Maximal Extensions of Differential Posets

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Posets

Definition

A *partially ordered set*, or poset, is a set $P$ following the properties:

1. Certain elements $x, y \in P$ are relatable under the binary relation $\leq$.
2. If $x \leq y$ and $y \leq x$ then $x = y$.
3. If $x \leq y$, and $y \leq z$, then $x \leq z$.

Definition

In a poset $P$, an element $y$ covers an element $x$ if $x \leq y$, and there doesn’t exist a distinct element $z$ such that $x \leq z \leq y$. We write $x \lessgtr y$. 

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Posets can be represented in diagrams called *Hasse diagrams*, which appear like directed graphs. An arrow points from the smaller element to the larger element. In this example, the relation $\leq$ is equivalent to the inclusion relation $\in$.

**Figure:** The Hasse diagram of the set of subsets of $(x, y, z)$.
Example: Young’s lattice

Young’s lattice $Y$ is the poset of integer partitions, non-increasing ordered tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$. These are represented visually by upper-left justified sets of boxes.

An element of $Y$ is greater than another element of $Y$ if each row is at least as large as the equivalent row in the other element.

**Figure:** The Hasse diagram of Young’s lattice $Y$ up to rank 5.
Definition (Stanley)

An \( r \)-differential poset \( P \) is a poset satisfying the following:

1. \( P \) is locally finite, graded, and has a unique minimal element \( \hat{0} \).

2. For every two elements \( x, y \in P \), the number of elements covering both \( x \) and \( y \) is the same as the number of elements covered by both \( x \) and \( y \).

3. If an element \( x \in P \) covers \( d \) elements, then \( r + d \) elements cover \( x \).
Example: Young’s lattice

Young’s lattice $Y$ is a 1-differential poset. $Y^r$ is the $r$-differential poset form of Young’s lattice, which is the set $Y \times Y \times Y \times \ldots \times Y$. An $r$ times element in $Y^r$ is an ordered $r$-tuple of elements of $Y$. Stanley conjectured that $Y^r$ is the smallest $r$-differential poset by size.

Figure: The Hasse diagram of Young’s lattice $Y$ up to rank 5.
Example: Fibonacci Lattices

The $r$-Fibonacci poset, notated by $Z(r)$, is the differential poset defined by the reflection-extension construction.

Figure: The Hasse diagram of the Fibonacci lattice $Z(2)$, a 2-differential poset, up to rank 3.
Fibonacci Reflection-Extension Construction

**Figure**: Reflecting the element in row 0 onto row 2

**Figure**: Extending every element of row 1 twice
Fibonacci Reflection-Extension Construction

**Figure:** Reflecting row 1 onto row 3

**Figure:** Extending each element in row 2 twice
Enumerative identities

Definition

Define \( e(x) = \sum_{y \prec x} e(y) \). Equivalently, \( e(x) \) equals the number of paths up from \( \hat{O} \) to \( x \).

Many combinatorial and enumerative properties of Young’s lattice apply to differential posets in general, making them interesting to study.

For example, the Robinson-Schensted bijection applied to Young’s lattice tells us that \( \sum_{x \in P_n} e(x)^2 = n! \) for \( x \in Y \). However, \( \sum_{x \in P_n} e(x)^2 = r^n n! \) for any \( r \)-differential poset \( P \).
Enumerative Identities Example: Young’s Lattice

The $e(x)$’s for the elements of row 5 of $Y$ are 1, 4, 5, 6, 5, 4, 1. Therefore,

$$\sum_{x \in Y_5} e(x)^2 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120 = 1^5 \times 5!$$
The $e(x)$'s for the elements of row 3 of $Z(2)$, the 2-differential Fibonacci poset, are 1, 1, 1, 4, 1, 2, 2, 1, 4, 1, 1, 1. Therefore, $\sum_{x \in Z(2)_3} e(x)^2 = 1 + 1 + 1 + 16 + 1 + 4 + 4 + 1 + 16 + 1 + 1 + 1 = 48 = 2^3 \times 3!$
Definition

The *rank* of an element in a differential poset is the number of steps taken to reach $\hat{0}$.

Definition

Define $p_n$ to be the number of elements in rank $n$ of a differential poset $P$. 
Definition

The \textit{$r$-Fibonacci numbers} $F_r(x)$ satisfy $F_r(0) = 1$, $F_r(1) = r$, and
$F_r(x) = r \cdot F_r(x - 1) + F_r(x - 2)$.

Note that if $r = 1$, we just get the regular Fibonacci numbers. Since the reflection-extension construction of the $r$-Fibonacci poset consists of reflecting the second to last row, and extending $r$ elements per element in the last row, the rank sizes of the $r$-Fibonacci poset are indeed the $r$-Fibonacci numbers.
Theorem (Byrnes 2012)

For any $r$-differential poset $P$ we have:

$$p_n \leq r \sum_{i=0}^{n} p_i - (p_{n-1} - 1),$$

and therefore $p_n \leq F_r(n)$.

The $r$-Fibonacci numbers satisfy Byrnes’ inequality, and some induction is sufficient to show $F_r(n)$ is the maximum rank size of rank $n$. 
Uniqueness of the maximal extension

Now, we move on to new results:

**Theorem**

In a differential poset $P$, if $p_n = F_r(n)$ for some particular $n$, then the partial $r$-differential poset $P_{[0,n]}$ is isomorphic to the $r$-Fibonacci poset $Z(r)_{[0,n]}$. 
Future directions

From the fact that the Fibonacci poset is the largest differential poset, Byrnes hypothesized that the reflection-extension construction will also give the maximal extension for any partial differential poset. Equivalently:

**Conjecture (Byrnes 2012)**

*In a differential poset,*

\[ p_n \leq rp_{n-1} + p_{n-1} \]
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Are there any questions?