quotient is small, the behavior of C_b follows the same behavior independent of n.

This paper is organized as follows. Section 2 of this paper outlines some of the basic theory behind partitions, as well as an overview of b-cores, b-quotients, and their relationship. Section 3 then defines Mullineux involution and begins to describe the effect of the involution on the b-core and -quotient. Then, Bezrukavnikov's conjecture is given, and the function C_b^2 is defined with reference to the conjecture. Finally, Section 4 details some of our conjectures about C_b^2 for both the general case and for the case where there are at most two b-ribbons.

2 Partitions

2.1 Basic definitions

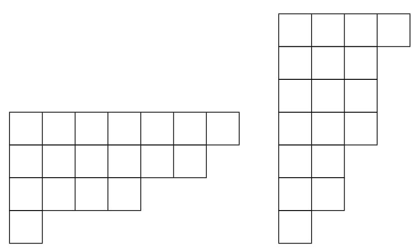
In this paper, a *partition* will be written as an ordered tuple of positive and non-increasing integers.

$$\lambda = (\lambda_1, \ldots, \lambda_k), \text{ where } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$$

Given a partition λ, it has *size* $|\lambda|$ equal to the sum of its parts. Throughout this paper, we will identify a partition with its *Young tableaux*, a collection of boxes whose centers are lattice points in the following set.

$$\{(i,j) \in \mathbb{N}^2 : i+1 \leq \lambda_{j+1}\}$$

We will draw Young tableauxs using English notation, where the axes go down and to the right. An example of the Young tableaux $(7,6,4,1)$ and its transpose is shown below.



All partitions have a *transpose* λ^t, which can be formed by taking the Young tableaux of λ and reflecting over the line $y = x$. The transpose of can also be determined directly from the numerical data of a partition.

$$\lambda_t = \mu, \text{ where } \mu_i = |\{j \in \mathbb{N} : \lambda_j \geq i\}|$$

Given a partition λ, we will occasionally abuse notation and say that it is comprised of *cells* $(i,j) \in \lambda$ if (i,j) is a lattice point in the Young tableaux of λ. The *content* of a cell (i,j) is then defined to be $c(i,j) = i - j$. The *arm* of a cell is all the cells to its right and the *leg* consists of all cells below it.

Finally, the *rim* of a partition refers to the cells in bottom-right portion of a partition. Equivalently, they can be thought of as the following set.

$$\lambda_{\text{rim}} = \{(i,j) \in \lambda : (i+1, j+1) \notin \lambda\}$$

2.2 Cores and Quotients

In this section, we will describe the cores and quotients of a partition. After defining them, some basic facts and results will be described and proven. For the rest of this discussion, we fix some natural number $b \geq 2$.

Definition 2.1. A *b-ribbon* of a partition λ is a connected subset $S \subset \lambda_{\text{rim}}$ such that $|S| = b$ and $\lambda \setminus S$ is a partition. A partition μ is a *b-core* of λ if it has no b-ribbons and can be obtained from λ by successively removing b-ribbons.

For example, the following partition shows two example of finding a 3-core from the same partition.

$$\begin{array}{|c|c|c|c|c|}\hline & & 3 & 1 & 1 \\ \hline & 3 & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline & & 3 & 3 & 3 \\ \hline & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} & 1 & \multicolumn{3}{c}{} \\ \cline{1-2} 1 & 1 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

Note that both examples remove different 3-ribbons but end up with the same 3-core. In fact, this result holds more generally, as described in [**?**].

Proposition 2.2. *The b-core of a partition λ is independent of the choice of ribbons removed. Thus, the b-core* $\text{Core}_b(\lambda)$ *is well-defined.*

For every partition λ we will also define a corresponding b-tuple of partitions which encodes the ribbons whose length is a multiple of b. To do this, for every cell $x = (i,j) \in \lambda$, let $\rho(x)$ be the content of the cell at the end of the arm of x and let $\gamma(x)$ be the content of the cell at the end of the leg of x, both modulo b. Then, for every residue class r modulo b, consider the cells $x \in \lambda$ such that $\rho(x) = r$ and $\gamma(x) = r - 1$. Note that each such cell corresponds to a path along the rim of λ, starting at the end of the arm and going to the end of the leg of x, whose length is a multiple of b. These will necessarily form an exploded Young tableaux inside of λ. Let λ^r denote the corresponding partition.

Definition 2.3. The b-quotient of a partition λ is defined to be

$$\text{Quot}_b(\lambda) = (\lambda^0, \ldots, \lambda^{b-1})$$

The 3-quotient of the partition $(5,4,2,2)$ is $(\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square)$, as is worked out below.

			2	
1				
1				

Given the b-core and b-quotient of a partition λ, one can rebuild λ with the information provided, as discussed in [**?**]. This is explained more concretely below.

Proposition 2.4. *Fix some b-core ν. Then, the map $\phi : \lambda \mapsto \mathrm{Quot}_b(\lambda)$ is a bijection from partitions with b-core ν to b-tuples of partitions.*

Remark 2.5. Proposition **??** provides another way to visualize a partition. Rather than viewing it as a Young tableaux, it can be thought of as starting with the b-core and building up in a way described by the b-quotient. This method of building up is explicitly described in the next theorem.

Theorem 2.6. *Given a partition λ and some partition λ' formed by adding a b-ribbon to λ, then the tuple $\mathrm{Quot}_b(\lambda')$ is formed by adding a cell to one of the partitions in $\mathrm{Quot}_b(\lambda)$.*

Proof. To check this, we will want to observe what happens to the exploded partitions when adding a ribbon to λ. For this proof, a cell being labelled means that it is part of an exploded partition in the p-quotient. Let R be the set of cells contained in the ribbon. Now, suppose that a cell $x = (i_0, j_0)$ is labelled in λ'. This can be divided into four cases based on whether the arm and leg of x end in R.

If neither arm nor leg ends in R, then both ends are also present in λ, which means that x is labelled by the same number in λ. If both the arm and leg are present in λ, then the corresponding path on the rim must lie entirely inside R. However, since R has b cells, the path must be exactly of length b and correspond to the entire ribbon. So, exactly one cell will be in this case and its content will be determined by the content of the top-right cell in the rim.

We will now deal with the case where the end of the arm is not in R but the leg is, as the final case can be treated very similarly. Let $(i_1, j_0) \in R$ be at the end of x's leg. Now, check whether $(i_1, j_0 - 1) \in R$. If it is not, then since we know that $(i_1 + 1, j_0) \notin R$, it follows that (i_1, j_0) is the bottom-left cell in the ribbon. Now, consider the top-right cell in the ribbon $y = (i_2, j_2)$. Th is cell must be below row i_0, as we know that the arm of x does not hit R. Furthermore, the cell right above this cell has the same content modulo b as (i_1, j_0). So, from this we know that (i_0, j_2) is labelled in λ with the same label that (i_0, j_0) is labelled with in λ'.

Next, assume that $(i_1, j_0 - 1) \in R$. Then, we know that $(i_1 - 1, j_0 - 1) \notin R$ and has the same content as (i_1, j_0), so we know that λ has a cell labelled at $(i_0, j_0 - 1)$. Furthermore, the same argument shows that the converse is true. If a cell in λ shares only a column with a cell in R, then this shift must happen. Note that this shift can be described as follows. For every cell in λ that moves, it will shift cyclically to the right in the columns that the ribbon is present in.

So, it follows that each labelled cell in λ' except for one corresponds to a labelled cell in λ. Furthermore, we also know explicitly that the exploded shape cannot change, as the ribbon has one cell of each possible content, meaning that different columns cannot get shifted around each other. So, the only change to the p-quotient is the single block added whose arm and leg both end in R. That is, the only change is that one block is added to one of the partitions, as desired. □

To see an example of this theorem, consider the following partitions, where a 3-ribbon is added.

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & 2 & \ \\ \hline 1 & \ & \ & \ \\ \cline{1-4} 1 & \ \\ \cline{1-2} \ & \ \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline \ & \ & 2 & \ & \ \\ \hline 1 & \ & \ & \ \\ \cline{1-4} \ & \ & 2 & \ \\ \cline{1-4} 1 & \ & \ \\ \cline{1-3}\end{array}$$

This theorem inductively shows the relationship between the sizes of λ, $\mathrm{Core}_b(\lambda)$, and $\mathrm{Quot}_b(\lambda)$, as also stated in [?].

Corollary 2.7. *For a partition λ, let ν be the b-core and let $(\lambda^0, \ldots, \lambda^{b-1})$ be the b-quotient.*

$$|\lambda| = |\nu| + b \sum_{i=0}^{b-1} |\lambda^i|$$

3 Mullineux Involution

In this section, we begin with a description of Mullineux involution, as described by Mullineux in [?]. We then describe some basic results and relate them to the b-core and b-quotient. In this section, b will be fixed as before. The case where b is prime is of particular interest due to its connections with representation theory.

3.1 Defining the Involution

To define Mullineux Involution, it is first necessary to introduce the concept of a b-regular partition, as well as that of the b-edge.

Definition 3.1. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is *b-regular* if does not have b parts of the same size.

Now consider the following subset of the cells on the rim of a partition. Starting at the top right of the rim, take the first b-elements on the rim. Then, move to the right-most cell on the rim in the next row of the partition, and take the next b-elements of the rim. Continue this until the end of the rim, where in the last segment it is possible that fewer than b cells are chosen. An example of this process is shown below when $b = 3$. The number in a cell indicates that it has been picked and indicates the order in which cells are chosen.

				2	1
				3	
	1				
3	2				
1					

Definition 3.2. The algorithm described above produces the *b-edge* of a partition.

With these tools in hand, it is now possible to describe the algorithm for Mullineux involution.

Starting with λ, continue removing b-edges until the empty partition is reached, giving a sequence of partitions $\lambda = \lambda_p, \ldots, \lambda_1, \lambda_0 = \emptyset$. For each $i = 1, \ldots, p$, let a_i be the number of cells in the b-edge of λ_i and let r_i be the number of rows in the same partition. From this, we define s_i as follows.

$$s_i = \begin{cases} a_i - r_i & \text{if } b|a_i \\ a_i - r_i + 1 & \text{otherwise} \end{cases}$$

Now, starting from the empty partition, define a new sequence of partitions $\nu_0 = \emptyset, \nu_1, \ldots, \nu_p = \nu$ such that ν_i has s_i rows and its b-edge has a_i cells. Then, ν is the Mullineux involution of λ, written $\nu = M_b(\lambda)$.

Theorem 3.3. *The algorithm described above is well-defined and is an involution on the set of b-regular partitions.*

Proof. See [?]. □

Remark 3.4. As written, nothing in the algorithm for Mullineux involution suggests that it is necessary for the starting partition to be b-regular. However, it proves to be impossible to carry out the algorithm on a p-irregular partition, as one may end up getting $s_i = 0$ or some other impossible set of parameters a_i, s_i.

Remark 3.5. The motivation for this involution comes from representation theory. When b is prime, it is well known that the irreducible representations over $\mathbb{F}_b$ of S_n, the symmetric group, are indexed by the b-regular partitions. Furthermore, there is a natural action on these representations given by tensoring with the sign representation, which is nontrivial for $b > 2$. Mullineux involution is the map induced onto the p-regular partitions by this correspondence, as was proven in [**?**]

3.2 Relationship with Cores and Quotients

We will now more closely examine the relationship between Mullineux involution and its effect on the b-core and b-quotient. In the case of the b-core, the answer turns out to be surprisingly simple and was proven by Mullineux in [**?**].

Theorem 3.6. *Let λ be a partition. Then, the b-core of $M_b(\lambda)$ is the transpose of the core of λ.*

$$\mathrm{Core}_b(M_b(\lambda)) = \mathrm{Core}_b(\lambda)^t$$

With this clean relationship between the b-core and Mullineux involution, a natural question to then ask is what the map induced by this involution on the b-quotient is.

Question 3.7. Fix some b-core ν. For λ with $\mathrm{Core}_b(\lambda) = \nu$, describe the map $\mathrm{Quot}_b(\lambda) \mapsto \mathrm{Quot}_b(M_b(\lambda)^t)$. Note that by composing with the transpose, this becomes a map that fixes the b-core and is thus a bijection.

Unfortunately, this question turns out to be difficult. With the aid of a computer, this map on b-quotients was worked out for $n \leq 40$ and for different values of b. Despite this, the map is quite complicated, even in simple cases. In the case of a core and one ribbon, this map permutes the entries of the b-quotient, but this permutation is dependent on the choice of b-core. At two ribbons, it already becomes much more complicated, and there is little correlation between the structure of the b-quotient of λ and $M_b(\lambda)^t$.

One particular difficulty in answering this question is that Mullineux involution is defined only for b-regular partitions, whereas the p-quotient applies for all partitions, regardless of regularity. Moreover, we were unable to find a way to tell from a b-quotient and the b-core whether or not the corresponding partition is b-regular without first recovering the original partition. So, for the rest of the paper, we will attempt to solve this by extending Mullineux involution to apply to all partitions.

3.3 A Conjecture

In this section, we will introduce an unpublished conjecture of Professor Bezrukavnikov, which implicitly defines an extension of Mullineux involution. Before giving the conjecture, though, we will need to define several more operations on partitions.

For partition λ, μ, let $\lambda \cup \mu$ be the multiset union. Then, for an integer b, let $b\lambda = (b\lambda_1 \cdots, b\lambda_k)$ and let $b * \lambda = \lambda \cup \cdots \cdots \lambda$ (b times). P_n will be used to denote the set of partitions of n.

Conjecture 3.8. Fix some positive integer n. The fractions with denominator at most n, in simplified form, partition $[0, 1]$ into intervals. For each interval I, define two bijections $B_I^1, B_I^2 : P_n \to P_n$ and two functions $D_I^1, D_I^2 : P_n \to \mathbb{Z}$. These functions will be defined by the following properties:

1. For $I = [0, \frac{1}{n}]$, $B_I^1(\lambda) = B_I^2(\lambda)^t = \lambda$
2. Let I, I' be adjacent intervals such that $I \cap I' = \frac{a}{b}$ and I' comes after I. Then, write $B_I^1(\lambda) = \nu \cup b\mu$, where ν has no parts divisible by b. From this, set

$$B_{I'}^1(\lambda) = \nu \cup b(\mu^t),\ D_I^1(\lambda) = b\,|\mu|$$

3. For I, I' as described above, write $B_I^2(\lambda) = \nu \cup b * \mu$, where ν is b-regular. Then,

$$B_{I'}^2(\lambda) = \left(M_b(\nu) \cup b * (\mu^t)\right)^t,\ D_I^2(\lambda) = b\,|\mu|$$

Then, for all intervals I, we have $D_I^1 = D_I^2$.

In working on this conjecture, it is useful to define two auxiliary functions C_b^1, C_b^2 such that if I, I' are adjacent with common endpoint $\frac{a}{b}$, then $B_{I'}^* = C_b \circ B_I^*$. That is, in the first type of transformation, C_b^I writes a partition as the union $\nu \cup b\mu$ and returns $\nu \cup b(\mu^t)$. In this way, the conjecture becomes a question about the composition of these C_b^* functions in an order depending on the sequence of denominators in the partitioned interval $[0, 1]$.

Of particular interest is C_b^2, as for b-regular partitions, $C_b^2(\lambda) = (M_b(\lambda))^t$. Furthermore, note that $b * \mu$ is a collection of $|\mu|$ b-ribbons , and so the b-core does not change under C_b^2. With this, we see that C_b^2 extends the definition of Mullineux involution while still maintaining the b-core, thus allowing us to continue looking at the b-quotient.

4 Conjectures on C_b^2

In this section, we will explore certain properties of the C_b^2 map defined in the previous section. Unfortunately, we are not able to offer proof, though in all cases the results have been tested empirically for a wide range of partitions. The analysis will begin with general properties of the map, and will then transition into looking at more specific examples.

When applying C_b^2, one of the first noticeable things is that, for the most part, the action of C_b^2 seems to shift blocks to the bottom left of the partition away from the top right. Applying C_b^2 multiple times, this continues until eventually, a large amount of cells move back to the top right. This qualitative observation then leads to the following conjecture.

Conjecture 4.1. Let $c(\lambda), r(\lambda)$ denote the number of columns and rows in λ, respectively. Then, if λ is b-regular, then the following inequalities hold.

$$c(\lambda) \geq c(C_b^2(\lambda)), r(\lambda) \leq r(C_b^2(\lambda))$$

Remark 4.2. One might also hope that the converse of Conjecture **??** is also true. However, this is not the case, as for example when $\lambda = (8, 8, 7, 3, 2, 1, 1, 1, 1, 1, 1)$ and $b = 6$. Despite this, the converse holds very frequently, suggesting that there is perhaps some further condition on when the number of columns decreases and the number of rows increases.

As before, determining the exact action of C_b^2 is difficult in general, though for certain smaller cases the answer appears to be relatively simple. To tackle this, we will impose restrictions on the size of the b-quotient and thus how close to a b-core the partitions are. In the case where the quotient is always empty, the partitions in question are the b-cores, for which the behavior of C_b^2 is already understood. So, the remainder of the paper will focus on the case where the b-quotient is of size at most 2.

First, consider the case where there is only one ribbon. Given a fixed b-core ν, there are exactly b possible ways to add a ribbon to ν, corresponding to each entry in the b-quotient being $\square$. In this case, the simplest possible story is that C_b^2 cycles through all b possibilities. In fact, this seems to be the case.

Conjecture 4.3. When restricted to partitions λ whose quotient has size exactly 1, $(C_b^2)^n(\lambda) = \lambda$ if and only if $b|n$.

Put another way, C_b^2 breaks these partitions into cycles of length b. One can also ask what happens to partitions with 2 ribbons. That is, we can write all such partitions as the disjoint unions of cycles created by C_b^2 and attempt to characterize the resulting distribution. An example of this for $n = 40, 60$ and $b = 9$ is shown below.

Surprisingly, the lengths of the cycles all cluster in two areas, one around 5 and one around 11. Furthermore, the distributions in these two examples appear to be similar. This, then, leads to the following surprising conjecture.

Conjecture 4.4. Fix some integer $b \geq 5$. Define a sequence of probability distributions $X_{2b}, X_{2b+1}, \ldots$, where X_n is the distribution of the lengths of cycles C_b^2 induces on partitions

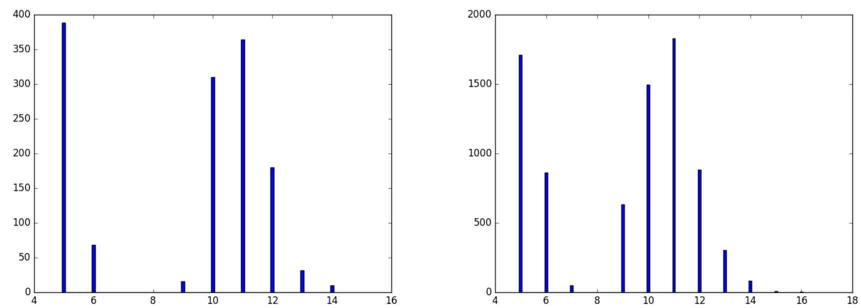


Figure 1: The number of cycles of certain length for $b = 9$. Left: $n = 40$. Right: $n = 60$

of size n with exactly 2 ribbons. Then, $\{X_n\}_{n \geq 2b}$ converges weakly to a bimodal distribution with peaks at around $0.5b$ and $1.2b$.

Remark 4.5. This conjecture has been tested for $n \leq 80$ and $b \leq 10$, above which point the code runs extremely slowly. As a result, there is not enough data to feel fully confident about the conjecture, though there is consistent behavior which seems to back it up. Furthermore, it is similarly unclear whether stronger notions of convergence can be applied, again due to the lack of data and somewhat slow convergence.

There are several other avenues of further research available. We would be very curious to see progress on any of the conjectures mentioned in this paper. In particular, better understanding of the case with at most 2 ribbons could lead to a proof of Bezrukavnikov's original conjecture on the interval $[0, \frac{3}{n})$, where each partition has at most 2 ribbons. Additionally, further exploration for more than 2 ribbons would yield more light on the distribution of orders of cycles, and whether similar results to Conjecture 4.4 generalize.

5 Acknowledgements

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References

[1] Haiman M., *Combinatorics, Symmetric Functions, and Hilbert Schemes.* https://math.berkeley.edu/ mhaiman/ftp/cdm/cdm.pdf

[2] Mullineux G. *Bijections of p-regular partitions and p-modular irreducibles of symmetric groups.* J. London Math. Soc. (2) 20 (1979) 60-66

[3] Jacon N., Lecouvey C. *On the Mullineux involution for Ariki-Koike algebras* arXiv:math.RT/0804.0478

[4] Mullineux G., *On the p-cores of p-regular diagrams.* J. London Math. Soc. (2) (1979) 222-226