# COMMON KNOWLEDGE:

## *GAMES OF LOGIC*

Matvey Borodin, Aidan Duncan, Joshua Guo, Kunal Kapoor, Anuj Sakarda, Jerry Tan, Armaan Tipirneni, Max Xu, and Kevin Zhao
Mentored by Tanya Khovanova

Presented at the 2019 MIT PRIMES conference

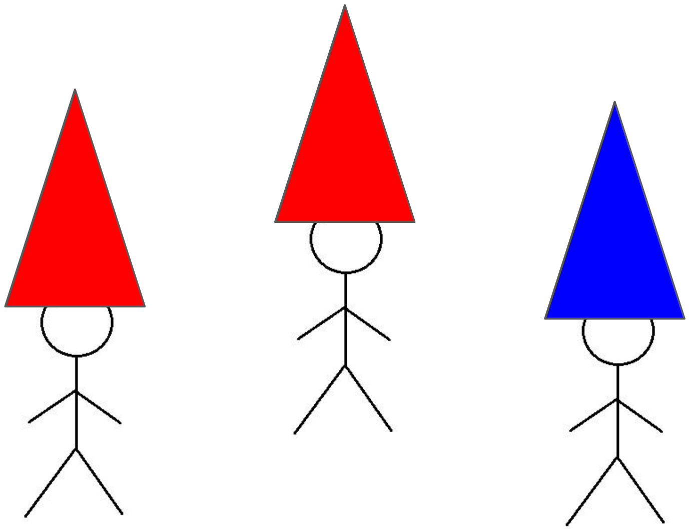
# Let’s start with a classical hat puzzle.

- The characters:
- The Sultan

The Three Sages

# How does this game work?

- The sultan shows everyone his 3 red hats and 2 blue hats
- 1 hat on each sage. They cannot see their own hats.
- The sages can see each other’s hats.
- **Game:** Every minute, the sages say whether or not they know the color of their own hat
- Once someone says yes, the game is over.

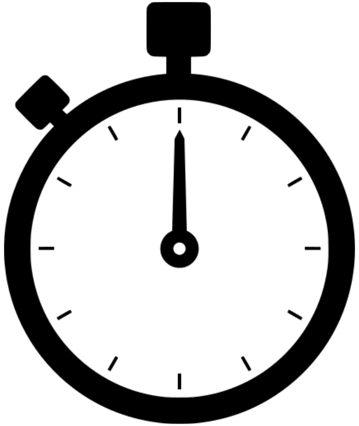


# What happens in this puzzle?

- **First minute:** If a sage sees 2 blue hats, they know their hat must be red
- **Second minute:** if no sage said yes the first round, then every sage knows that there are at least 2 red hats.
- If a sage sees one blue hat, then they know that their hat is red.
- **Third minute:** no one has said yes, there must be 3 red hats.
- They all know this and say that they know their hat color.

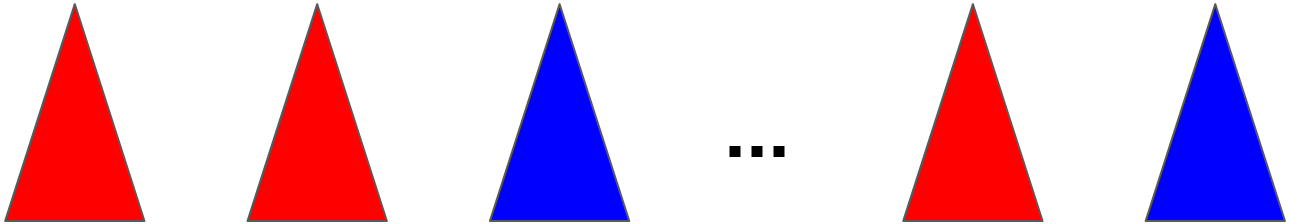
# Some things to note

- If they didn't agree to talk once every minute, would the game change?
- How did knowing that there were 2 blue hats and 3 red hats help the sages?



# Generalization with more sages, more hats, and more opportunities for people to be decapitated

- N sages, N red hats, N-1 blue hats
- First Round: If sage sees N-1 blue hats, they are red
- Second Round: They all know there are at least 2 red hats. If anyone sees N-2 blue hats, they know they have a red hat.
- $K^{th}$ Round ($1<K<N+1$): They all know that there are at least K red hats. If anyone sees N-K blue hats, they know they have a red.
- Done in N rounds



# Some Important Distinctions

COMMON KNOWLEDGE

VS

MUTUAL KNOWLEDGE

# Some More Important Distinctions

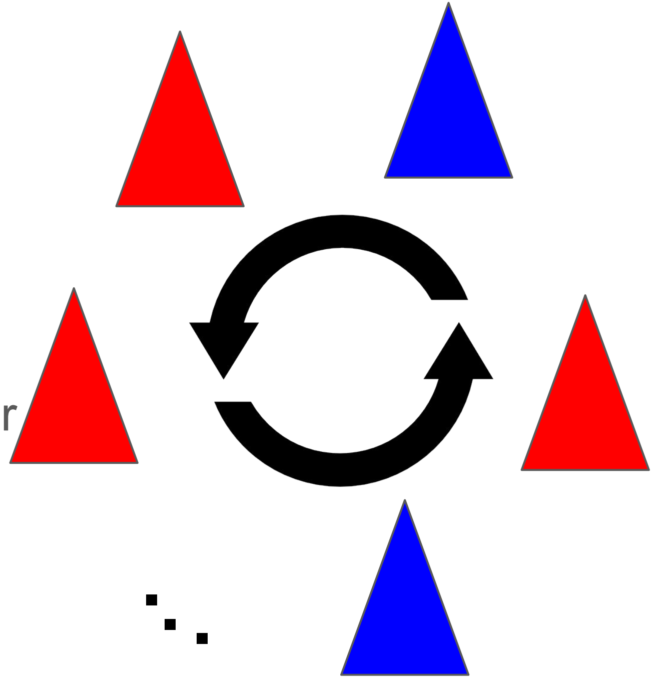
Circular Game

VS

Simultaneous Game

# Hats: the circular edition!!

- Sages who see red hats after them say no.
- The last sage with a red hat says yes.
- All the following blues say yes.
- Round 2: Nothing new happens
- Everyone before the last red never figures out their color



# Alice the Blind Person

- Now we have a circular game with Alice, a blind sage
- It is common knowledge that there exists a red hat and that Alice is blind
- *N* people, each person has a red or blue hat
- Question: How many people are guaranteed to figure out their color?

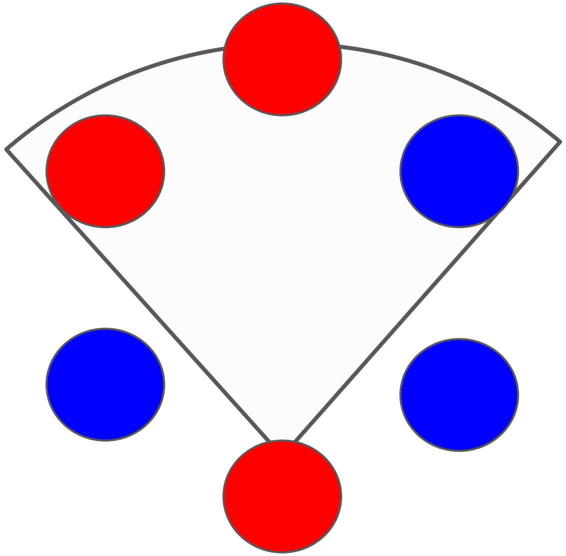
# Alice the Blind Person - What will happen?

- If someone before Alice, say Max, is the last red, he will say yes and everyone after him, knows they are blue.
- If no one before Alice says yes, and Alice is the last to speak, she knows she must be red.
- If Alice is blue the game proceeds as normal. The last red will say yes, and in round 2 Alice knows she is blue
- If Alice is red, no one after her knows anything. On round 2, Alice knows she is red.
- Ironically, the blind person is the only person to always know his/her color.

# Far-sighted People

Definition: Far-sighted people can see the hats of everyone **except** their immediate neighbors.

- 9 far-sighted people, simultaneous game
- Announcement: Exactly three red hats
- In the first round everyone says NO
- What will they say in the second round?



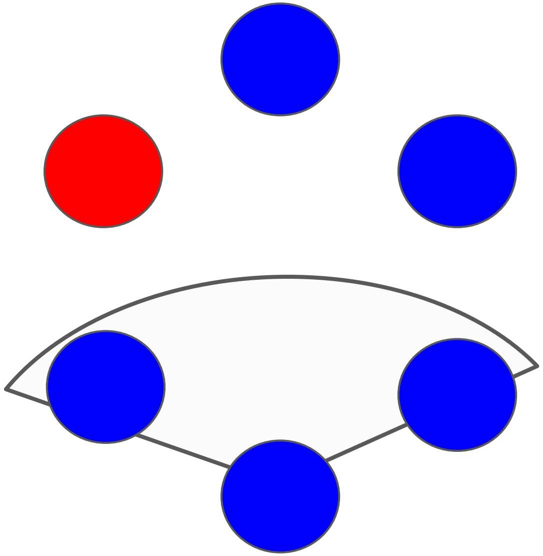
# Far-sighted People - What will happen?

- If anyone sees 3 reds they know they are blue
- No one said yes in round 1, so no one saw 3 reds
- Thus everyone is either red or next to a red
- So we can’t have 3 blues in a row
- The configuration must be R-B-B-R-B-B-R-B-B
- Every person knows this and sees 2 reds, so everyone says YES in the second round

# Nearsighted People

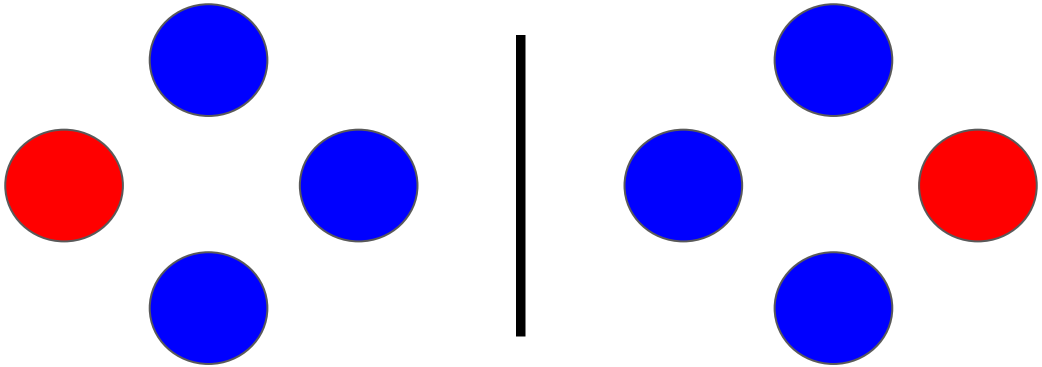
Definition: Nearsighted people only see the hats of their immediate neighbors

- N nearsighted people, simultaneous game
- Announcement: Exactly one red hat
- Some people never figured out their color
- What is N?



# Nearsighted People - What will happen?

- If N ≤ 3 everyone figures out their color
- If N > 3, the 2 neighbors of the red hat say YES in round 1
- In round 2, if N > 4, everyone knows the red is between the 2 that said YES, so everyone says YES round 2
- The exception is when N = 4, since they don't know which side has the red hat



# From colors to numbers

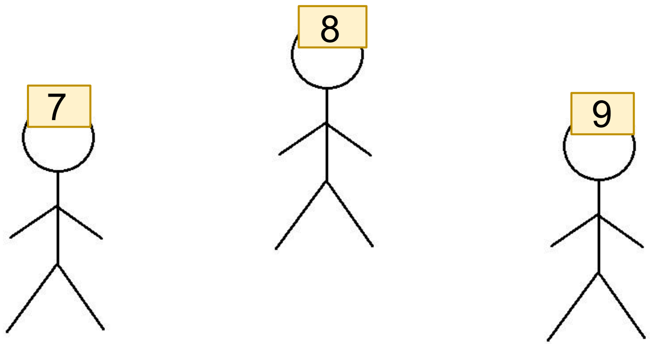
In the next puzzle

- There has been a shortage of hats
- Now, sticky notes are used
- On the sticky notes numbers are written
- The handwriting is good enough to be read by all of the people


(Example sticky note)

# *N* people and *N* sticky notes

- Simultaneous game
- *N* people, *N* sticky notes
- Each sticky note has a number
- The sages are told that their numbers are consecutive non-negative integers



# *N* people and *N* sticky notes - What will happen?

Case 1: The maximum = the range

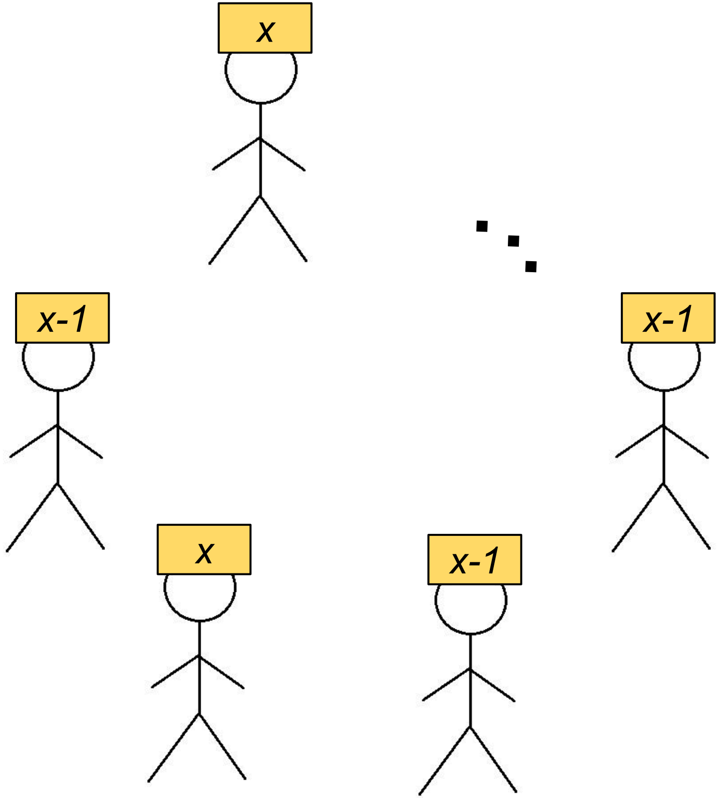
- First round: everyone except the 0 figures out their number
- Second round: person with 0 figures out their number

Case 2: The maximum ≠ the range

- First round: people who aren’t the maximum or minimum figure out their numbers
- In the second round the max and min figure out their numbers

# More Sticky Notes

- $N$ non-negative integers
- They are told that the difference between the maximum and minimum numbers on their foreheads is 1
- The game is simultaneous



# More Sticky Notes - What will happen?

- $P$ = number of people with the maximum, $M$ = max
- First round people figure out their number if $P=M=1$
- Second round, if there are 2 ones they both say yes
- If there are $r$ ones, on round $r$ they say yes
- On round $N-1$ there are no zeros, if someone sees $N-1$ ones, they must have 2
- After round $2(N-1)$ they realize there are no ones
- After round $m(N-1)$ where $m$ is the minimum, everyone figures out there are no numbers less than $m$.
- Game is over in round $m(N-1)+P$

# Acknowledgements

Thanks to:

- Tanya for being a great mentor
- The PRIMES Step program for organizing this

Any Questions?