

Crossing Numbers on the Disk with Multiple Holes

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Abstract

In this paper we look at the crossing number, pair crossing number, and odd crossing number of graphs on a disk with multiple holes. We provide an upper bound $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ for the odd crossing number of graphs on the annulus with n edges. For graphs on a disk with multiple holes, we develop polynomial time algorithms to find the number of crossings between two edges, given their homotopy classes. We describe each edge as a word referring to its homotopy class. The algorithms also proved the crossing number, pair crossing number, and odd crossing number for graphs on a disk with multiple holes.

Summary

In this paper, we look at the number of crossings on graphs on disks with holes. Edges on this graph do not go through holes, but around holes. A disk with one hole is an annulus, for which we find the upper bound for the odd crossing number of a graph with n edges. For the graphs on a disk with multiple holes, we describe each edge as a word and using these words, find out whether two edges cross and how many times they cross.

1 Introduction

First, we define a *drawing* D of a graph G as a mapping of the vertices and edges of the graph to the Euclidean plane. The edges are continuous curves, and they can intersect each other several times. In this paper, we use Pelsmajer, Schaefer, and Stefankovic's [1] restriction on edges:

- An edge does not contain any endpoints of other edges in its interior
- No edges touch (i.e. intersect without crossing)
- At most two edges go through any given point
- Any two edges intersect at finitely many points.

Now we define the *crossing number*, *pair crossing number*, and *odd crossing number* of a graph G . First the *crossing number* of a drawing D , denoted as $\text{cr}(D)$, is the number of crossings in the drawing. The *pair crossing number* $\text{pcr}(D)$ is the number of pairs of edges that cross each other at least once. Finally, the *odd crossing number* of a drawing $\text{ocr}(D)$ is the number of pairs of edges that cross an odd number of times. The crossing number of a graph $\text{cr}(G)$ is then defined to be the minimum of the $\text{cr}(D)$ for all drawings D of G . Likewise, the *pair crossing number* $\text{pcr}(G)$ of G , and *odd crossing number* $\text{ocr}(G)$ of G are defined to be the minimum of the $\text{pcr}(D)$ for all drawings D of G and the minimum of the $\text{ocr}(D)$ for all drawings D of G , respectively.

Crossing numbers were first introduced during World War II when Turan posed the question: what is the minimum possible number of crossings in a drawing of a complete bipartite graph? Turan was working in a brick factory where wagons were taking bricks from kilns to storage sites. However, where the wagon paths intersected, wagons had a difficult time getting across the crossing. More recently, Tóth [2] analyzed different ways of counting the crossing number of a graph.

In Section 2, we introduce the definitions and set up of graphs on the annulus and on a disk with multiple holes. In Section 3, we prove an upper bound for the odd crossing number of graphs with n edges on the annulus. In Section 4, we consider different ways of counting odd crossing number, pair crossing number, and crossing number for graphs on the disk with multiple holes. In Section 5, the we look at possible extensions of our results.

2 Background and Definitions

In this section, we give a brief overview of known inequalities relating $\text{cr}(G)$, $\text{pcr}(G)$, and $\text{ocr}(G)$. We also look at the definitions and set ups of graph on the annulus and the disk with multiple holes. The following inequality,

$$\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G),$$

follows directly from the definition because each pair of edges that cross an odd number of times in the $\text{ocr}(G)$ is counted in the $\text{pcr}(G)$, and that pair of edges cross at least once.

Pelsmajer, Schaefer, and Stefankovic [3] proved that for any ε greater than 0 there exists infinitely many graphs satisfying the following inequality

$$\text{ocr}(G) < \left(\frac{\sqrt{3}}{2} + \varepsilon \right) \cdot \text{cr}(G).$$

Tóth [2] proved a stronger inequality that for any ε greater than 0, there exists infinitely many graphs satisfying

$$\text{ocr}(G) < \left(\frac{3\sqrt{5} - 5}{2} + \varepsilon \right) \cdot \text{pcr}(G).$$

Pelsmajer, Schaefer, and Stefankovic [3] conjectured that there is a constant $c > 0$ such

that

$$\text{cr}(G) < c \cdot \text{ocr}(G).$$

Pach and Tóth [4] formulated a weaker version of this conjecture

$$\text{cr}(G) \leq 2\text{ocr}(G)^2.$$

For finding the crossing number of a graph, Schaefer, Sedgwick, and Stevankovic [5] provide an $O(m^{11})$ time algorithm, where m is the number of edges in the graph. We improve the algorithm time for graphs on the disk with multiple holes.

2.1 Dehn Twist on a Graph

A *Dehn twist* is a twist in a graph such that we create a ring around a certain area and cause all the edges that pass through the ring to have additional curves. To be more specific, the ring has an outside boundary and an inside boundary. Any parts of edges inside the inside boundary are not affected by the Dehn twist, but any parts of edges between the two boundaries are affected by the Dehn twist. The Dehn twist can be used in either direction for multiple loops. For instance, for an edge that goes through the Dehn twist ring, a Dehn twist that goes clockwise around the ring causes the edge to have 2 more loops overall, 1 clockwise where the edge enters the Dehn Twist ring and 1 counterclockwise where the edge exits the Dehn twist ring, as seen in Figure 1. For edges that have incident vertices within the inside boundary of the Dehn twist, the edge only gets 2 clockwise twists. Dehn twists do not create any crossings. In fact, they preserve the number of crossings, because all of the edges are distorted in the same manner. A Dehn twist is analogous to taking two straight lines that intersect perpendicularly, and turning the 4 rays from the intersection point into 4 non-intersecting spirals that spiral out in the same direction and at the same rate.

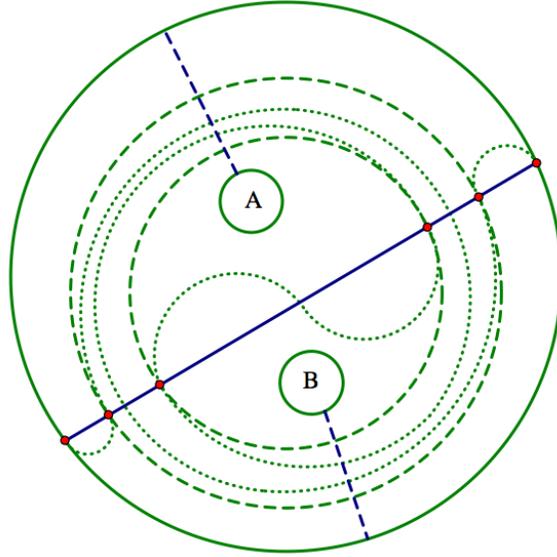


Figure 1: The straight solid line segment is the original edge. The dashed lines are the rings for the Dehn twist. The dotted curvy edge is the edge after the Dehn twist is applied to the region. Note that the edge does not intersect itself.

2.2 Crossing Numbers on the Annulus

A slightly different approach to crossing numbers on graphs starts by looking at graphs in the annulus. The *annulus* is a disk on the plane with a hole. We start with n points on the outside disk and n points on the hole, and we create n edges that start from the outside disk of the annulus to the hole, such that each vertex is connected to exactly one edge.

We first label the vertices on the disk with $a_0, a_1, a_2, \dots, a_{n-1}$ starting from an arbitrary vertex and going clockwise. Then, we label the edges by the index of the vertex on the outside ring. Thus, edge 0 has the vertex a_0 . To label the vertices on the hole, we start with the vertex that is incident to edge 0, and we label it as b_0 . We then go clockwise around the hole to label the rest of the vertices as $b_1, b_2, b_3, \dots, b_{n-1}$.

For the sake of a reference, we let edge 0 be a straight line segment. We can always achieve this by twisting the hole until a_0 and b_0 align. While twisting the hole might make the edges a bit more complicated, it will not create any more crossings. Another way of looking at this

is a Dehn twist where the hole is in between the inside and outside boundaries of the Dehn twist. We make the necessary number of twists to make edge 0 straight.

2.2.1 Permutation on the Annulus

Given a graph on the annulus, another way of defining the edges is as a *permutation*. In the permutation π , we permute the b_i mapped to a_i . For example, a permutation with 6 vertices could be $b_3 \rightarrow a_0$, $b_5 \rightarrow a_1$, $b_4 \rightarrow a_2$, $b_2 \rightarrow a_3$, $b_0 \rightarrow a_4$, and $b_1 \rightarrow a_5$. In this permutation, π , we let π_i be the index j of b_j which is connected by edge i to vertex a_i .

2.2.2 Adding Twists to the Permutation

The *number of twists* for each edge is the number of times the edge crosses edge 0 in a clockwise manner minus the number of times the edge crosses edge 0 in a counterclockwise manner. Since in our drawing we have edge 0 as a straight line segment, we define edge 0 to have no twists. For edge i , we define the *homotopy class* on the annulus, x_i , to be the number of twists. Homotopy classes are equivalence classes, such that two edges with the same homotopy class are equivalent. Pelsmajer, Schaefer, and Stefankovic [1] proved the formulas for the crossing numbers on the annulus for a permutation graph, given the number of twists for each edge to be:

$$\text{cr}(G) = \min \left\{ \sum_{i < j} |x_i - x_j + [\pi_i > \pi_j]| : x_i \in \mathbb{Z}, i \in [n] \right\},$$

$$\text{pcr}(G) = \min \left\{ \sum_{i < j} [x_i - x_j + [\pi_i > \pi_j] \neq 0] : x_i \in \mathbb{Z}, i \in [n] \right\},$$

$$\text{ocr}(G) = \min \left\{ \sum_{i < j} [x_i - x_j + [\pi_i > \pi_j] \not\equiv 0 \pmod{2}] : x_i \in \mathbb{Z}, i \in [n] \right\}.$$

2.3 Disk with Multiple Holes

We start with an outside ring and n holes within the ring. We then draw reference lines from the outside ring to the holes, one reference line per hole, such that the reference lines do not intersect. In the following figure, the dotted lines are the reference lines. Each edge in the disk with multiple holes has vertices on the holes or the boundary of the disk, such that both vertices are not on the same hole or on the boundary.

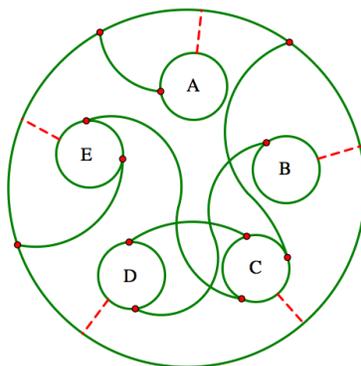


Figure 2: This graph on a disk with multiple holes has 5 holes: A, B, C, D, and E. The reference lines are the dotted lines. These reference lines are used for homotopy classes. Every edge drawn in this graph has a homotopy class of 1.

2.3.1 Homotopy Class on the Annulus with Multiple Holes

First we define a *homotopy class* as an equivalence class of homotopy. We say that two edges are *homotopic* if they can be continuously deformed to each other with their endpoints fixed. Given a graph on the annulus with multiple holes and reference lines to each hole, we define the edge connecting two vertices that does not cross any reference line to have a homotopy class of $[1]$. Every homotopy class is uniquely described by a word $[abc^{-1}ba^{-1}]$, and every such word gives a unique homotopy class. Each letter in the homotopy class word refers to the reference line that is crossed. If the reference line to hole I is crossed clockwise, the homotopy class word has i^{-1} in it. If the reference line to hole I is crossed counterclockwise,

the homotopy class word has i in it. The word is simply a list of such letters in order of the reference lines crossed from one vertex to the other.

One important note about the homotopy class words is that any ii^{-1} can be removed as long as they are consecutive in the homotopy class. This follows because crossing a reference line clockwise and then counterclockwise immediately after is homotopic to not crossing the reference line at all. We can simply pull the edge until it no longer crosses the reference line.

2.3.2 Minimal Edges for Homotopy Classes and Twists

A *minimal edge* for a homotopy class is an edge that has a word which cannot be simplified. In other words, there are no ii^{-1} or $i^{-1}i$ for all i . Likewise, a *minimal edge* for the number of twists is an edge that does not cross a reference line clockwise and then counterclockwise in succession.

3 Upper Bound for Odd Crossing Number on the Annulus

We can represent the number of odd crossings in a drawing by creating a new graph, *permutation graph*, where the vertices of the new graph are the edges from the permutation. If two edges, i and j , cross in the simple permutation drawing, then there is an edge connecting vertices i and j in the new graph. In the permutation graph, color all of the vertices either red or blue. We color a vertex i red if x_i is odd and blue otherwise. If two red vertices are connected by an edge, then the corresponding edges in the permutation graph cross an odd number of times. Likewise, if two blue vertices are connected by an edge, the corresponding edges in the permutation graph cross an odd number of times. If a red vertex and a blue vertex have no edge between them in the colored graph, then they intersect an odd number of times in the permutation graph.

Theorem 1. *The odd crossing number of a graph on the annulus with n edges is at most:*

$$\text{ocr}(G) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

Proof. We proceed by induction on n . Start with just 1 edge. In this case we do not have any crossings because we have one edge only. Now, assume that for a graph on the annulus with $n-1$ edges there are at most $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ crossings. If we add in an n th edge with vertices a_{n-1} and π_{n-1} , such that a_{n-1} is on the perimeter and π_{n-1} is on the hole, then there are at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ crossings created. To show this, we look at the parity of the crossings. When edge n has homotopy class 0, either $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges cross edge n an odd number of times or $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges cross edge n an even number of times. In the first case, we are done. In the second case, we then take edge n to have a homotopy class of 1, and we see that all the edges that crossed an odd number of times cross an even number of times and vice versa. Then we have what we desired, which is adding at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ to the odd crossing with the addition of the n th edge. We see that there can be at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges that cross either edge n with pseudo homotopy class a or edge n with pseudo homotopy class b . Then, for our graph with $n+1$ edges, we see that it has at most $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$ edges. This equals $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ because we have:

$$\left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \left\lfloor \frac{2n-2}{4} \right\rfloor \leq \left\lfloor \frac{(n-1)^2 + 1}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

We use casework to show equality. If n is odd, then $\frac{(n-2)^2}{4}$ has a fractional part of $\frac{1}{4}$ and $\frac{n-1}{2}$ has no fractional part, so $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$ necessarily equals $\left\lfloor \frac{(n-1)^2 - 1}{4} \right\rfloor$, which has a fractional part of $\frac{1}{4}$, so this also necessarily equals $\left\lfloor \frac{n^2}{4} \right\rfloor$. If n is even, then $\frac{(n-2)^2}{4}$ has no fractional part and $\frac{n-1}{2}$ has a fractional part of $\frac{1}{2}$. Since the two fractional parts do not sum to be more than 1, we also get that $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$ necessarily equals $\left\lfloor \frac{(n-1)^2 + 1}{4} \right\rfloor$, which equals $\frac{(n-1)^2}{4}$. \square

4 Crossing Numbers for a Graph on a Disk with Multiple Holes

There are multiple ways to draw a given graph on the plane. With each drawing, we can associate a homotopy class word to each edge. Using these homotopy classes for edges, we can take the edges pairwise and see if that pair of edges cross and if that pair crosses an odd number of times. Though it is not important for edges to be non-self-intersecting for odd crossing number, it is quite necessary for edges to be non-self-intersecting for the pair crossing number. Undoing any self-intersections is simple, but it changes the homotopy class. Also, the number of letters in the homotopy class of any edge on a graph with n vertices is at most 2^n [6].

Proposition 1. *Self-intersections in an edge can be removed without changing any crossing numbers*

Proof. In order to do so, we follow the edge until we reach the point where it self intersects. Following the given direction of the edge, we let the part of the edge before the intersection be a and the part of the edge after the intersection be b for both edge segments in the intersection. Then we erase just the intersection and connect the two a parts of the intersection and connect the two b parts of the intersection. \square

4.1 Odd Crossing Number for a Graph on a Disk with Multiple Holes

Given two edges and their homotopy classes, we can determine if they intersect each other an even or an odd number of times. First, we let i be a reference line and e be an edge. Then, $f(i, e)$ is the number of times i and i^{-1} appear in the homotopy class of e . Let our two edges be e_i and e_j . Also, let the vertices incident to e_i be on holes a and b . Let the vertices

incident to e_j be on holes c and d . The holes a, b, c , and d are not necessarily distinct, but a and b are distinct and c and d are distinct from each other, respectively. Let $g(e_i, e_j)$ equal 1 if e_i and e_j cross when both have homotopy class 1. Otherwise, $g(e_i, e_j)$ equals 0. Then, we get the following equation.

Theorem 2. *The two edges e_i and e_j cross an odd number of times if*

$$f(a, e_j) + f(b, e_j) + f(c, e_i) + f(d, e_i) + g(e_i, e_j) \equiv 1 \pmod{2}$$

Proof. In the two homotopy classes for the edges, all of the reference lines to holes that do not contain any of the vertices incident to the edges results in an even number of crossings. This follows because we can always deform one of the edges to stay strictly outside of the other edge if both edges cross the same reference line. Given two homotopy classes for the two edges, any drawing of the edges will still result in the same number of crossings mod 2. Thus, we can redraw our two edges by having two concentric circles in the middle of the disk. Each letter of the homotopy class can be represented as a loop that starts from one of the concentric circles and ends in the same concentric circle. We let e_i always come from and go to the inner concentric circle for each letter of the homotopy class and we let e_j always come from and go to the outer concentric circle for each letter of the homotopy class. This means for any letter in the homotopy class of edge e_i that isn't a hole which contains the vertices incident to edge e_j , the letter will cause e_i to cross e_j an even number of times. However, for any letter in the homotopy class of edge e_i that is a hole which contains the vertices incident to edge e_j , e_i will cross e_j an odd number of times. □

4.2 Pair Crossing Number for Graphs on the Disk with Multiple Holes

Given the drawing of the disk with multiple holes, we now flatten all the holes and the reference lines. The way we do this is by splitting all the reference lines so that there are now two reference lines to each hole, as seen in Figure 3 left.

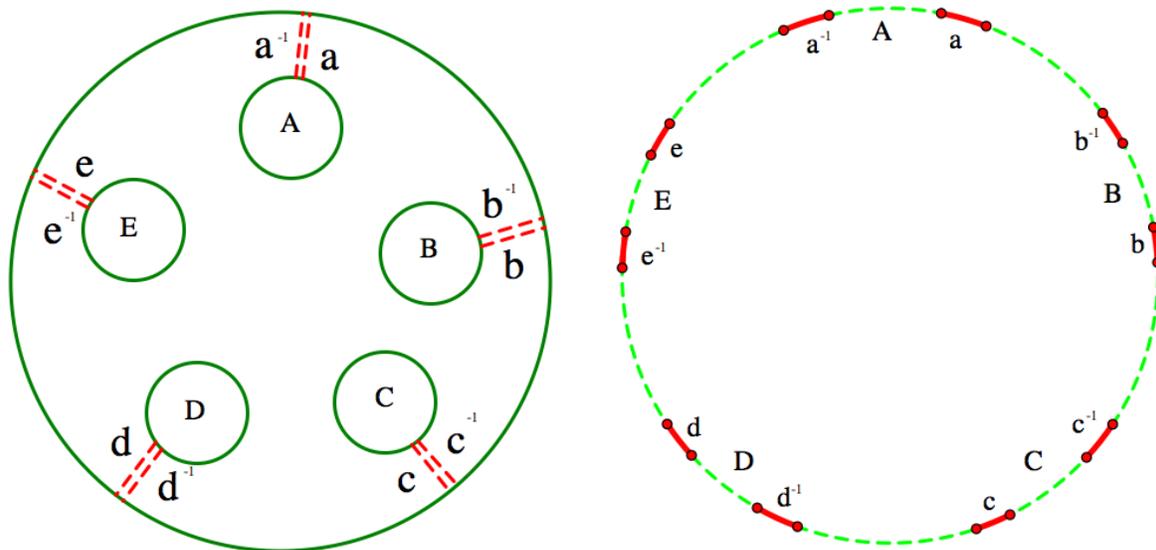


Figure 3: *Left*: This image shows the splitting the reference lines into two reference lines. Here, we split the reference lines such that there are no vertices in between the two split reference lines. *Right*: Flattening the reference lines and the holes from figure on the left into one huge circle. The reference lines are solid while the outside boundary and the holes are dotted lines. The upper case letters refer to the holes while the lower case letters refer to the reference lines and the direction of the reference lines. Note that in a homotopy class with letter e^{-1} , the edge would first enter through the reference line e^{-1} and then exit out of the reference line e in both figures. In the second figure, the edge would briefly exit the circle, but in the actual drawing, it is still within the boundaries.

We then flatten out the reference lines and the holes until they all become one circle as seen in Figure 3 right. The holes are denoted with capital letters and the reference lines now have two parts, i and i^{-1} . The reference lines still are used to denote the homotopy classes of edges. For instance, an edge with homotopy class a will go into a and pop out of a^{-1} , where we have an imaginary dotted part from a to a^{-1} outside of the circle.

Here, we want all of the edges to be self non-intersecting. We can always change the homotopy class of a self-intersecting edge such that it no longer self-intersects, without changing the crossing number of the graph and drawing. We can draw the edge as being made up of only straight line segments because it does not self-intersect.

Theorem 3. *The order for determining whether two edges with homotopy class word lengths of m and n is $O(m + n)$.*

Proof. Drawing the graph and keeping track of the sections and how they connect, both through boundaries and through reference lines, for one of the edges with the split reference lines is $O(m)$. Then, we go through each letter of the second edge and check to see if the edge is forced to cross a boundary in the graph. Since we check each letter of the homotopy class of the second edge, this is $O(n)$. □

4.3 Crossing Number for Graphs on the Disk with Multiple Holes

We use the same construction for the crossing number as we did for the pair crossing number. We start with two non-self-intersecting edges, or if we have edges that are not self-intersecting, we remove the self-intersections. The first edge is drawn with straight line segments, and these line segments partition the graph into different sections. We represent the edge and the different sections it creates with a tree. The vertices of the tree are the different sections. Then, we draw an edge between two vertices if the two sections are adjacent to each other. We draw a dotted edge between two vertices if a reference line can take an edge from one section to another without crossing the original edge.

Proposition 2. *All the sections that can access a reference line i create a path. A path with the same length is created by the sections that can access the reference line i^{-1} , such that corresponding vertices in the path have a dotted edge between them in the tree representation.*

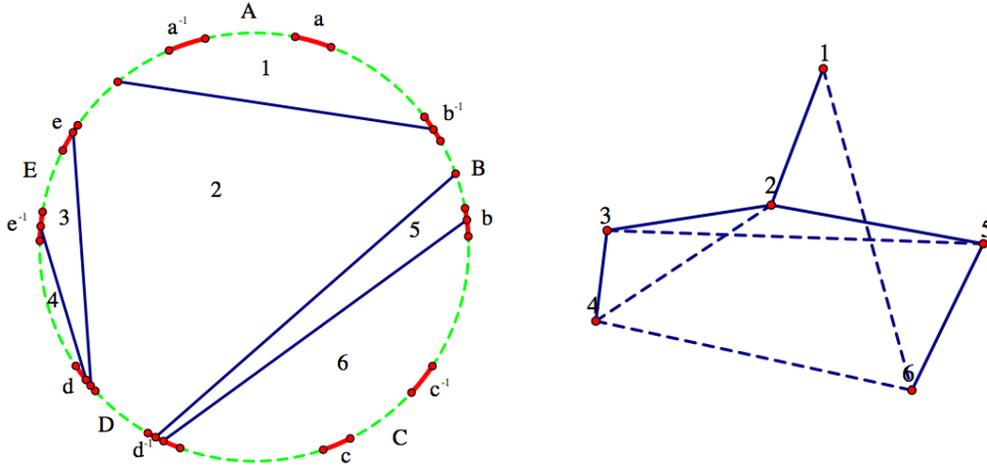


Figure 4: *Left*: We have the edge from the boundary of the disk to hole B , with homotopy class $b^{-1}d^{-1}e^{-1}d$. Then, we number all of the sections created by partitioning the graph with the first edge. *Right*: We make a tree with the solid edges such that the vertices of the tree are the numbered sections from the graph on the left. The dashed edges refer to crossing a reference line from one incident vertex to another in the tree or from one numbered section to another in the graph.

Proof. This follows since all of the sections that can access a reference line have a boundary part of the initial edge that stems from the reference line, and each boundary represents an edge in the tree. Figure 4 illustrates this property. \square

Using the proposition above, we note that if our second edge has b in its homotopy class, going through reference line b at the first opportunity and then going along the edges of the tree to a different section, i , that has access to reference line b^{-1} is the same as going to the section that connects to i in b^{-1} and then going through reference line b . The path to the vertex that first accesses the reference line minimizes the number of crossings.

Proposition 3. *The minimal crossing for an edge through the edge already drawn in the circle consists of straight line segments from reference lines to reference lines.*

Proof. This is easy to see using the tree. In order to get from one part of the graph to another part, as partitioned by the first edge, the edge needs to travel along the edges of the tree from one part to another, and the shortest path on the tree corresponds to a straight line

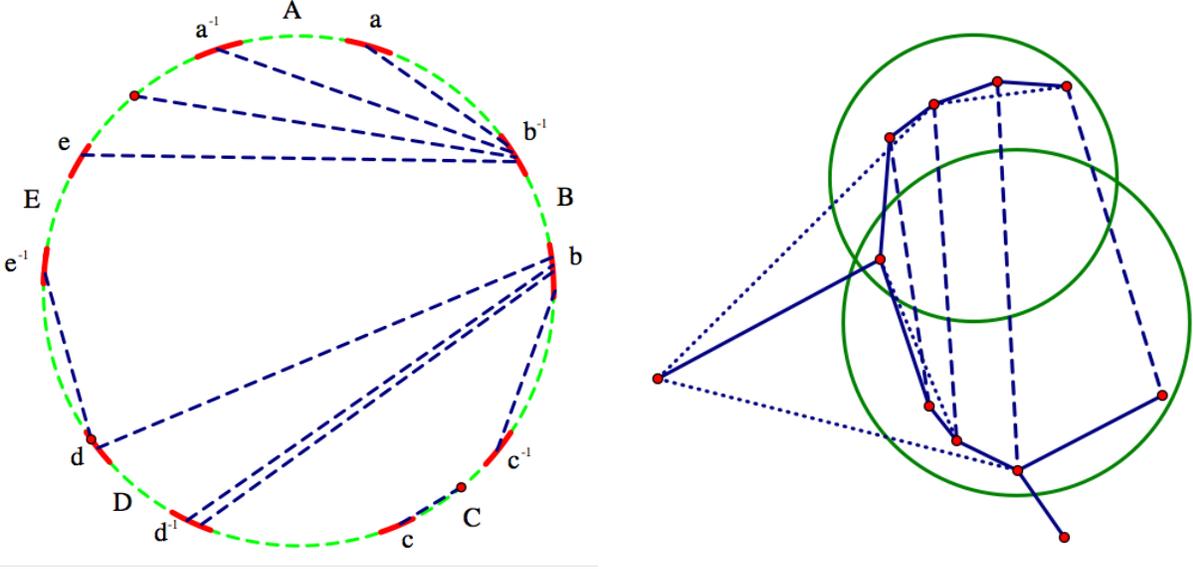


Figure 5: *Left:* Here we look at an edge that starts on the boundary of the disk and ends on hole C . The homotopy class of this edge is $b^{-1}d^{-1}bedba^{-1}b^{-1}$. We then look at the sections partitioned by the parts of the edge that go through reference lines b and b^{-1} . *Right:* The two circled parts refer to the two paths created by the sections that have access to reference line b^{-1} , the upper circle, and the sections that have access to reference line b , the lower circle. Note that the dashed edges in the tree are all non-intersecting and follow the two paths.

between the two sections on the graph, because the boundaries of the sections are straight lines and the original edge is non-self-intersecting. \square

With these two propositions, we now find the number of times two non-self-intersecting edges with known homotopy classes cross. First, we create a graph and the tree using one of the edges.

Theorem 4. *There is an $O(m \cdot n)$ time algorithm for computing the number of crossings given 2 edges and their homotopy classes.*

Proof. Each time the second edge traverses an edge on the tree, it goes from one section to another section, which means it crosses the first edge. Thus, counting all of the edges traversed on the tree is the same as counting the number of crossings. For each letter in the

2nd edge, we compute the minimal number of crossings to reach each reference line, as given by Proposition 3. □

This algorithm improves the n^{11} time algorithm for computing the number of crossings between two edges with their homotopy classes, given by Schaefer, Sedgwick, and Stefankovic [5]. However, our algorithm is restricted to graphs on disks with multiple holes, while their algorithm works for any surface.

We use Schaefer and Stefankovic's [6] lemma to find the crossing number of a graph:

Lemma 1. *Suppose that a graph is drawn in the plane, and edge e is crossed by m other edges. If there are at least 2^m crossings on e , then the drawing can be modified such that (i) the number of crossings between any two edges does not increase, and (ii) the number of crossings on e decreases.*

Using this lemma, we know that the homotopy class of each edge can be expressed as a word with at most 2^{m+k} letters, where m is the number of edges and k is the number of holes. We then use Theorem 4 and find the minimum crossing number of all possible words. Since the edges are drawn with straight line segments, they can be drawn minimally simultaneously.

5 Conclusion

We have now developed algorithms for finding various types of crossing numbers on disks with multiple holes. These algorithms could potentially lead to finding a smaller ratio between pair crossing number and odd crossing number than the currently known ratio. Future work could aim to create a faster, more efficient algorithm for the crossing number for graphs on a disk with multiple holes, not just the number of crossings between two edges.

A possible extension of our algorithms involves expressing each vertex as a hole. This is done by extending each vertex into a loop to form the hole. All of the edges that the vertex

is incident to now stems from the hole, and these edges can be permuted in order around the hole. Then we can draw reference lines from each hole to a boundary loop and run through our algorithms. However, this is very inefficient, and it would be interesting to see a faster algorithm to find crossing numbers for general graphs.

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