#### Oscillating near circles

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# Oscillating Near Circles at Intermediate Reynolds Numbers

Alex Zhao Mentored by Dr. Nick Derr

> PRIMES 2023 October 14

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# Outline

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- Numerical Analysis is the application of computers to numerically create approximate solutions to complex problems.
- Fluid Dynamics is the branch of physics that models fluid motion using differential equations in multiple variables.

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### Steady Action Driven by Oscillation

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Klotsa et al., PRE 2009





Collis et al., JFM 2017

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### **Problem Geometry**



# Non-Dimensionalized Problem Geometry



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### xample (Hinch)

Solve 
$$x^2 - \varepsilon x - 1 = 0$$
 for the positive solution.  
Expanding  $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$  yields

$$x_0^2 - 1 + \varepsilon (2x_0x_1 - x_0) + \varepsilon^2 (2x_0x_2 + x_1^2 - x_1) + \cdots = 0.$$

Setting each coefficient to 0, and solve the resulting equations. Then  $x = 1 + \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \dots$ , agreeing with  $\frac{1}{2}\varepsilon + \sqrt{1 + \frac{1}{4}\varepsilon}$ .

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Example (Hughes)

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# Find the solution u to u'' = -f on [0, 1] where u(0) = u(1) = 0.

In this problem, u'' + f = 0 is the field equation and u(0) = u(1) = 0 is the boundary condition. Equivalently,  $\int u'' w + fw \, dx = 0$  for all w. We call w the terms function.

If w(0) = w(1) = 0, integrating by parts yields the weak form

$$\int fw \, dx = \int u' w' \, dx.$$

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Example (Hughes)

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### Restrict w and u to piecewise linear functions.



Using  $\phi_i$  to denote basis functions,  $u = \sum_i u_i \phi_i$  and  $w = \sum_i w_i \phi_i$ .

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Conclusion and Acknowledgement References Then the coefficients  $u_i$  form a vector.

Recall

$$\int fw \ dx = \int u' w' \ dx.$$

If *u* satisfies this equation for *w* equalling each basis function  $\phi_i$  then *u* is a solution.

$$\forall i, \int f \phi_i \, dx = \sum_j u_j \left( \int \phi'_j \phi'_i \, dx \right).$$

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This now becomes a linear matrix problem

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The length of the red line is  $\varepsilon_2 \sin^2 \Phi$ , to first order in  $\varepsilon_2$ .

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There is no velocity on the ellipse boundary; Taylor expand this condition from the circle.

On the circle boundary,

 $\vec{u} - \varepsilon_2 \sin^2(\Phi) (\vec{r} \cdot \nabla) \vec{u} = 0.$ 

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# We will use the expansion

$$\vec{u} = \varepsilon_1 e^{it} (\hat{u}_0 + \varepsilon_2 \hat{u}_1) + \varepsilon_1^2 (\bar{u}_0 + \varepsilon_2 \bar{u}_1).$$

- **()** Solving for time-independent velocity fields  $\hat{u}_0, \hat{u}_1, \bar{u}_0, \bar{u}_1$ .
- At order ε<sub>1</sub> the velocity is driven by oscillation, necessitating the e<sup>it</sup> term.
- (a) At order  $\varepsilon_1^2$  the velocity self-interferes, creating a steady flow.
- We have additional expansions in  $\varepsilon_2$  at both orders to examine the contribution from eccentricity.

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### Leading Oscillatory Flow Plots



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Results



### Leading Steady Flow Plots

### Alex Zhao

Results







(b)



(c)



(d)



Re=1



(h)

Re=20







(i) Re=300

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In this talk, we

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- Discussed concepts in numerical fluid dynamics such as the perturbation method and the finite element method
- Introduced geometry and equations for our research problem
- Presented pressure and velocity streamline plots Many thanks to
  - My mentor, Dr. Nick Derr
- MIT PRIMES organizers, Dr. Gerovitch and Dr. Khovanova

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• My parents

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