Enumerating permutations with singleton double descent sets

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Definitions and Terminology

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Example.

Permutations in $\mathfrak{S}_5$ with descents (bolded) at indices 1, 3, and 4:
21**54**3, **4**1**53**2, **5**1**43**2, **3**1**54**2, **5**3**42**1, **4**3**52**1, **5**2**43**1, **4**2**53**1, **3**2**54**1
$\Rightarrow d(\{1, 3, 4\}; 5) = 9$.

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  - bounds on roots of $d(I;n)$ for certain sets I

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  - Recursion?
  - Is it a polynomial? If not, can we study asymptotics?

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- New goal: study permutations with singleton double descent sets

Singleton double descent sets (cont'd)

Theorem.

Let $I = \{m\}$ be a singleton set. Then we have

$$
\begin{aligned}
dd(I; n+1) = & \sum_{k=m+1}^{n} \binom{n}{k} \cdot dd(I; k) \cdot b_{n-k} \\
& + \binom{n}{m-2} \cdot dd(\emptyset; m-2) \cdot \big(dd(\emptyset; n-m+2) - b_{n-m+2} \big) \\
& + \sum_{k=0}^{m-4} \binom{n}{k} \cdot dd(\emptyset; k) \cdot c(\{m-1-k\}; n-k)
\end{aligned}
$$

where $c(I; n)$ denotes the number of permutations in $\mathfrak{S}_n$ with an initial ascent and double descent set I, and b_n denotes $c(\emptyset; n)$.

Patterns in data

Size of $\mathfrak{S}_n$	{2}	{3}	{4}	{5}	{6}	{7}	{8}	{9}	{10}	{11}	{12}	{13}
3	1	0	0	0	0	0	0	0	0	0	0	0
4	3	3	0	0	0	0	0	0	0	0	0	0
5	15	11	15	0	0	0	0	0	0	0	0	0
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Conjecture.

$\{dd(\{i\}; n)\}_{i=1}^{n}$ is asymptotically equidistributed. Namely, for fixed

$$0 < \alpha < \beta < 1, \quad \sum_{\alpha n < i < \beta n} dd(\{i\}; n) \sim (\beta - \alpha) \sum_{i=2}^{n-1} dd(\{i\}; n).$$

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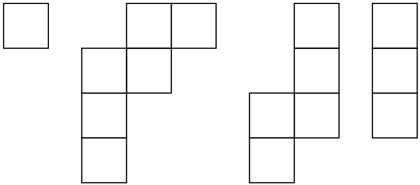
Given a fixed $n \in \mathbb{N}$, the numbers $dd(\{i\}; n)$ for $2 \le i < \left\lceil \frac{n}{2} \right\rceil$ follow a "down up down up" pattern. Namely, $dd(\{i\}; n) > dd(\{i+1\}; n)$ if i is even, and $dd(\{i\}; n) < dd(\{i+1\}; n)$ if i is odd.

Rim hooks: an approach for asymptotics of $dd(I;n)$

- Rim hooks:



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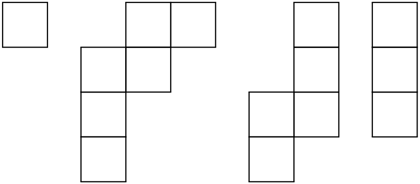

Definition.

A *rim hook tableau* *is a filling of a rim hook with the numbers* 1 *through* n*, where* n *is the length of the rim hook, satisfying the following rule: numbers must be arranged in the squares decreasing from bottom to top and increasing from left to right.*

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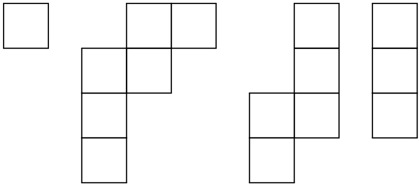

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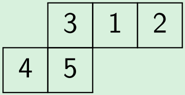
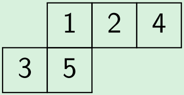


Valid tableau:



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- Permutation can be written as a rim hook tableau:

$632415 \in \mathfrak{S}_6$ corresponds to $\begin{ytableau} \none & 1 & 5 \\ 2 & 4 \\ 3 \\ 6 \end{ytableau}$, a tableau of $\begin{ytableau} \none & \; & \; \\ \; & \; \\ \; \\ \; \end{ytableau}$

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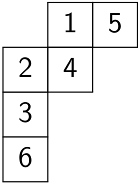
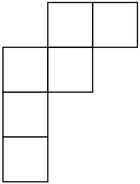
$632415 \in \mathfrak{S}_6$ corresponds to 

, a tableau of 

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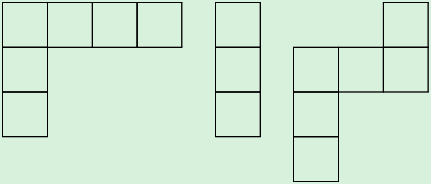
Example.

Double descent set $\{2\}$: 

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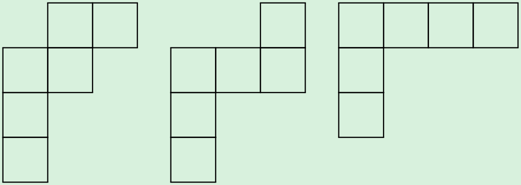

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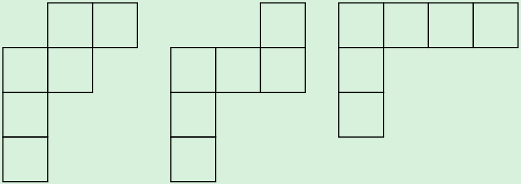

- This provides us with another way to express $dd(I;n)$:

$$dd(I;n) = \sum_{\mathfrak{r} \in \mathcal{R}_I(n)} f^{\mathfrak{r}}$$

where $f^{\mathfrak{r}}$ denotes the number of valid tableaux for a rim hook $\mathfrak{r}$.

Using rim hooks to estimate asymptotic growth

Theorem.

$\#\mathcal{R}_{\{m\}}(n) = F_{n-m}F_{m-1}$, where F_n is the nth Fibonacci number.

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- This gives us a Fibonacci "mutliplication table"

m \ n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
2	1	1	2	3	5	8	13	21	34
3	0	1	1	2	3	5	8	13	21
4	0	0	2	2	4	6	10	16	26
5	0	0	0	3	3	6	9	15	24
6	0	0	0	0	5	5	10	15	25
7	0	0	0	0	0	8	8	16	24
8	0	0	0	0	0	0	13	13	26

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Acknowledgements

I would like to thank the following:

- My mentor Pakawut Jiradilok
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- My parents

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