AN INTEGRAL FORMULA FOR TRIPLE LINKING IN HYPERBOLIC SPACE

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ABSTRACT. We provide a geometrically natural formula for the triple linking number of 3 pairwise unlinked curves in three dimensional hyperbolic space.

CONTENTS

1. INTRODUCTION

In a three-dimensional space, two closed curves that are disjoint with each other may have nontrivial positioning relationship, in which neither of the curves can be shrunk to a point without touching the other. Such phenomenon is called linking, and two curves are linked to each other if they cannot be "pulled apart" in the above sense. C.F. Gauss (1833) once gave an integral formula for the linking number of two closed curves in the 3-dimensional Euclidean space ($\mathbb{R}^3$), which represents the number of times one curve wraps the other.

For 3 closed curves, there is an analogous number called the Milnor invariant of a triple link, developed by John Milnor (1954). Like in the 2-component case, the Milnor invariant can also be represented by an integral formula, but only in the case when the pairwise linking numbers all vanish.

The linking numbers (both for 2 curves and for 3 curves) generalize naturally to other 3-dimensional manifolds with nice topological properties, such as S^3 and H^3. D. DeTurck, et al. (2011) investigated the integral formulas for the cases of $\mathbb{R}^3$ and S^3, and in this paper, we will state an integral formula for the case of H^3, the 3-dimensional hyperbolic space, which is geometrically natural.

2. BACKGROUND

2.1. **Double Linking.** We begin with the double linking case. It is well known that two oriented closed curves in $\mathbb{R}^3$ have a unique integer that describes the linking of them, namely

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the linking number, which describes the number of times one curve wraps around the other. It can also be shown that the link-homotopy classes of all 2-component links in $\mathbb{R}^3$ are in bijective relationship with the set of all integers, via the linking number.

An interesting fact about the linking number is that it can be represented by an integral formula, as stated in the following theorem of Gauss:

Theorem 2.1. *The linking number of two curves $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$, is given by*

$$Lk(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{d\gamma_1}{ds} \times \frac{d\gamma_2}{ds} \cdot \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|^3} dsdt$$

The reason that this formula is true is due to a degree argument as follows. We first introduce the concept of the configuration space:

Definition 2.2. *For a topological space X, let the configuration space $Conf_n(X)$ denote the space of n-tuples of distinct points in X. Formally, $Conf_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^n : \forall 1 \leq i < j \leq n, x_i \neq x_j\}$.*

The configuration space has the subspace topology of X^n.

Lemma 2.3. *Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be two closed curves, then the linking number of γ_1 and γ_2 is equal to the degree of a map $G \circ e_L : T^2 \to S^2$ where $e_L = \gamma_1 \times \gamma_2 : (T^2 =) S^1 \times S^1 \to Conf_2(\mathbb{R}^3)$, and $G : Conf_2(\mathbb{R}^3) \to S^2$ is defined by*

$$G(x, y) = \frac{x - y}{|x - y|}$$

Proof. Since the linking number and the degree of a map are both homotopy invariant, we can assume without loss of generality that γ_1 is at a standard position, which is a unit circle on a horizontal plane, oriented counter-clocewise (otherwise we can apply a homotopy to place it this way). Next, we consider the vertical cylinder below this circle. It follows that the linking number of γ_1 and γ_2 can be computed by counting the signed intersection number of γ_2 with this cylinder (outwards as negative, inwards as positive). This is because the linking number of γ_1 and γ_2 is equal to the signed intersection number of γ_2 with any surface bounded by γ_1, with the orientation coherent with γ_1. Put a bottom face on the cylinder at low enough altitude, and we will get a surface bounded by γ_1, so that the intersection number of γ_2 with this cylinder is exactly the linking number of γ_1 and γ_2.

As shown in figure 1 below, the two pairs (Y_1, X_1) and (Y_2, X_2) are where the Gauss map covers $N \in S^2$. Both of them are negative, so the linking number of γ_1 and γ_2 is -2.

On the other hand, we prove that the signed intersection number of γ_2 with this cylinder is also the degree of $G \circ e_L$. Now we have put the two curves at a generic position, so that γ_2 and the cylinder below γ_1 intersect transversely. This in particular implies that the north pole N of S^2 is a regular value of $G \circ e_L$. Since $G \circ e_L(s, t) = N$ exactly when $\gamma_1(s)$ is right above $\gamma_2(t)$, the signed intersection number of γ_2 with the cylinder below γ_1 is also equal to the signed number of times that N is covered by $G \circ e_L$. By the property of the degree of a map, this is equal to the degree of $G \circ e_L$. □

We know that the degree of a map can be computed by the pull-back of a differential form: for a map $f : X \to Y$ where X, Y are manifolds of the same dimension, and ω being any differential form on Y of full dimension,

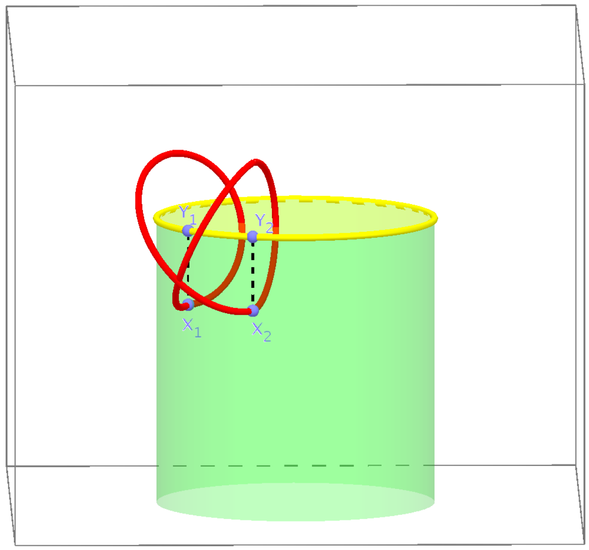

FIGURE 1

$$\deg(f) \cdot \int_Y \omega = \int_X f^* \omega,$$

Now we let $X = T^2, Y = S^2, f = G \circ e_L$, and ω being the standard area form on S^2, then by calculation, we get exactly the Gauss' linking integral.

Another thing to notice about Gauss' formula is that the Gauss map above is geometrically motivated, that is, it is equivariant under the isometries of $\mathbb{R}^3$, which means the action of the isometry group of $\mathbb{R}^3$ is commutative with the map G (if we let translations of $\mathbb{R}^3$ act trivially on S^2).

2.2. **Triple Linking.** In the case of triple linking, even if the three pairwise linking numbers are 0 (which means that if we ignore any of the three curves, the remaining two can be shrunk to constant maps), the three curves together may still form a link. The famous example for this case is the Borromean rings (figure 2):

For the triple linking case, John Milnor (1954) discussed this phenomenon in details in his paper *Link Groups*, in which he describes a link (with arbitrarily many components) in terms of the fundamental group of the complement of the link. Specifically, there is also a number called the Milnor μ-invariant of a link, which is invariant under link homotopy, and defined up to mod an integer depending on the pairwise linking numbers. However, in the case where the three pairwise linkings are trivial, the Milnor invariant is defined as a numerical integer, and DeTurck et al. (2011) found an integral formula for the Milnor μ-invariant in this case.

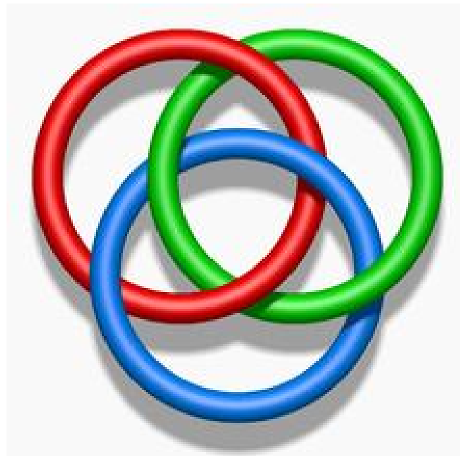

FIGURE 2

Their proof proceeds by two steps. First they proved the Milnor invariant is related to the Pontryagin invariant of a map $g_L : T^3 \to S^2$, just as the pairwise linking number is related to the degree of a map from $T^2 \to S^2$. In the case the pairwise linking numbers are 0, we have that $deg(g_L|_{T^2}) = 0$ for all the subtori $T_2 \subset T_3$, and there exists an integral formula for the Pontryagin ν-invariant of g_L:

$$\nu(g_L) = \int_{T^3} d^{-1}(g_L^*(\omega)) \wedge g_L^*(\omega) \tag{1}$$

where ω is the standard area form on S^2. Similar to the map $G \circ e_L$, their map is given as the composition of two maps

$$T^3 \to Conf_3(\mathbb{R}^3) \to S^2.$$

The first map is given by product of the three curves, and the second map is given by $F_{\mathbb{R}^3} = G_{\mathbb{R}^3}/|G_{\mathbb{R}^3}|$ where

$$G_{\mathbb{R}^3}(A,B,C) = \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{a \times b}{|a||b|} + \frac{b \times c}{|b||c|} + \frac{c \times a}{|c||a|} \tag{2}$$

$$= \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + (\sin\alpha + \sin\beta + \sin\gamma)n \tag{3}$$

(For notations, A, B, C denote three distinct points in $\mathbb{R}^3$, and a is the vector from B to C, etc. The vector n is the unit normal vector to the oriented plane determined by A, B, C, which is undefined when the three points are on the same plane.) Note that similar to the double linking case, F is again isometry-equivariant.

3. Triple Linking in Hyperbolic Space

In order to find a geometrically meaningful integral formula for triple linking in H^3, we need to find an $F_{H^3} : Conf_3(H^3) \to S^2$ which is $Isom(H^3)$ equivariant. We now construct such an F_{H^3}.

Throughout the following we will use the ball model of hyperbolic space. The advantage of the ball model is that the boundary of the space, which consists of points at infinity, is an S^2, so that the isometry group of H^3, namely $Isom(H^3)$, has a natural action on S^2 by extending to points at infinity. In particular, elements of $Isom(H^3)$ act as conformal maps on the sphere at infinity.

Our first intuition is to define F_{H^3} in the same way as in $\mathbb{R}^3$, which will inevitably give us the answer, because H^3 is diffeomorphic to $\mathbb{R}^3$, which means topologically there is nothing different between them. However, the problem is that this definition is not geometrically natural.

Next, we seek a variant of the map F_{H^3} which respects the geometry of H^3. For this intention, it is natural to consider the geodesic triangle formed by three points, because geodesics are natural in the geometry of H^3, and they are isometry-invariant (which means any isometry of H^3 takes a geodesic to another geodesic). However, we again meet a problem that we cannot compare vectors in different tangent spaces, or do any computation with them. Then we had an idea that we can parallel transport the vectors at each vertex to another point uniquely associated with them in a geometric way, and do computations at that point. Fortunately this idea gives us a meaningful formula.

The special point we found is the incenter of the geodesic triangle formed by the three points.

Theorem 3.1. *The three angle bisectors of a geodesic triangle in H^3 intersect at one point, which we call the* ***incenter*** *of the geodesic triangle. (shown in figure 3)*

Proof. This is a classical result in hyperbolic geometry. The bisector of an angle in the hyperbolic space is the locus of points with the same (hyperbolic) distance to the two sides of the angle, and the intersection point of two angle bisectors satisfies that it has the same distance to the three sides of the geodesic triangle, so that it is on the third angle bisector.

It can also be shown that the incenter lies on the totally geodesic submanifold passing through the three vertices (which, in the ball model, is the unique sphere passing through the three vertices and perpendicular to the sphere at infinity at their intersection), due to the fact that the bisector of an angle lies in the same plane as the two sides. □

One of the most important reasons that we choose the incenter of the geodesic triangle is that the incenter depends continuously on the three vertices, along with the three angle bisectors. Even if the geodesic triangle is degenerate (where the three vertices lie on the same geodesic), the incenter of this triangle makes sense, which is exactly the vertex that is between the other two. (Other points associated with the geodesic triangle do not have this property, such as the circumcenter.)

Now we are ready to give the map F_{H^3}. We define F_{H^3} through the following process.

Let I be the hyperbolic incenter of x, y and z. Let a_g denote the unit vector at y pointing towards z along the geodesic between them. (Similarly, let b_g is the unit vector at z pointing

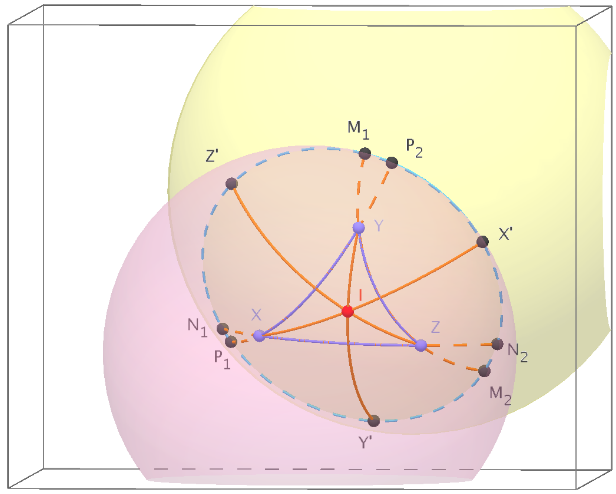

FIGURE 3

towards x, and c_g the from x to y, where the subscript g stands for geodesic.) In addition let the three angles of the geodesic triangle formed by x, y, z be $\alpha_g, \beta_g, \gamma_g$ respectively, and let n_g be the unit vector normal to the unique totally geodesic submanifold determined by x, y, z (which is a sphere perpendicular to the boundary S^2 when they are not on the same geodesic, and the direction depends on the ordering of the three vertices, using the right hand rule).

Parallel translate a_g, b_g and c_g to I along geodesics (from y to I, etc.), and call them $\tilde{a_g}, \tilde{b_g}$ and $\tilde{c_g}$ respectively.

Then, construct a vector in $T_I H^3$ (the tangent space at I), as:

$$V(x, y, z) = \tilde{a_g} + \tilde{b_g} + \tilde{c_g} + (\sin\alpha_g + \sin\beta_g + \sin\gamma_g)n_g$$

When the three vertices x, y, z are on the same geodesic, n is undefined but V is still defined since $\sin\alpha_g = \sin\beta_g = \sin\gamma = 0$. We note that V is never zero. The reason for this is that the second part of its expression, namely $(\sin\alpha_g + \sin\beta_g + \sin\gamma_g)n_g$ is nonzero whenever x, y, z are not on the same geodesic, while the first part, namely $\tilde{a_g} + \tilde{b_g} + \tilde{c_g}$ is nonzero when x, y, z are distinct points on the same geodesic. Noting that the two parts are perpendicular to each other, since $\tilde{a_g} + \tilde{b_g} + \tilde{c_g}$ is tangent to the totally geodesic submanifold.

Finally, we draw a geodesic ray from I in the direction of V

$$F_{H^3}(x, y, z) = \lim_{t\to\infty} exp_I(tV) \tag{4}$$

so that $F_{H^3}(x, y, z)$ is a point on the sphere at infinity. (shown in figure 4 below)

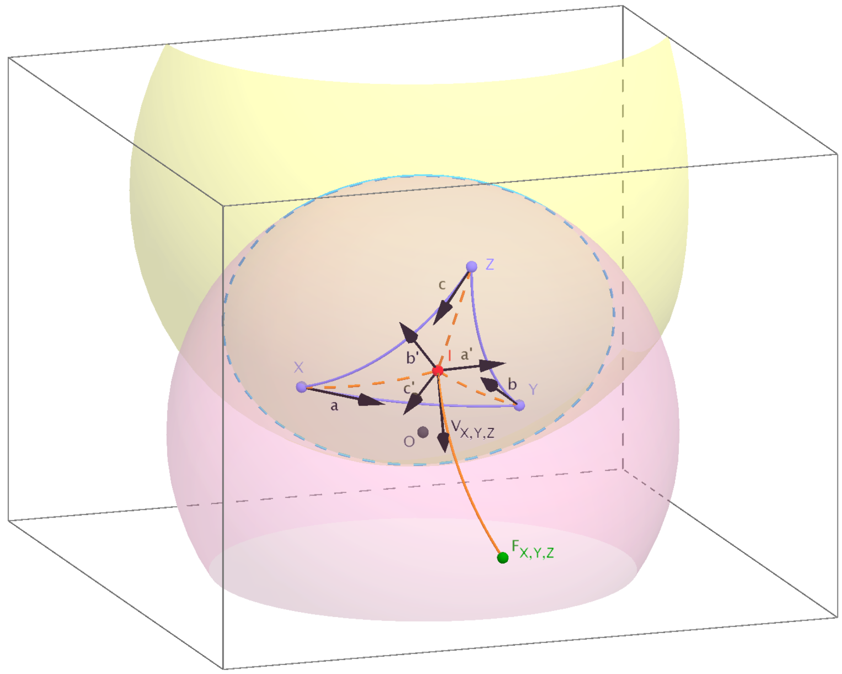

FIGURE 4

Then we have the following:

Theorem 3.2. *With F_{H^3} defined by Equation 4, and a link L with characteristic map $e_L : T^3 \to Conf_3(H^3)$ such that the pairwise linking numbers are 0,*

$$\mu(L) = \frac{1}{2}\nu(F_{H^3} \circ e_L).$$

Proof. The point of the proof is that the map F_{H^3} is related to map F_{H^3} in the Euclidean case (when the three vertices are considered to be points in the unit sphere in $\mathbb{R}^3$). In particular, they are homotopic.

There is an explicit homotopy between them. Let

$$f_t(x, y, z) = F_{H^3}(tx, ty, tz), t \in (0, 1] \tag{5}$$

We note that the incenter of tx, ty, tz depend continuously on t, and V also depends continuously on the three vertices (hence it depends continuously on t). Therefore, f_t is a homotopy.

When $t = 1$, $f_1(x, y, z) = F_{H^3}(x, y, z)$.

When $t \to 0^+$, we have $\lim_{t\to 0^+} f_t(x, y, z) = F_{\mathbb{R}^3}(x, y, z)$.

The intuition of this statement is that near the origin, the hyperbolic space H^3 is isometric to the Euclidean space up to the first order. This means that as $t \to 0^+$, $F_{H^3}(tx, ty, tz)$ and $F_{\mathbb{R}^3}(tx, ty, tz)$ are very close to each other. On the other hand, since G is scale invariant, we have $F_{\mathbb{R}^3}(tx, ty, tz) = F_{H^3}(x, y, z)$, so that $\lim_{t\to 0^+} f_t(x, y, z) = F_{\mathbb{R}^3}(x, y, z)$.

We will prove this statement in the next lemma, and now we assume this is true. Then with the homotopy relationship between the two maps, we are also aware that the Pontryagin ν-invariant is fixed by homotopy, and by definition the Milnor invariant of a link in H^3 is the same as the Milnor invariant of this link when it is considered to be inside the unit ball in $\mathbb{R}^3$. Therefore, we have $\mu(L) = \frac{1}{2}\nu(F_{\mathbb{R}^3} \circ e_L) = \frac{1}{2}\nu(F_{H^3} \circ e_L)$, as stated in the theorem.

□

Lemma 3.3. *With the notations above, we have*

$$\lim_{t \to 0^+} f_t(x, y, z) = G(x, y, z)$$

for all $(x, y, z) \in Conf_3(H^3)$.

Although this lemma is straightforward to understand and one can easily be convinced, to explain it very clearly is harder than expected. In order to prove it, however, we need to make use of the properties of the ball model (in the Euclidean space). Throughout the following, a tangent vector

Lemma 3.4. *For a point X in the unit ball $D^3 \subset \mathbb{R}^3$, if a sphere P passes through X and is perpendicular to the unit sphere $S^2 = \partial D^3$, then the radius of the sphere P is larger than $(\frac{1}{2|X|} - 1)$ (where $|X|$ is the length of X, or the Euclidean distance from X to the origin).*

Proof. Note that P is perpendicular to S^2. Let C be its center, and r be its radius, then necessarily $|C|^2 = r^2 + 1$. This implies that the shortest distance from the origin to the sphere P is $|C| - r = \frac{1}{r+\sqrt{r^2+1}} > \frac{1}{2(r+1)}$. The distance from X to the origin must be greater than this, so that $|X| > \frac{1}{2(r+1)}$, or $r > \frac{1}{2|X|} - 1$. □

Lemma 3.5. *In H^3, if a vector v is parallel transported along a geodesic Γ from x_1 to x_2 to a vector v' ($x_1, x_2 \in \Gamma$), then the angle (in the ball model) between v and v' does not exceed the angle between γ and γ'.*

Proof. Actually we can prove a stronger statement, which completely characterizes the parallel transport along geodesics in H^3.

The parallel transport in H^3 (with the ball model) along geodesics goes as follows: let $X, Y \in H^3, v \in T_X H^3$, and let $P_{X,Y}(v)$ be the vector obtained by parallel transporting the vector v along the geodesic from X to Y, and let C be the center of the geodesic arc connecting X and Y, then rotating by the angle $\angle XCY$ induces an isometry of $\mathbb{R}^3$, hence acting on vectors in $\mathbb{R}^3$. The vector $P_{X,Y}(v)$ can be obtained by rotating with angle $\angle XCY$, and then scaling properly so that $P_{X,Y}(v)$ has the same length as v in respective metrics (because parallel transport preserves metric).

Let's see figure 5 below. We know that the statement above is true when v is the tangent vector of the geodesic from X to Y, because this is the definition of a geodesic (whose tangent vectors remain parallel along itself). Now consider the vector $v \in T_{H^3}X$ being the unit vector perpendicular to γ inside the plane XCY (which is also a plane passing through the origin), pointing $\overrightarrow{CX}$. Since parallel transport preserves metric, we know that $P_{X,Y}(v)$ must be perpendicular to γ'. On the other hand, by symmetry $P_{X,Y}(v)$ must also lie inside the plane XCY (otherwise reflecting with respect to this plane is an isometry that does not preserve the parallel transport), by preservation of length $P_{X,Y}(v)$ is a unit vector, and by continuity it points in the direction $\overrightarrow{CY}$. Therefore, we can see that $P_{X,Y}(v)$ is exactly the vector determined as stated above.

Finally, let u be the unit vector at X which is perpendicular to both γ and v, and such that (γ, v, u) form a positively oriented triple. Then $P_{X,Y}(u)$ is orthogonal to the plane XCY, and by continuity, the vector $P_{X,Y}(u)$ points to the same side of the plane XCY as u. Therefore, the parallel transport of u also follows the rule above. Since (γ, v, u) is a basis of $T_X H^3$, and parallel transport is linear, other vectors also follow the rule above. In particular, this means that angle between a vector and its translated vector does not exceed the rotation angle of the transform, which is exactly the angle between γ and γ'. $\square$

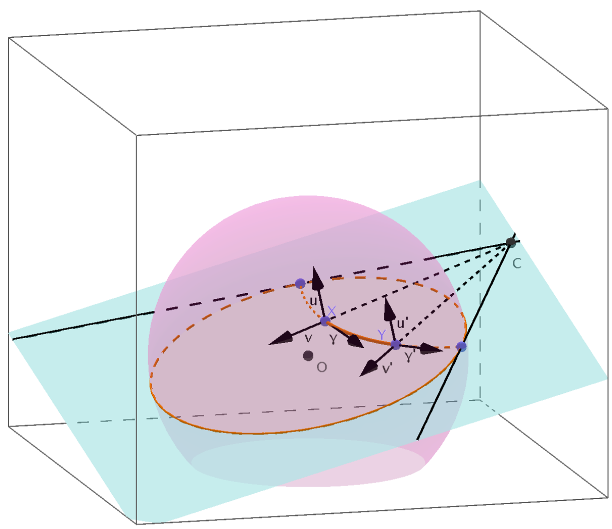

FIGURE 5

Proof of lemma 3.3. Let us consider the difference between $V(tx, ty, tz)$ and $G(tx, ty, tz)$.

First, recall that the vector $\frac{a}{|a|}$ is the unit vector from ty to tz, while the vector a_g is the unit vector from ty to tz along the geodesic from ty to tz. Suppose that the radius of the geodesic from ty to tz is r, then the angle between $\frac{a}{|a|}$ and a_g is equal to $\arcsin(\frac{|y-z|}{2r})$. According to lemma 3.4, it tends to 0 as $t \to 0^+$. On the other hand, the angle between a_g and $\tilde{a_g}$ does not exceed the angle between the tangent vectors of the geodesic from x to I at respective points, which is equal to $\arcsin(\frac{|x-I|}{2R})$, where R is the radius of the geodesic from x to I. Since this angle also tends to 0 by lemma 3.4, we know that as $t \to 0^+$, the difference between $\frac{a}{|a|}$ and $\tilde{a_g}$ tends to 0. The argument is similar with b and c.

For the second part, note that as $t \to 0^+$, the angles of the geodesic triangle (tx, ty, tz) also tend to respective angles of the Euclidean triangle (x, y, z). This means that $(\sin \alpha_g +$

$\sin\beta_g + \sin\gamma_g$) also tends to $(\sin\alpha + \sin\beta + \sin\gamma)$. Note that the radius of the totally geodesic submanifold (a sphere) passing through (tx, ty, tz) tends to infinity, so that the unit normal vectors do not vary too much on the sphere when t is small, and we have $\lim_{t\to 0^+} n_g = n$, so that the second part of $V(tx, ty, tz)$ tends to the second part of $G(tx, ty, tz)$, hence $\lim_{t\to 0^+}(V(tx, ty, tz) - G(x, y, z)) = 0$.

Finally, as $t \to 0^+$, the distance from I to the origin (i.e. $|I|$) tends to 0, so that the geodesic from I in the direction of V has a large radius for a small t be lemma 3.4, so that the angle between $V(tx, ty, tz)$ and $F(tx, ty, tz)$ tends to 0 as t tends to 0.

Overall, we have $\lim_{t\to 0^+} F(tx, ty, tz) = G(x, y, z)$.

□

Combining theorem 3.2 with equation 1, we can now give the formula for the Milnor invariant of a pairwise-unlinked triple linking in H^3:

Theorem 3.6. *With usual notations, we have*

$$\mu(L) = \frac{1}{2}\int_{T^3} d^{-1}(g_L^*\omega) \wedge g_L^*\omega$$

where $g_L = F \circ e_L$ is given by equation 4.

Remark 3.7. *In their paper, DeTurck et al. used analytic tools (such as the fundamental solution of the Laplacian on the torus) to find the exterior antiderivative of $g_L^*\omega$, and hence giving a more explicit formula. In this paper, we simply state the integral in a descriptive form because it is clearer for readers to understand where this formula comes from.*

References

[1] Brody, N. & Schettler, J. Rational Hyperbolic Triangles and a Quartic Model of Elliptic Curves. arXiv:1406.0467v3.

[2] Cannon, J.W. et al. Hyperbolic Geometry. *Flavors of Geometry*, MSRI Publications, Vol. 31, 1997.

[3] DeTurck, D. et al. Pontryagin invariants and integral formulas for Milnor's triple linking number. arXiv:1101.3374v1.

[4] Milnor, J. Link Groups, *Annals of Mathematics*, Vol 59 No 2 (1954) 177-195.

[5] Pohl, W.F. The Self-Linking Number of a Closed Space Curve. *J. Math. Mech.* 17, 975-985, 1968.

[6] Ricca, R.L. & Nipoti, B. Gauss' Linking Number Revisited. *Journal of Knot Theory and Its Ramifications*, Vol. 20, No. 10 (2011) 1325-1343.

[7] Shonkwiler, C. The Gauss Linking Integral in S^3 and H^3.
Retrieved from http://www.math.colostate.edu/∼clayton/research/talks/linkingintegrals.pdf.