A Variation of the Erdős-Turán Conjecture and the Minimality of Multiplicative Bases

Benjamin Kang
Under the Direction of Yichi Zhang
Massachusetts Institute of Technology

Research Science Institute
July 30, 2019
Abstract

One problem that has been unsolved for nearly a century is the Erdős-Turán Conjecture, an important problem in additive number theory. It states that the efficiency of additive bases of order two of positive integers is always infinite. In this paper, we work towards a solution to the multiplicative analog of this problem. First, we prove that the Erdős-Turán Conjecture implies its multiplicative analog. Then, we introduce the density of a set and prove that finite efficiency is only possible if the basis has density zero. Furthermore, we provide examples of bases and calculate their densities, where one of our bases has density zero. Lastly, we consider a partially ordered set of the bases, and we consider which bases are minimal elements in this set. If a basis is not minimal, we attempt to discover the smallest subset which remains a basis along with its density.

Summary

In this paper, we explore a variation of a famous conjecture in additive number theory proposed by Erdős and Turán. We explore the relationship between the Erdős-Turán Conjecture and its multiplicative analog, the variation we consider. We define a basis to be a set of fundamental building blocks for the set of positive integers and its density to be the fraction of positive integers it contains. Then we prove a relation between the density and another property of a multiplicative basis. Lastly we introduce some examples of bases, calculate their densities, and determine whether or not they are minimal. A basis is minimal if we cannot remove building blocks while preserving all properties. If the basis is not minimal, we attempt to discover the smallest set of building blocks and calculate the new density.
1 Introduction

One interesting conjecture that has been extensively studied in the past century is the Erdős-Turán Conjecture, an additive number theory problem. This problem was motivated by Lagrange’s Theorem, which states that all positive integers can be expressed as the sum of at most four perfect squares, and Vinogradov’s Theorem, which states that all sufficiently large odd integers can be expressed as the sum of at most three primes. This leads to the question of whether these sets are efficient at representing positive integers as sums of their elements.

Let $A$ be a subset of the positive integers. For each positive integer $n$, consider the number of times $n$ can be written as the sum of two not necessarily distinct elements of $A$. The conjecture states that if this number is greater than 0 for all sufficiently large $n$, then it approaches infinity as $n$ approaches infinity.

Any set $A$ so that every sufficiently large integer may be written as a sum of at most $k$ elements of $A$ is called an additive basis of order $k$. Some examples include the set of odd numbers, which has order 2, the set of perfect squares, which has order 4 by Lagrange’s Theorem, and the set of primes, which has order 3 if Goldbach’s Conjecture is true.

We consider a variation of the Erdős-Turán Conjecture. Rather than use an additive basis of order 2, we use a multiplicative basis of order 2. This means that each positive integer except those in a finite set can be expressed as the product of two not necessarily distinct elements of a subset $A$. We conjecture that if $A$ is a multiplicative basis of order 2, then the number of times a number $n$ can be expressed as a product of two elements of $A$ approaches infinity as $n$ approaches infinity.

First, in Section 2, we define terms relating to the Erdős-Turán Conjecture and its multiplicative analog. In Section 3, we prove that the Erdős-Turán Conjecture implies its multiplicative analog. In Section 4, we first prove a major component of our conjecture relating
to the densities of bases. Multiplicative bases have been studied much less rigorously than additive bases, so we then investigate a few properties. We construct some examples of bases and calculate their density. Lastly, in Section 5, we consider the set of all bases as a partially ordered set, and each set such that there is no set less than it is called a minimal set. We then determine whether our examples of multiplicative bases are minimal and if not, try to find a subset that is minimal.

2 Definitions and Background

First, we provide a mathematical definition for an additive basis as given by Erdős and Turán [1].

**Definition 2.1.** A subset $A \subseteq \mathbb{N}$ is an additive base of order $k$ if all positive integers except a finite set can be written as a sum of $k$ elements of $A$.

Next, we define the additive representation function and additive efficiency of a sequence.

**Definition 2.2.** The representation function of a subset $A \subset \mathbb{N}$ and $h \in \mathbb{N}$ is $R_{A,h}(n) = \#\{(a_1, a_2, \ldots, a_h) \in A^h | a_1 + a_2 + \cdots + a_h = n\}$.

The function $R_{A,h}(n)$ represents the number of times a positive integer $n$ can be expressed as the sum of $h$ not necessarily distinct elements of $A$.

**Definition 2.3.** The additive efficiency is given by

$$E(A) = \limsup_{n \to \infty} R_{A,h}(n).$$

The additive efficiency represents the smallest upper bound on the function $R_{A,h}(n)$ over all positive integers $n$. We now give the statement of the Erdős-Turán Conjecture.

**Conjecture 2.1** (Erdős-Turán [2], 1941). The additive efficiency of any additive basis of order 2 is infinite.
Now we introduce our variation of this conjecture. We define a multiplicative basis as given by Xiao [3].

**Definition 2.4.** A subset $A \subseteq \mathbb{N}$ is a *multiplicative base of order* $k$ if all positive integers except a finite set can be written as a product of $k$ not necessarily distinct elements of $A$.

**Corollary 1.** A multiplicative basis $A$ of order $k$ must contain 1 and all sufficiently large primes.

*Proof.* By the definition, we know all sufficiently large primes can be written as a product of $k$ elements of $A$. A prime $p$ can only be expressed in this way by $1^{k-1} \cdot p$, which means that both 1 and $p$ must be elements of $A$. \quad \square

Similarly, we also define a multiplicative representation function and multiplicative efficiency for a given sequence.

**Definition 2.5.** Let the representation function of a set $A$ and a number $h \in \mathbb{N}$ be $r_{A,h}(n) = \#\{(a_1, a_2, \ldots, a_h) \in A^h | a_1 a_2 \ldots a_h = n\}$.

The function $r_{A,h}(n)$ represents the number of times a positive integer $n$ can be expressed as the product of $h$ not necessarily distinct elements of $A$.

**Definition 2.6.** The multiplicative efficiency is given by

$$e(A) = \limsup_{n \to \infty} r_{A,h}(n)$$

The multiplicative efficiency represents the smallest upper bound on the function $r_{A,h}(n)$ over all positive integers $n$.

### 3 Variation of Erdős-Turán Conjecture

We introduce the main problem that is the focus of this paper.

**Conjecture 3.1.** The multiplicative efficiency of any multiplicative basis of order 2 is infinite.
3.1 Relation to Erdös-Turán Conjecture

Conjecture 3.1 is essentially the multiplicative case of Conjecture 2.1. Furthermore, the two conjectures are intrinsically related.

First, we introduce two operations, the sum-set and product-set on two sets $A$ and $B$.

**Definition 3.1.** $A \oplus B = \{a + b | a \in A \cup \{0\}, b \in B \cup \{0\}\}$

The sum-set of two sets $A$ and $B$ is the set containing all elements of either $A$ or $B$ and positive integers that can be expressed as the sum of an element of $A$ and an element of $B$.

**Definition 3.2.** $A \otimes B = \{ab | a \in A, b \in B\}$

The product-set of two sets $A$ and $B$ is the set containing all elements that can be expressed as the product of an element of $A$ and an element of $B$. Now we proceed with the relationship between the two conjectures given by Dr. Khovanova [4].

**Claim 3.2.** *If the Erdös-Turán Conjecture is true, then Conjecture 3.1 is true.*

**Proof.** Consider any multiplicative basis $A$ of order 2. Construct two sequences $B$ and $C$ such that $B = A \cap \{2^n | n \geq 0\}$ and $C = \log_2 B$. We know that $A \otimes A$ contains all positive integers except for a finite set. Powers of 2 can only be formed by the product of 2 powers of 2, which means that all pairs of elements of $A$ whose product is a power of 2 are also elements of $B$. Hence, $B \otimes B$ contains all powers of 2 except for a finite set, and $C \oplus C$ contains all positive integers except a finite set. If the Erdös-Turán Conjecture is true, then the additive efficiency of $C$ is infinite. This would mean that the multiplicative efficiency of $B$ is infinite, and hence infinite for $A$ as well. This means that the Erdös-Turán Conjecture implies Conjecture 3.1.

4 Efficiency with Positive Analytic Density

Given a subset $A$ of $\mathbb{N}$, let $f_A$ be the characteristic function of $A$. 


4
Definition 4.1.

\[ f_A(a) = \begin{cases} 
1 & \text{if } a \in A \\
0 & \text{if } a \not\in A 
\end{cases}. \]

Now we introduce the Dirichlet series of a sequence \( A \).

Definition 4.2.

\[ L_A(s) = \sum_{n=1}^{\infty} f_A(n) n^{-s} = \sum_{n \in A} n^{-s}. \]

Lastly, we define the analytic density of a sequence \( A \).

Definition 4.3.

\[ d(A) = \lim_{s \to 1} \frac{L_A(s)}{\zeta(s)} \quad \text{where} \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \left( 1 - p^{-s} \right)^{-1}. \]

Theorem 4.1. Given a multiplicative basis \( A \) of order 2, its analytic density must be 0 if it has finite efficiency.

Proof. We have

\[ L_A(s)^2 = \sum_{n=1}^{\infty} \left( \sum_{d|n} f_A(d) f_A \left( \frac{n}{d} \right) \right) n^{-s}. \]

Notice that because \( f_A(d) f_A \left( \frac{n}{d} \right) = 1 \) if and only if both \( d \) and \( \frac{n}{d} \) are in \( A \), we have

\[ r_{A,2}(n) = \sum_{d|n} f_A(d) f_A \left( \frac{n}{d} \right). \]

Let \( B = A \otimes A \). Because \( r_{A,2}(n) > 0 \) for all sufficiently large \( n \), there exists a set \( C \) with a finite number of elements such that \( B = \mathbb{N} \setminus C \). It follows that

\[ L_B(s) = \sum_{n \in \mathbb{N} \setminus C} n^{-s}. \]

Suppose for the sake of contradiction that there exists a finite positive integer \( k \) such that \( k \geq r_{A,2}(n) \) for all \( n \). We then see that

\[ \limsup_{n \to \infty} r_{A,2}(n) = k \implies kL_B(s) + k \sum_{n \in C} n^{-s} \geq L_A(s)^2. \]

Using the fact that the density of any sequence is at most 1, we have

\[ k \geq kd(B) = \lim_{s \to 1} \frac{kL_B(s)}{\zeta(s)} \geq \lim_{s \to 1} \frac{L_A(s)^2 - k \sum_{n \in C} n^{-s}}{\zeta(s)} \geq 0. \]
However, $C$ is a finite set, so the numerator is finite while the denominator is infinite,
\[
\lim_{s \to 1} k \sum_{n \in C} n^{-s} \zeta(s) = 0.
\]
This implies that
\[
k \geq \lim_{s \to 1} \frac{L_A(s)^2}{\zeta(s)} \geq 0 \implies \lim_{s \to 1} \frac{L_A(s)^2}{\zeta(s)^2} = 0 \implies d(A) = 0.
\]
Therefore, for a subset to have finite multiplicative efficiency, it must have density 0. \qed

Now we must determine whether or not there exists a multiplicative basis of density 0.

### 4.1 Examples of Multiplicative Bases

First, we introduce some examples of multiplicative bases. Let $S_1$ be the set containing 1 and all numbers $p_1^{e_1} \ldots p_n^{e_n}$ such that $e_1 + e_2 + \ldots + e_n$ is odd as given by Dr. Khovanova [4]. Let $S_2$ be the set containing 1 and all numbers $p_1^{e_1} \ldots p_n^{e_n}$ such that each of $e_1, e_2, \ldots, e_n$ is odd as given by Dr. Khovanova [4]. Let $S_3$ be the set containing 1 and all numbers $p_1^{e_1} \ldots p_n^{e_n}$ such that $n$ is odd. For our last example, we first split the sequence of all primes into subsequences $P_i$ such that each prime is in exactly one subsequence, a subsequence contains consecutive primes, and
\[
\prod_{p \in P_i} \left(1 - \frac{1}{p}\right) \leq \frac{1}{2}.
\]
For example, we have $P_1 = \{2\}$, $P_2 = \{3, 5, 7\}$, and so on. It is well known that
\[
\prod_p (1 - \frac{1}{p}) = 0,
\]
which implies that there must be an infinite number of sets $P_i$. Let
\[
A_p = \bigcup_{i=0}^{\infty} P_{2i+1} \quad \text{and} \quad B_p = \bigcup_{i=0}^{\infty} P_{2i+2}
\]
Now let $A$ and $B$ be the sets of all numbers which have all prime factors as elements of $A_p$ and $B_p$ respectively. Let $S_4 = A \cup B$ as given by [5].

We wish to calculate the analytic densities of these two sequences. First, we define another type of density, the asymptotic density.
Definition 4.4.

\[
\partial(A,n) = \frac{|A \cap \{1,2,\ldots,n\}|}{n} \quad \text{and} \quad \partial(A) = \lim_{n \to \infty} \partial(A,n)
\]

Given this new type of density, we relate it to the analytic density through the following result.

**Theorem 4.2** (Dirichlet-Dedekind [6]). If \( \partial(A) \) exists, then \( d(A) = \partial(A) \).

This means that the asymptotic density and analytic density are equal as long as the asymptotic density exists.

### 4.1.1 Properties of \( S_1 \)

First, we will show that \( S_1 \) is a multiplicative basis.

**Claim 4.3.** \( S_1 \) is a multiplicative basis.

**Proof.** Since \( 1 \in S_1 \), all elements of \( S_1 \) are elements of \( S_1 \otimes S_1 \). Therefore, we now just need to show that all numbers greater than 1 where the sum of the exponents in its prime factorization is even will be in \( S_1 \otimes S_1 \). For any \( x \) of this form, we know \( x > 1 \), so there exists a prime \( p \mid x \). We have \( p \in S_1 \) and \( \frac{x}{p} \in S_1 \), so \( x \in S_1 \otimes S_1 \). Therefore, since \( S_1 \otimes S_1 = \mathbb{N} \), \( S_1 \) is a multiplicative basis. \( \square \)

Now we solve for the density of \( S_1 \). First, we used a program to find the asymptotic density up to the first one million terms. Our results are shown in Table 1.
Table 1: Density of the sequence $S_1$ over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 1, the asymptotic density of $S_1$ approaches 0.5 and is already very close with small values of $n$. We will now show analytically that this is indeed true using [7].

Claim 4.4. $d(S_1) = \frac{1}{2}$

Proof. For any positive integer $x$, we let the function $\Omega(x)$ be the sum of the exponents in its prime factorization. This means that

$$\frac{1 - (-1)^{\Omega(x)}}{2} = f_{S_1}(x) \implies \partial(S_1) = \lim_{x \to \infty} \sum_{i=1}^{x} \frac{1 - (-1)^{\Omega(i)}}{2x}.$$ 

It is well known that

$$\sum_{i=1}^{x} (-1)^{\Omega(i)} = o(x) \implies d(S_1) = \partial(S_1) = \frac{1}{2} - \lim_{x \to \infty} \sum_{i=1}^{x} \frac{(-1)^{\Omega(i)}}{2x} = \frac{1}{2}.$$ 

Hence, the density of $S_1$ is $\frac{1}{2}$.

4.1.2 Properties of $S_2$

First, we will show that $S_2$ is a multiplicative basis.

Claim 4.5. $S_2$ is a multiplicative basis.

Proof. For any positive integer $x$ we express it as $p_1^{e_1} \ldots p_n^{e_n} q_1^{f_1} \ldots q_n^{f_n}$ such that all $e_i$ are even while all $f_i$ are odd. We take

$$x_1 = \prod_{i=1}^{n} p_i \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=1}^{m} q_i^{f_i}.$$

8
Both $x_1$ and $x_2$ are elements of $S_2$. Therefore, since $S_2 \otimes S_2 = \mathbb{N}$, $S_2$ is a multiplicative basis.

<table>
<thead>
<tr>
<th>n</th>
<th>$\partial(S_2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.7</td>
</tr>
<tr>
<td>1000</td>
<td>0.707</td>
</tr>
<tr>
<td>10000</td>
<td>0.7055</td>
</tr>
<tr>
<td>100000</td>
<td>0.70457</td>
</tr>
<tr>
<td>1000000</td>
<td>0.704464</td>
</tr>
</tbody>
</table>

Table 2: Density of the sequence $S_2$ over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 2, the asymptotic density of $S_2$ approaches approximately 0.7044 and that it does not change much between small and large values of $n$. We will now show analytically that this is indeed true using [8]. First, we define a multiplicative sequence $A$.

**Definition 4.5.** A sequence $A$ is multiplicative if for any two $a, b \in \mathbb{N}$ such that gcd($a, b$) = 1, $f_A(a)f_A(b) = f_A(ab)$.

Now we define the Euler product expansion of a Dirichlet Series of a multiplicative sequence $A$.

**Definition 4.6.**

$$L_A(s) = \sum_{n=1}^{\infty} f_A(n)n^{-s} = \prod_p \left( \sum_{n=0}^{\infty} f_A(p^n)p^{-ns} \right).$$

We can use the Euler product expansion to calculate the density of $S_2$.

**Claim 4.6.** $d(S_2) = \frac{K_1 \pi^2}{6} \approx 0.7044422$ where $K_1$ is the carefree constant.
Proof. Notice that $S_2$ is a multiplicative sequence, and $f_{S_2}(p^n) = 1$ if and only if $n$ is odd or 0. The Euler product expansion for $S_2$ is

$$L_{S_2}(s) = \prod_p \left(1 + \frac{p^{-s}}{1 - p^{-2s}}\right).$$

Using the expression for $\zeta(s)$ given in Definition 4.3,

$$d(S_2) = \lim_{s \to 1} \frac{L_{S_2}(s)}{\zeta(s)} = \lim_{s \to 1} \prod_p \left(\frac{1 + p^{-s} - p^{-2s}}{1 + p^{-s}}\right) = \prod_p \left(\frac{1 - 2p^{-2} + p^{-3}}{1 - p^{-2}}\right)$$

$$= \prod_p \left(1 - \frac{2p - 1}{p^3}\right) \prod_p (1 - p^{-2})^{-1}.$$

The first product is known as the carefree constant which is approximately 0.4282495, while the second product is $\frac{1}{\zeta(2)} = \frac{\pi^2}{6}$. Hence, we obtain $d(A) = \frac{K_1 \pi^2}{6} \approx 0.7044422.$

4.1.3 Properties of $S_3$

Claim 4.7. $S_3$ is a multiplicative basis.

Proof. For any positive integer $x$ we express it as $p_1^{e_1} \ldots p_n^{e_n}$. If $n$ is odd, then $x \in S_3 \implies x \in S_3 \otimes S_3$. If $n$ is even, we take

$$x_1 = p_n^{e_n} \text{ and } x_2 = \prod_{i=1}^{n-1} p_i^{e_i}.$$

Both $x_1$ and $x_2$ are elements of $S_3$. Therefore, since $S_3 \otimes S_3 = \mathbb{N}$, $S_3$ is a multiplicative basis. \qed

<table>
<thead>
<tr>
<th>n</th>
<th>$\partial(S_3, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.44</td>
</tr>
<tr>
<td>1000</td>
<td>0.469</td>
</tr>
<tr>
<td>10000</td>
<td>0.5009</td>
</tr>
<tr>
<td>100000</td>
<td>0.50361</td>
</tr>
<tr>
<td>1000000</td>
<td>0.500955</td>
</tr>
</tbody>
</table>

Table 3: Density of the sequence $S_3$ over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.
As shown in Table 3, the asymptotic density of $S_3$ starts below 0.5 but then approaches 0.5 from above. It is not very close to 0.5 for small values of $n$, but approaches it when $n$ is larger. We were unable to prove this, but it is supported by our numerical calculations.

4.1.4 Properties of $S_4$

Claim 4.8. $S_4$ is a multiplicative basis.

Proof. For any positive integer $x$ we express it as $p_1^{e_1} \ldots p_n^{e_n} q_1^{f_1} \ldots q_n^{f_n}$ such that all $p_i \in A_p$ and all $q_i \in B_p$. Notice that $x_1 = p_1^{e_1} \ldots p_n^{e_n} \in A \subset S_4$ and $x_2 = q_1^{f_1} \ldots q_n^{f_n} \in B \subset S_4$. Both $x_1$ and $x_2$ are elements of $S_4$. Therefore, since $S_4 \otimes S_4 = \mathbb{N}$, $S_4$ is a multiplicative basis.

Claim 4.9. $d(S_4) = 0$.

Proof. Notice that $A$ and $B$ are both multiplicative sequences, and $f_A(p^n) = 1$ if and only if $p \in A_p$ and $f_B(p^n) = 1$ if and only if $p \in B_p$. The Euler product expansion for $A$ is

$$L_A(s) = \prod_{p \in A_p} (1 - p^{-s})^{-1}.$$  

Furthermore, we know that $A_p$ and $B_p$ together contain all the primes without any overlap, so using the expression for $\zeta(s)$ given in Definition 4.3,

$$d(A) = \lim_{s \to 1} \frac{L_A(s)}{\zeta(s)} = \lim_{s \to 1} \prod_{p} (1 - p^{-s}) \prod_{p \in A_p} (1 - p^{-s})^{-1} = \lim_{s \to 1} \prod_{p \in B_p} (1 - p^{-s})$$

$$= \prod_{n=0}^{\infty} \prod_{p \in P_{2n+2}} (1 - p^{-1}) \leq \prod_{n=0}^{\infty} \frac{1}{2} = 0 \implies d(A) = 0.$$

Similarly, we may obtain that $d(B) = 0$. Since $S_4 = A \cup B$ and $A \cap B = \emptyset$,

$$L_{S_4}(s) = \sum_{n \in S_4} n^{-s} = \sum_{n \in A} n^{-s} + \sum_{n \in B} n^{-s} = L_A(s) + L_B(s)$$

$$\implies d(S) = d(A) + d(B) = 0.$$

Hence, the density of $S_4$ is 0.

Corollary 2. There exists a multiplicative basis with density 0 for any order greater than 1.
Proof. We provide an example of a multiplicative basis with density 0 for order $k > 1$. We create the $k$ sets

$$Q_{ip} = \bigcup_{n=0}^{\infty} P_{kn+i}.$$ 

These $k$ sets are disjoint and cover all primes. Let $Q_i$ be the set of all numbers which have all prime factors as elements of $Q_{ip}$. We obtain

$$d(Q_i) = \lim_{s \to 1} \prod_{p \not\in Q_{ip}} (1 - p^{-s}) = \prod_{n \not\equiv i \pmod{k}} \prod_{p \in P_n} (1 - p^{-1})$$

$$\leq \prod_{n \not\equiv i \pmod{k}} \frac{1}{2} = 0 \implies d(Q_i) = 0.$$ 

Notice that the set

$$Q = \bigcup_{n=1}^{k} Q_i$$

is a basis of order $k$ and

$$d(Q) = \sum_{n=1}^{k} d(Q_i) = 0$$

We have constructed a basis of order $k$ with density 0, so we are done. \qed

5 Minimality of Multiplicative Bases

Consider a partially ordered set (poset) containing all multiplicative bases. We consider a set $A$ less than a set $B$ if $A \subset B$ and there are an infinite number of positive integers that are elements of $B$ and not $A$. We call a basis $A$ minimal if there are no bases that are less than $A$. For this poset, there may be infinitely many minimal sets.

Previously, we found a basis of order 2 with density 0, so a natural question is to consider whether minimal bases are all of density 0 and if not, find other minimal bases based on the bases we found.

Theorem 5.1. $S_1$ is a minimal set.

Proof. Suppose we remove an element $k \in S_1$ and call the new set $S'_1$. We claim that $k \not\in S'_1 \otimes S'_1$. If $k = 1$, then this is clearly true since the only way to achieve 1 is through $1 \cdot 1$.  

12
Now consider some other $k \in S_1$. Assume for the sake of contradiction that $k = xy$ where $x, y \in S'_1$. If $x, y \neq 1$, then the sum of the exponents in the prime factorization of $xy$ is even. This means that $k \neq xy$. If $x = 1$, then $y = k \notin S'_1$, and similarly for $y = 1$. Hence, we have reached a contradiction, so $k \notin S'_1 \otimes S'_1$.

Suppose there exists a basis $S$ that is less than $S_1$. There must be an infinite number of elements of $S_1$ that are not in $S$, which means an infinite number of elements of $S_1$ are not in $S \otimes S$. However, this must imply that $S$ is not a basis of order 2, which is a contradiction. Hence, $S_1$ is a minimal basis.

This also shows that not all minimal bases have density 0.

**Theorem 5.2.** $S_2$ is not a minimal set.

**Proof.** We claim that we can remove all elements of the form $p_1^{e_1}p_2^{e_2}...p_n^{e_n}$ such that $n$ is even and the number of exponents which are 1 is less than $\frac{n}{2}$ to form a new basis $S'_2$.

We will show that any positive integer can be formed by the product of two not necessarily distinct elements of $S'_2$. We can express any positive integer as $k = p_1^{e_1}p_2^{e_2}...p_n^{e_n}q_1^{f_1}q_2^{f_2}...q_m^{f_m}$ such that $e_1, e_2, ..., e_n$ are even while $f_1, f_2, ..., f_m$ are odd. We want to find two numbers $x_1$ and $x_2$ that multiply to $k$ such that both $x_1$ and $x_2$ are in $S'_2$. This means that both $x_1$ and $x_2$ must be divisible by $p_1, p_2, ..., p_n$ while only one of the two is divisible by each of $q_1, q_2, ..., q_n$. We now split this proof into four different cases.

Case 1: $m + n$ is even and $n$ is even

If $m = 0$, then we take

$$x_1 = \prod_{i = 1}^{\frac{n}{2}} p_i \prod_{i = \frac{n}{2} + 1}^{n} p_i^{e_i - 1} \quad \text{and} \quad x_2 = \prod_{i = 1}^{\frac{n}{2}} p_i^{e_i - 1} \prod_{i = \frac{n}{2} + 1}^{n} p_i.$$

Both $x_1$ and $x_2$ have an even number of prime factors, but the number of exponents that are 1 in both numbers is at least $\frac{n}{2}$, so both $x_1$ and $x_2$ are elements of $S'_2$. 

This also shows that not all minimal bases have density 0.
If $m > 0$, then we find odd positive integers $a$ and $b$ such that $a + b = m$. Take 
\[ x_1 = \prod_{i=1}^{n} p_i^{a} \prod_{i=1}^{m} q_i^{f_i} \] and 
\[ x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=a+1}^{m} q_i^{f_i}. \]
Both $x_1$ and $x_2$ have an odd number of prime factors, so they are both elements of $S'_2$.

Case 2: $m+n$ is even and $n$ is odd

Take the two numbers 
\[ x_1 = q_1^{f_1} \prod_{i=1}^{n} p_i \] and 
\[ x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=2}^{m} q_i^{f_i}. \]
Notice that $x_1$ has an even number of prime factors but $n \geq 1$, so the number of exponents that are 1 is at least $\frac{n+1}{2}$, and $x_2$ has an odd number of prime factors, so they are both elements of $S'_2$.

Case 3: $m+n$ is odd and $n$ is even

Take the two numbers 
\[ x_1 = \prod_{i=1}^{n} p_i \] and 
\[ x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=1}^{m} q_i^{f_i}. \]
Notice that $x_1$ has an even number of prime factors but $n \geq 0$, so the number of exponents that are 1 is at least $\frac{n}{2}$, and $x_2$ has an odd number of prime factors, so they are both elements of $S'_2$.

Case 4: $m+n$ is odd and $n$ is odd

Take the two numbers 
\[ x_1 = \prod_{i=1}^{n} p_i \] and 
\[ x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=1}^{m} q_i^{f_i}. \]
Both $x_1$ and $x_2$ have an odd number of prime factors, so they are both elements of $S'_2$.

We have now considered all possible parity cases for $n$ and $m+n$, and every positive integer is able to be constructed from elements in $S'_2$. Therefore, $S'_2$ is a multiplicative basis.

It is still possible to remove more elements, but we were unable to find a systematic way to do so. Using the new set, we numerically estimated the density.
As shown in Table 4, the asymptotic density of $S'_2$ is barely different from $S_2$. Removing the elements we found unnecessary barely changed the densities. This suggests that the removed set has very few elements compared to the positive integers and has density 0.

**Theorem 5.3.** $S_3$ is not a minimal set.

*Proof.* We claim that we can remove all elements of the form $p^{e_1}_1p^{e_2}_2\ldots p^{e_n}_n$ where $n$ is not of the form $2^k - 1$ and $e_1, e_2, \ldots, e_n > 1$ to form a new basis $S'_3$.

We will show that for all natural numbers $x = p^{e_1}_1p^{e_2}_2\ldots p^{e_n}_n$ it is possible to express $x = x_1x_2$ such that $x_1, x_2 \in S'_3$. We now split this proof into three cases.

**Case 1:** $e_1, e_2, \ldots, e_n > 1$

First we find the $k$ such that $2^k - 1 \leq x < 2^{k+1} - 1$. Now let $y = x + 1 - 2^k$ and find the $l$ such that $2^{l-1} - 1 \leq x + 1 - 2^k < 2^l - 1$. Let $z = 2^k + 2^l - x - 2$. We can express $x_1 = \prod_{i=1}^{z} p_i^{z_i - 1} \prod_{i=z+1}^{2^k-1} p_i^{e_i}$ and $x_2 = \prod_{i=1}^{z} p_i^{e_i - 1} \prod_{i=2^k}^{x} p_i^{z_i}$. Notice that $x_1$ has $2^k - 1$ prime factors and $x_2$ has $2^l - 1$ prime factors, so $x_1, x_2 \in S'_3$.

**Case 2:** $\min(e_1, e_2, \ldots, e_n) = 1$ and $n$ is odd

These elements themselves are already elements of $S'_3$ since they were originally in $S_3$ and not removed.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\partial(S'_2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.7</td>
</tr>
<tr>
<td>1000</td>
<td>0.704</td>
</tr>
<tr>
<td>10000</td>
<td>0.7045</td>
</tr>
<tr>
<td>100000</td>
<td>0.70423</td>
</tr>
<tr>
<td>1000000</td>
<td>0.70437</td>
</tr>
</tbody>
</table>

Table 4: Density of the sequence $S'_2$ over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.
Case 3: $\min(e_1, e_2, \ldots, e_n) = 1$ and $n$ is even

Without loss of generality, suppose that $e_1 = 1$. Take

$$x_1 = \prod_{i=1}^{n-1} p_i^{e_i} \text{ and } x_2 = p_n^{e_n}.$$ 

We know $x_1 \in S'_3$ since it has an odd number of prime factors and at least one exponent is one. Hence, $x_1, x_2 \in S'_3$.

We have considered every possible case for the exponents in the prime factorization of $x$. Every positive integer is able to be expressed as the product of two elements of $S'_3$, so therefore $S'_3$ is a multiplicative basis.

While it is possible to remove more elements, we were unable to find a systematic way to do so. We numerically estimated the density of the new set. We found that the values were the exact same as with $S_3$, so the set of numbers we removed does not contain any of the first million positive integers. This suggests that our removed set contains very few numbers compared to the set of all positive integers and has density 0.

**Theorem 5.4.** $S_4$ is a minimal set.

**Proof.** By our definition of $S_4$, we know that there is no element $x \in S_4$ such that it has prime factors in both $A_p$ and $B_p$. Hence, the only way to express such a number $x = x_1x_2$ such that $x_1, x_2 \in S_4$ is if $x_1 \in A$ and $x_2 \in B$ or vice versa.

Without loss of generality, suppose we remove an element $y > 1$ of $A$ from $S_4$ to get $S'_4$. For any element $z > 1$ of $B$, the only way to form $yz$ from $S_4$ is from $y$ and $z$. However, $y \not\in S'_4$, so it is impossible to form $yz$ from two elements of $S'_4$. There are infinitely many elements of $B$, so there are infinitely many positive integers not in $S'_4 \otimes S'_4$. Similarly, $S'_4$ is not a basis if we remove an element of $B$.

Suppose there exists a basis $S$ that is less than $S_4$. There must be elements in $S_4$ that are not in $S$, which means an infinite number of positive integers are not in $S \otimes S$. Therefore, $S$ is not a basis of order 2, so we have reached a contradiction. Hence, $S_4$ is a minimal basis. □
6 Conclusion and Future Work

In summary, we have worked towards the multiplicative variation of the Erdős-Turán Conjecture. We proved that it is only possible for a basis to have finite efficiency if its density is 0. To explore this last case more, we considered some examples of multiplicative bases and calculated their density both numerically and analytically. However, we were not able to disprove the case of density 0 bases as we successfully found a construction for one. As a basis with density 0 existed, we investigated whether all bases had a subset of density 0 that was also a basis. For our bases, we proved whether they were minimal and tried to find a minimal subset of it if it was not minimal.

In the future, we plan on continuing our work on our conjecture. Although we were unable to prove the case with density 0, our conjecture is not disproved by the example. Furthermore, we will investigate more examples of multiplicative bases to understand more about them as well as the set of minimal bases.

7 Acknowledgments

I would like to thank Yichi Zhang for being a great mentor on this project and Dr. Tanya Khovanova for teaching me how to write a paper and giving me many new ideas and directions for the project. I would like to thank my tutor Dr. John Rickert for giving me helpful feedback on my papers and presentations. I would like to thank Sean Elliott and Alex Zhu for providing me feedback on my paper. I would like to thank Dr. Amy Sillman and Dr. Slava Gerovitch for being the director of RSI and RSI Math respectively. I would like to thank Lockheed Martin, Mr. and Mrs. Vito J. Germinario, Sandra C. Eltringham, Ms. Dana M. Ervin, and The Honorable Thomas R. Pickering for sponsoring me and the Center for Excellence in Education and the Massachusetts Institute of Technology for providing me with this amazing opportunity.
References


