Classifying bipolynomial Hopf algebras over graded local rings

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Abstract

Hopf algebras have been a very important area of research for much of the past century, with people observing and studying such structures in a wide range of fields. Ravenel and Wilson proved that certain bipolynomial Hopf algebras are isomorphic to the Witt Hopf algebra W_R , but only when the underlying rings are $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$. We generalize this isomorphism over graded local rings, which creates new possibilities in algebraic topology and other areas of mathematics.

Summary

A Hopf algebra is a complicated algebraic structure that occurs in many different areas of mathematics. We build on previous research to show how different types of Hopf algebras share the same structure for a wider range of conditions. This allows us to simplify and regularise our work by considering more well-studied Hopf algebras rather than less wellunderstood Hopf algebras, which has implications in different fields of current research.

1 Introduction

Hopf algebras are a type of algebraic structure with applications in different fields of mathematics, such as quantum groups, algebraic geometry and algebraic topology. A Hopf algebra is simultaneously an algebra and a coalgebra [1], which gives them many interesting relationship properties, such as when dualising.

A Hopf algebra H is considered to be bipolynomial if both it and its dual are polynomial algebras. Ravenel and Wilson [2] proved that any bipolynomial Hopf algebra H is isomorphic to the tensor product of Witt Hopf algebras W_R (see Definition 2.5) over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$ for prime p, where $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } \gcd(b, p) = 1\}$ is the ring of integers localised at p.

Theorem 1.1 (Ravenel and Wilson [2]). For a graded bicommutative Hopf algebra H over the ring $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{F}_p$, if there are algebra isomorphisms $H \cong R[x_0, x_1, x_2, ...]$ and $H^* \cong R[y_0, y_1, y_2, ...]$, where the polynomial algebras $R[x_0, x_1, x_2, ...]$ and $R[y_0, y_1, y_2, ...]$ have generators x_i, y_i with deg $x_i = p^i$ and deg $y_i = -p^i$, then $H \cong W_R$.

The above theorem is a special case of Ravenel and Wilson's theorem, but their proof does not immediately generalise if R is a graded local ring (see Definition 2.1), because the dual H^* does not behave well. Over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$, there is a Hopf algebra isomorphism (see Lemma 5.1) between the dual $R[x]^*$ of the Hopf algebra R[x] and the divided power Hopf algebra $\Gamma_R[x^*]$ (see Example 2.1) generated by the dual x^* , but $R[x]^* \cong \Gamma_R[x^*]$ does not hold over graded local rings R in general.

Theorem 1.2. For a graded bicommutative Hopf algebra H over a graded local ring R, if $H \cong R[x_0, x_1, x_2, \ldots]$ as algebras and $H \cong \Gamma_R[z_0, z_1, z_2, \ldots]$ as coalgebras, where generators x_i, z_i have deg $x_i = \text{deg } z_i = p^i$, then $H \cong W_R$.

We generalise Theorem 1.1 to Theorem 1.2 over graded local rings R, such as $R = \mathbb{Z}_{(p)}[u]$ and $R = \mathbb{F}_p[u]$ where deg u = 1. Rather than consider the algebra isomorphism $H^* \cong R[y_0, y_1, y_2, \ldots]$ as Ravenel and Wilson did, we instead consider the coalgebra isomorphism $H \cong \Gamma_R[z_0, z_1, z_2, \ldots]$ over graded local rings R. By using induction on different degrees, we show that generators of the same degree in H and W_R map to each other, leading to the Hopf algebra isomorphism $H \cong W_R$.

2 Preliminaries

We define some mathematical terminology involving structures such as algebras, coalgebras and dual spaces. Throughout this paper, we consider only graded structures, and the graded local ring R is assumed to be connected.

Definition 2.1 (Graded local ring). A graded local ring R can be decomposed into the direct sum $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ where $R_0 = \mathbb{Z}_{(p)}$ or $R_0 = \mathbb{F}_p$ and $R_i R_j \subseteq R_{i+j}$.

Let M^* be the dual space of the *R*-module *M*.

2.1 Hopf algebras

Definition 2.2 (Algebras [1]). We refer to unital associative algebras as algebras. An algebra A is an R-module with multiplication $\mu : A \otimes A \to A$ and unit $\eta : R \to A$ that satisfy the commutative diagrams:

$$\begin{array}{cccc} A \otimes A \otimes A \xrightarrow{\mu \otimes \mathrm{id}} A \otimes A & & \\ & & \downarrow^{\mathrm{id} \otimes \mu} & \downarrow^{\mu} & \text{and} & \\ & & A \otimes A \xrightarrow{\mu} & A & \end{array} \qquad A \otimes R \xrightarrow{\mathrm{id} \otimes \eta} A \otimes A \xleftarrow{\eta \otimes \mathrm{id}} R \otimes A \\ & & \downarrow^{\mu} & \downarrow^{\mu} & \\ & & & \downarrow^{\mu} & \cong & A & \end{array}$$

An augmented algebra A has a morphism of algebras (the counit) $\varepsilon : A \to R$, and the augmentation ideal I is the kernel of ε .

The indecomposables are the elements of the quotient space I/I^2 .

Definition 2.3 (Coalgebras [1]). A coalgebra C is an R-module with comultiplication ϕ : $C \to C \otimes C$ and counit $\varepsilon : C \to R$ that satisfy the commutative diagrams:

$$C \xrightarrow{\phi} C \otimes C$$

$$\downarrow^{\phi} \qquad \downarrow^{\mathrm{id} \otimes \phi} \qquad \mathrm{and} \qquad \overbrace{C \otimes C}^{\cong} C \xrightarrow{\varphi} \downarrow^{\phi} \xrightarrow{\cong} C$$

$$C \otimes C \xrightarrow{\phi \otimes \mathrm{id}} C \otimes C \otimes C \qquad C \otimes R \xleftarrow{id \otimes \varepsilon} C \otimes C \xrightarrow{\varphi \otimes \mathrm{id}} R \otimes C$$

An augmented coalgebra C has a morphism of coalgebras (the unit) $\eta : R \to C$. The primitives are the elements of the set $\{h \in H \mid \phi(h) = h \otimes 1 + 1 \otimes h\}$.

Definition 2.4 (Hopf algebra [1]). A Hopf algebra H is defined as a R-module that is both an algebra and a coalgebra.

Example 2.1 (Divided power Hopf algebra [3]). The divided power Hopf algebra $\Gamma_R[x]$ has a basis $\gamma_k(x)$ where $\gamma_0(x) = 1$, $\gamma_i(x) = \frac{x^i}{i!}$ and $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$, with comultiplication $\phi(\gamma_i(x)) = \sum_{j=0}^i \gamma_j(x)\gamma_{i-j}(x)$.

In morphisms between Hopf algebras, primitives map to primitives linearly and indecomposables map to indecomposables linearly [3]. We can use such maps to determine morphisms between Hopf algebras, as shown in the following lemma.

Lemma 2.1. For a coalgebra C, an R-module map $g : C \to R\{x_0, x_1, x_2, ...\}$ uniquely determines a coalgebra map $G : C \to \Gamma_R[x_0, x_1, x_2, ...]$.

Proof. Consider any element $c \in C$. The comultiplication ϕ is coassociative, so $\phi^{n-1}(c) = \sum_{i=1}^{m} c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$ where $c_{1i}, c_{2i}, \ldots, c_{ni} \in C$ for positive indices *i* up to non-negative integer *m*, and

$$g(\phi^{n-1}(c)) = g\left(\sum_{i=1}^{m} \bigotimes_{j=1}^{n} c_{ji}\right) = \sum_{i=1}^{m} \bigotimes_{j=1}^{n} g(c_{ji}).$$

For a tensor product $c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$, if any $g(c_{ji})$ is non-linear, then the tensor product is considered degenerate as it does not contribute towards G(c).

Let $g_n(c) = \sum \prod g(c_{ji})$ for non-degenerate tensor products $c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$ be the element(s) in the divided power coalgebra $\Gamma_R[x_0, x_1, x_2, \ldots]$ induced by $g(\phi^{n-1}(c))$. We write the comultiplication $\phi(c)$ as $\phi(c) = c \otimes 1 + 1 \otimes c + \sum c' \otimes c''$. Since

$$G(\phi(c)) = G(c \otimes 1 + 1 \otimes c) + G\left(\sum c' \otimes c''\right)$$

= $(g_1(c) + g_2(c) + \cdots) \otimes 1 + 1 \otimes (g_1(c) + g_2(c) + \cdots)$
+ $g_1\left(\sum c' \otimes c''\right) + g_2\left(\sum c' \otimes c''\right) + \cdots,$

and

$$\phi(G(c)) = \phi(g_1(c) + g_2(c) + \cdots + g_n(c)) = \phi(g_1(c)) + \phi(g_2(c)) + \cdots$$

= $\phi(g_1(c) \otimes 1 + 1 \otimes g_1(c)) + \phi(g_2(c) \otimes 1 + 1 \otimes g_2(c) + \cdots) + \cdots ,$

we prove that $G(\phi(c)) = \phi(G(c))$ by rearranging. Thus, G is a coalgebra map.

2.2 Witt Hopf algebras

Definition 2.5 (Witt Hopf algebra). The Witt Hopf algebra $W_{Z_{(p)}}$ has generators y_i of deg $y_i = p^i$, with comultiplication $\phi(z_i) = z_i \otimes 1 + 1 \otimes z_i$ for primitives

$$z_{0} = y_{0},$$

$$z_{1} = py_{1} + y_{0}^{p},$$

$$z_{2} = p^{2}y_{2} + py_{1}^{p} + y_{0}^{p^{2}},$$

$$\vdots$$

$$z_{i} = \sum_{j=0}^{i} p^{j}y_{j}^{p^{p-j}},$$

and $W_R = W_{Z_{(p)}} \otimes R$ for the ring R.

Let $W_R(n)$ be the sub-Hopf algebra with generators $y_0, y_1, y_2, \ldots, y_n$.

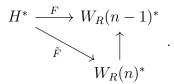
A basis of $W_R(n)$ is the set of monomials $y_0^{i_0}y_1^{i_1}\cdots y_n^{i_n}$ for non-negative exponents i_j , so that there are inclusion maps $W_R(0) \to W_R(1) \to W_R(2) \to \cdots$, and their dual maps $W_R(0)^* \to W_R(1)^* \to W_R(2)^* \to \cdots$ are all onto.

For the graded local ring R, its maximal ideal is $I_m = (p) \oplus R_1 \oplus R_2 \oplus R_3 \oplus \cdots$ where (p) represents the multiples of p. Thus, if deg r > 0 for some element $r \in R$, then $r \in I_m$. According to Nakayama's Lemma (see Lemma 5.2), we construct the quotient ring $R/I_m = \mathbb{F}_p$ to help prove the Hopf algebra isomorphism $H \cong W_R$ over the graded local ring R.

3 Proof of Theorem 1.2

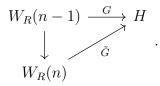
3.1 Key Lemma

Ravenel and Wilson [2] proved the following lemma over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$ for the algebra $H = R[x_0, x_1, x_2, \dots]$ where generators x_i have deg $x_i = p^i$. **Lemma 3.1** (Ravenel and Wilson [2]). Given an surjective algebra map $F : H^* \to W_R(n-1)^*$ in degrees $\leq p^n$, there is an algebra map $\tilde{F} : H^* \to W_R(n)^*$ that is isomorphic in degrees $\leq p^n$, such that the following diagram commutes:



However, because the duals of Hopf algebras do not behave well over graded rings R, we avoid taking duals by using coalgebra maps to prove the following lemma for the coalgebra $H = \Gamma_R[x_0, x_1, x_2, ...]$ where generators x_i have deg $x_i = p^i$.

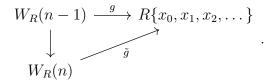
Lemma 3.2. Given a coalgebra map $G: W_R(n-1) \to H$ that is injective in degrees $\leq p^n \mod I_m$, we can find a coalgebra map $\tilde{G}: W_R(n) \to H$ that is isomorphic in degrees $\leq p^n \mod I_m$, such that the following diagram commutes:



Proof. For $y \in W_R(n-1)$, we let $G(y) = \sum a_i x_0^{i_0} x_1^{i_1} x_2^{i_2} \cdots$ where $a_i \in R$ and i_j are nonnegative integers for each $j = 0, 1, 2, \ldots$. We construct a linear map $g : W_R(n-1) \rightarrow R\{x_0, x_1, x_2, \ldots\}$ by taking the linear terms of G(y) such that $i_0 + i_1 + i_2 + \cdots = 1$ and letting $g(y) = b_0 x_0 + b_1 x_1 + b_2 x_2 + \cdots$ for $b_i \in R$. As $W_R(n-1)$ is generated by $y_0, y_1, \ldots, y_{n-1}$ while $W_R(n)$ is generated by y_0, y_1, \ldots, y_n , we obtain a linear map $\tilde{g} : W_R(n) \rightarrow R\{x_0, x_1, x_2, \ldots\}$ by letting

$$\tilde{g}(y_0^{i_0}y_1^{i_1}\cdots y_n^{i_n}) = \begin{cases} g(y_0^{i_0}y_1^{i_1}\cdots y_n^{i_n}) & i_n = 0, \\ 0 & i_n > 0. \end{cases}$$

Thus,



Due to Lemma 2.1, we can define the coalgebra map $\tilde{G} : W_R(n) \to H$ from the linear map $\tilde{g} : W_R(n) \to H$. As G is injective, \tilde{G} is isomorphic in degrees $< p^n \mod I_m$. To prove the isomorphism in degree $p^n \mod I_m$, we need to show that y_n maps to $\gamma_{p^n}(x_0)$ while $p^{n-1 \text{ times}}$

other y_i do not map to it. However, rather than directly computing $x_0 \otimes x_0 \otimes \cdots \otimes x_0$ from the comultiplication $\phi^{p^n-1}(y_n)$, we can use a Hopf algebra mapping from the divided power coalgebra H to the symmetric polynomials to simplify our proof.

Definition 3.1 (Hopf algebra of symmetric polynomials [3]). The Hopf algebra S of symmetric polynomials in s_1, s_2, s_3, \ldots is $S = R[\sigma_1, \sigma_2, \sigma_3, \ldots]$ where generators σ_i are the elementary symmetric polynomials

$$\sigma_1 = \sum s_i,$$

$$\sigma_2 = \sum_{1 \le i < j} s_i s_j,$$

$$\vdots$$

$$\sigma_n = \sum_{1 \le k_1 < k_2 < \dots < k_n} s_{k_1} s_{k_2} \cdots s_{k_n},$$

with comultiplication $\phi(\sigma_i) = \sum_{j=0}^{i} \sigma_j \otimes \sigma_{i-j}$.

By Newton's identities, the *i*-th power sum symmetric polynomials $c_i = \sum_{j=1,2,3,\dots} x_j^i$ have comultiplication $\phi(c_i) = c_i \otimes 1 + 1 \otimes c_i$, so c_i are primitives.

According to Husemoller [3], the injective map $K: W_R \to S$ maps the primitives $z_i \in W_R$ to the primitives $c_{p^i} \in S$, so the degrees of elements in W_R and the degrees of elements in S agree with each other.

The primitives of W_R are $z_i = y_0^{p^i} + py_1^{p^{i-1}} + \cdots + p^i y_i$, and by Newton's identities, K sends y_n to $\sigma_{p^n} + h(\sigma_1, \sigma_2, \ldots, \sigma_{p^n-1})$, where the polynomial h is composed of monomials of degree p^n . Because comultiplication is preserved, we can apply ϕ^{p^n-1} for the elements y_n and $\sigma_1, \ldots, \sigma_{p^n-1}, \sigma_{p^n}$ on both sides:

$$\phi^{p^n-1}(\sigma_1) = \sigma_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \sigma_1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \sigma_1,$$

$$\vdots$$

$$\phi^{p^n-1}(\sigma_{p^n-1}) = 1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 + \sigma_1 \otimes 1 \otimes \cdots \otimes \sigma_1 + \cdots + \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes 1 + q_{p^n-1}(\sigma_1, \sigma_2, \dots, \sigma_{p^n-1}),$$

$$\phi^{p^n-1}(\sigma_n) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 + q_{p^n}(\sigma_1, \sigma_2, \dots, \sigma_{p^n}),$$

where $q_i(\sigma_1, \sigma_2, \ldots, \sigma_i)$ are degenerate tensors of degree *i*.

For a monomial $\sigma_1^{q_1} \sigma_2^{q_2} \cdots \sigma_{p^n-1}^{q_{p^n-1}}$ in the polynomial h, consider its comultiplication

$$\phi^{p^{n-1}}(\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_{p^{n-1}}^{q_{p^n-1}}) = (\phi^{p^{n-1}}(\sigma_1))^{q_1}(\phi^{p^{n-1}}(\sigma_2))^{q_2}\cdots(\phi^{p^{n-1}}(\sigma_{p^{n-1}}))^{q_{p^{n-1}}}$$

Note the cyclic structure of each comultiplication $\phi^{p^n-1}(\sigma_i)$, which is due to the sum of tensor products being invariant by permutation over σ_i . Only the comultiplication $\phi^{p^n-1}(\sigma_{p^n})$ con-

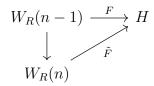
tains a single linear term $\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1$, while all other $\phi^{p^n-1}(\sigma_i)$ have permutations of tensor products that are multiples of p. Thus, in the comultiplication $\phi^{p^n-1}(\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_{p^n-1}^{q_{p^n-1}})$,

the coefficient of the linear term $\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1$ is a multiple of p because of the cyclic structure of the comultiplication for each σ_i .

The sum of the coefficients of $\overbrace{\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1}^{p^n-1 \text{ times}}$ in $\phi^{p^n-1}(K(x_n))$ is indivisible by the prime p, so y_n is the only term among all y_i which maps to σ_n . Thus, $\tilde{G}(x_n)$ contains $\gamma_{p^n}(x_0)$ with an unit coefficient, so G is a coalgebra isomorphism in degree p^n . \square

3.2 Induction on $W_R(n-1) \rightarrow W_R(n)$

Lemma 3.3. Given an Hopf algebra surjection $F: W_R(n-1) \to H$, there is a Hopf algebra surjection $\tilde{F}: W_R(n) \to H$, such that the following diagram commutes:



Proof. The Witt Hopf algebra $W_R(n-1)$ has generators $y_0, y_1, y_2, \ldots, y_{n-1}$, and we let $\tilde{F}(y_i) = F(y_i)$ for indices $0 \leq i < n$. Also, the generator $y_n \in W_R(n)$ maps to $\tilde{G}(y_n)$ as a coalgebra, and $\tilde{G}(y_n) = ce_n + f(e_0, e_1, e_2, \dots, e_{n-1})$ for some unit $c \in R$. Thus, $\tilde{F}(y_n) = \tilde{G}(y_n)$ is an algebra isomorphism in degrees $< p^n$, so \tilde{F} is a Hopf algebra surjection.

Conclusion 4

We generalised the first part of Ravenel and Wilson's proof of a Hopf algebra isomorphism between bipolynomial Hopf algebras H whose generators have degrees of prime powers and the Witt Hopf algebra W_R over graded local rings R, which has applications in algebraic topology and other fields of mathematics. A potential path of future research would be to follow through on the second part of Ravenel and Wilson's proof to show that any bipolynomial Hopf algebra H is isomorphic to the tensor product of Witt Hopf algebras W_R over graded local rings R.

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Appendix

Lemma 5.1. The dual of the bipolynomial Hopf algebra R[x] is the divided power Hopf algebra $\Gamma_R[x^*]$ for rings $R = \mathbb{Z}_{(p)}$ and \mathbb{F}_p .

Proof. By definition of duals, $x^*(x) = 1, (x^2)^*(x^2) = 1, (x^3)^*(x^3) = 1, \dots, (x^n)^*(x^n) = 1.$

To express the dual $(x^n)^*$, consider the comultiplication $\phi^{n-1}(x^n) = \sum x_1 \otimes x_2 \otimes \cdots \otimes x_n$.

For a tensor product $x_1 \otimes x_2 \otimes \cdots \otimes x_n$, if any x_i is non-linear, then the tensor product is considered degenerate. Thus,

$$\begin{split} \phi(x^2) &= 2 x \otimes x + \text{degenerate terms } x^2 \otimes 1 + 1 \otimes x^2, \\ \phi^2(x^3) &= 6 x \otimes x \otimes x + \text{degenerate terms,} \\ \vdots \\ \phi^{n-1}(x^n) &= n! \underbrace{x \otimes x \otimes \cdots \otimes x}_{n-1 \text{ times}} + \text{degenerate terms,} \\ \text{so } (x^2)^* &= \frac{(x^*)^2}{2}, (x^3)^* = \frac{(x^*)^3}{6}, \dots, (x^n)^* = \frac{(x^*)^n}{n!}. \end{split}$$

However, this dualisation does not hold for graded local rings R in general. Because of the following lemma, we mod by I_m so that the quotient space $R/I_m = \mathbb{F}_p$ is concentrated in degree 0, leading to better behavior than over R.

Lemma 5.2 (Nakayama's Lemma). If there is an isomorphism between finite free *R*-modules X and Y mod I_m , then there is an isomorphism between X and Y.