Bases for Quotients of Symmetric Polynomials

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Basic Ideas

**Definition**

*R* is a commutative ring with unity.

**Definition**

Ring of polynomials:

\[
R[x] = \left\{ r_0 + r_1 x + \cdots + r_d x^d \mid r_j \in R, r_d \neq 0 \right\}
\]

d is the degree of the polynomial.

**Example**

- \(2 - x + x^2 \in \mathbb{Z}[x]\), degree 2
- \(\pi + 2x^2 - ix^5 \in \mathbb{C}[x]\), degree 5
More Indeterminates

**Definition**

$k$ indeterminates $x_1, \ldots, x_k$:

$$R[x_1, \ldots, x_k] = R[x_1] \cdots [x_k] = \left\{ \sum_{j_1, \ldots, j_k \geq 0, \text{Finitely many terms}} r_{j_1, \ldots, j_k} x_1^{j_1} \cdots x_k^{j_k} \mid r_{j_1, \ldots, j_k} \in R \right\}$$

$$\max(j_1 + \cdots + j_k \mid r_{j_1, \ldots, j_k} \neq 0)$$ is the degree of the polynomial.

**Example**

- $x_1 x_2 + x_1 x_2 x_3 + 2x_1^5 x_3^2 \in \mathbb{Z}[x_1, x_2, x_3]$, degree 7
- $\pi + 2x_1^2 - ix_1 x_2 + x_4^3 \in \mathbb{C}[x_1, x_2, x_3, x_4]$, degree 3
Symmetric Polynomials

Definition

$S$ is the subset of $R[x_1, \ldots, x_k]$ of polynomials that remain unchanged when indeterminates are permuted.

Example

If $k = 2$, then

$$x_1 + x_2 \in S$$

since $x_2 + x_1 = x_1 + x_2$.

Example

If $k = 3$, then

$$x_1 + x_2 \notin S$$

since $x_2 + x_3 \neq x_1 + x_2$, but

$$x_1 + x_2 + x_3 \in S$$
Partitions

Definition

A **partition** \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \) is a decreasing sequence of positive integers, that is, \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0 \). The **Young diagram** of \( \lambda \) is the left-aligned grid of boxes with \( \lambda_i \) boxes in the \( i \)th row.

\( \text{Par}_k \) is the set of partitions with \( \ell(\lambda) \leq k \). \( \text{Par}_{k,n-k} \) is the set of partitions whose Young diagram fits inside of box of height \( k \) and length \( n-k \).

The conjugate of \( \lambda \), \( \lambda' \), is the partition whose Young diagram is the reflection of the Young diagram of \( \lambda \) across the main diagonal.

Example

Let \( \lambda = (3, 2) \). Then \( \lambda \in \text{Par}_2 \), \( \lambda \in \text{Par}_{2,3} \), \( \lambda \notin \text{Par}_{2,2} \), \( \lambda' = (2, 2, 1) \).

\[ \lambda : \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\end{array} \quad \lambda' : \quad \begin{array}{ccc}
\bullet & \bullet & \\
\bullet & & \\
\end{array} \]

Note that \( \lambda' \in \text{Par}_k \iff \lambda_1 \leq k \).
Homogeneous Symmetric Polynomials

Definition

\[ h_i = \sum_{j_1 + \cdots + j_k = i, j_1, \ldots, j_k \geq 0} x_1^{j_1} \cdots x_k^{j_k} \]

\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}} \]

Example

If \( k = 2 \):

\[ h_0 = 1 \]

\[ h_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 \]

\[ h_{(2,1)} = h_2 h_1 = (x_1^2 + x_1 x_2 + x_2^2)(x_1 + x_2) = x_1^3 + 2x_1^2 x_2 + 2x_1 x_2^2 + x_2^3 \]

Theorem (Enumerative Combinatorics Vol. 2)

\[ \{h_\lambda \mid \lambda' \in Par_k\} \text{ is a basis for } S \text{ over } R \]
### Definition

Let $\ell(\lambda) \leq k$. Then

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j=1}^{\ell(\lambda)}$$

### Example

If $k = 2$:

$$s_{(2,1)} = \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} = h_2 h_1 - h_3 h_0$$

$$= (x_1^2 + x_1 x_2 + x_2^2)(x_1 + x_2) - (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3)$$

$$= x_1^2 x_2 + x_1 x_2^2$$

### Theorem (Enumerative Combinatorics Vol. 2)

$$\{s_\lambda \mid \lambda \in \text{Par}_k\}$$ is a basis for $S$ over $R$. 


Motivation

- \( R = \mathbb{Z} \)
  Cohomology ring of the Grassmannian,
  \[
  H^*(Gr(k, n)) \cong S/\langle h_{n-k+1}, \ldots, h_n \rangle
  \]

- \( R = \mathbb{Z}[q] \)
  Quantum cohomology ring of the Grassmannian,
  \[
  QH^*(Gr(k, n)) \cong S/\langle h_{n-k+1}, \ldots, h_{n-1}, h_n + (-1)^k q \rangle
  \]

Theorem (Postnikov)

\[ \{s_\lambda \mid \lambda \in Par_{k,n-k}\} \]

is a basis (over \( R \)) for both quotients; that is, every member of \( S \) can written uniquely as

some member of the ideal + \( \sum c_\lambda s_\lambda \), \( c_\lambda \in R, \lambda \in Par_{k,n-k} \)
Theorem (Grinberg)

Let $a_i \in R$. Then

$$\{s_\lambda \mid \lambda \in \text{Par}_{k,n-k}\}$$

is a basis for

$$S/\langle h_{n-k+1} - a_1, \ldots, h_n - a_k \rangle$$
Example

If \( k = 2, \ n = 4 \):

\[
\{ s_\emptyset, s(1), s(1,1), s(2), s(2,1), s(2,2) \}
\]

\[
= \{ 1, x_1 + x_2, x_1 x_2, x_1^2 + x_1 x_2 + x_2^2, x_1^2 x_2 + x_1 x_2^2, x_1^2 x_2^2 \}
\]

is a basis for

\[
S/\langle h_3 - a_1, h_4 - a_2 \rangle
\]

\[
= S/\langle x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 - a_1, x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 - a_2 \rangle
\]

For instance:

\[
x_1^4 + x_2^4 = -(x_1 + x_2)(h_3 - a_1) + 2(h_4 - a_2) + 2a_2 s_\emptyset - a_1 s(1)
\]
Power Sums

**Definition**

\[ p_i = x_1^i + \cdots + x_k^i \]

\[ p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell(\lambda)} \]

**Example**

If \( k = 2 \):

\[ p_3 = x_1^3 + x_2^3 \]

\[ p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2)(x_1 + x_2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 \]

**Theorem (Enumerative Combinatorics Vol. 2)**

*If \( \mathbb{Q} \subseteq R \), then \( \{ p_\lambda \mid \lambda' \in \text{Par}_k \} \) is a basis for \( S \) over \( R \).*
Quotients with $p_i$'s

**Theorem (W)**

Let $\mathbb{Q} \subseteq R$. Then

$$\{s_\lambda \mid \lambda \in Par_{k,n-k}\}$$

is a basis for

$$S/\langle p_{n-k+1}, \ldots, p_n \rangle$$

**Example**

If $k = 2$, $n = 4$:

$$\{s_\emptyset, s_{(1)}, s_{(1,1)}, s_{(2)}, s_{(2,1)}, s_{(2,2)}\}$$

$$= \{1, x_1 + x_2, x_1 x_2, x_1^2 + x_1 x_2 + x_2^2, x_1^2 x_2 + x_1 x_2^2, x_1^2 x_2^2\}$$

is a basis for

$$S/\langle p_3, p_4 \rangle = S/\langle x_1^3 + x_2^3, x_1^4 + x_2^4 \rangle$$

For instance:

$$x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 = (x_1 + x_2)p_3 + s_{(2,2)}$$
Future Directions

- $a_i \not\in R$ for both $h_i$’s and $p_i$’s.
- Writing Pieri’s rule in the basis of the quotients:
  \[
  h_i s_{\lambda} = \sum_{\mu/\lambda \text{ has } i \text{ squares across } i \text{ columns}} s_{\mu} = \sum_{\mu \in \text{Par}, n-k} c_{\lambda,\mu} s_{\mu}
  \]
- What is $S$ mod other ideals of symmetric polynomials?
- Which other ideals of $S$ give the same basis when modded out?
- $s_{\lambda}$ and $p_{\lambda}$ are related by representation theory; is this usable?
Thank You

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References

