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Definition

A module $M$ over a ring $R$ is a set of elements that can be added together and multiplied by a scalar $\lambda \in R$. An algebra is a module equipped with a product between elements in $M$ that outputs another element in $M$.

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- The set of $n \times n$ square matrices in $R$
The Shuffle Algebra

For a ring $R$, the shuffle algebra $A^R$ is the subset of the set of symmetric rational functions in arbitrarily many variables with coefficients in $R$, generated by 1 variable functions.

The **shuffle product** takes a function in $k$ variables and a function in $l$ variables and “shuffles” their variables to get a function in $k + l$ variables:

$$F(a, b) \ast G(c, d) = F(a, b)G(c, d) + F(a, c)G(b, d) + F(a, d)G(b, c) + F(b, c)G(a, d) + F(b, d)G(a, c) + F(c, d)G(a, b).$$

Frank Wang

The Shuffle Algebra of the Hilbert Scheme of Points of the Plane

Montgomery High School
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The Integral Shuffle Algebra

Definition

The **integral shuffle algebra** is a subset of \( \bigoplus_{k \geq 0} \text{Sym}_R(z_1, \ldots, z_k) \) and is the shuffle algebra over the ring \( R = \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}] \).

The shuffle product is

\[
P(z_1, \ldots, z_k) \ast Q(z_1, \ldots, z_l) = \frac{1}{k! l!} \sum_{\text{sym}} P(z_1, \ldots, z_k)Q(z_{k+1}, \ldots, z_{k+l}) \prod_{1 \leq i < j \leq k+l} \frac{(z_i - q_1 q_2 z_j)(z_j - q_1 z_i)(z_j - q_2 z_i)}{z_i - z_j}.
\]

We want to find conditions to determine whether a given symmetric rational function is in the integral shuffle algebra.
The fractional shuffle algebra is the shuffle algebra over the ring \( K = \mathbb{C}(q_1, q_2) \) with the same shuffle product as the integral shuffle algebra.

**Theorem (Negut, 2014)**

A symmetric rational function \( p(z_1, \ldots, z_k) \) is in the fractional shuffle algebra if and only if it is a Laurent polynomial (\( p \in K[z_1^{\pm 1}, \ldots, z_k^{\pm 1}] \)) and it satisfies the wheel conditions:

\[
p(z_1, q_1 z_1, q_1 q_2 z_1, z_4, z_5, \ldots, z_k) = p(z_1, q_2 z_1, q_1 q_2 z_1, z_4, z_5, \ldots, z_k) = 0.
\]

These conditions are necessary but not sufficient for the integral shuffle algebra.
Ideals

Definition

An **ideal** of a ring $R$ is a subset of $R$ that is closed under addition and multiplication by elements of $R$. An ideal can be written as $(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are the generators of the ideal.

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Ideal form of wheel conditions: $p(z_1, q_1z_1, q_1q_2z_1, \ldots) = 0$ if and only if $p \in (q_1q_2z_1 - z_3, q_1z_1 - z_2, q_2z_2 - z_3)$ of the ring of Laurent polynomials.
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Ideals can also be thought of as \( R \)-modules that are contained in \( R \).
Definition

A quotient $R/I$ of a ring $R$ by an ideal $I$ is the ring of equivalence classes in the ring where two elements $a$ and $b$ are equivalent if $a - b \in I$.

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- $\mathbb{Z}_2 = \mathbb{Z}/(2)$ is the integers mod 2
- $R[x]/(x) = R$
- $R[x]/(x^2) = \{ax + b \mid a, b \in R\}$
The Hilbert scheme $\text{Hilb}_n$ of $n$ points in the plane is the set of ideals $I \subseteq \mathbb{C}[x, y]$ such that the dimension of $\mathbb{C}[x, y]/I$ as a vector space over $\mathbb{C}$ is $n$.

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The Hilbert scheme $\text{Hilb}_n$ of $n$ points in the plane is the set of ideals $I \subset \mathbb{C}[x, y]$ such that the dimension of $\mathbb{C}[x, y]/I$ as a vector space over $\mathbb{C}$ is $n$.

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$$\mathbb{C}[x, y]/(x, y) = \mathbb{C} \Rightarrow (x, y) \in \text{Hilb}_1$$
The Hilbert Scheme of Points in the Plane

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Examples:

- $\mathbb{C}[x, y]/(x, y) = \mathbb{C} \Rightarrow (x, y) \in \text{Hilb}_1$
- $\mathbb{C}[x, y]/(x^2, xy, y^2) = \{ax + by + c\} \Rightarrow (x^2, xy, y^2) \in \text{Hilb}_3$
- $\mathbb{C}[x, y]/(x) = \{a + by + cy^2 + \ldots\}$ so $(x) \notin \text{Hilb}_n$ for any $n$
Consider the equivariant $K$-theory group $K^T(Hilb_n)$ of the Hilbert scheme and let

$$L_R = \bigoplus_{n \geq 0} K^T(Hilb_n), \quad L_K = L_R \otimes_R K$$

where $R = \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}]$ and $K = \mathbb{C}(q_1, q_2)$.

Then $L_R$ is a module over the integral shuffle algebra and $L_K$ is a module over the fractional shuffle algebra.
Theorem

Let $A_k^R$ be the subset of the integral shuffle algebra consisting of functions in $k$ variables. Then the following hold:

$A_k^R$ is an ideal of $R[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$ for all $k$.

$A_2^R$ is the ideal $(z_1 * z_1^0, z_1^0 * z_1^0)$ of $R[z_1^{\pm 1}, z_2^{\pm 1}]$.

As an ideal of $R[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, $A_3^R$ is generated by the elements $z_1^{d_1} * z_1^{d_2} * z_1^0$ for $0 \leq d_1 \leq 2$, $0 \leq d_2 \leq 1$. 
Ongoing Work

Recall the ideal form of the wheel conditions:

\[ p \in (q_1q_2z_1 - z_3, q_1z_1 - z_2, q_2z_2 - z_3), \]

\[ p \in (q_1q_2z_1 - z_3, q_2z_1 - z_2, q_1z_2 - z_3). \]

We create a similar condition from the generators of \( A_2^R \):

**Theorem**

\( A_k^R \) is contained in the ideal

\[ (z_1 \ast z_1^0, z_1^0 \ast z_1^0) \]

of \( R[z_1^{\pm 1}, \ldots, z_k^{\pm 1}] \) for \( k \geq 2 \).
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• Use this to prove another general condition.
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- Use this to prove another general condition.
- Find a computer algebra system/algorithm to calculate ideals of $\mathbb{R}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, as hand calculations are not feasible:

$$P(z_1, \ldots, z_k) \ast Q(z_1, \ldots, z_l) =$$

$$\frac{1}{k!!l!!} \sum_{\text{sym}} P(z_1, \ldots, z_k) Q(z_{k+1}, \ldots, z_{k+l}) \prod_{1 \leq i < k \atop k < j \leq k+l} \frac{(z_i - q_1 q_2 z_j)(z_j - q_1 z_i)(z_j - q_2 z_i)}{z_i - z_j}.$$
Plans for Future Work

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- Try to prove that conditions are sufficient or find new ways to generate conditions that can be proven.
Difficulties of the Project: $z_1^0 \ast z_1^0 \ast z_1^0$

$$z_1^0 \ast z_1^0 \ast z_1^0 = 6q_1^3 q_2 z_1^2 + (-3q_1^2 q_2 - 3q_1 q_2^2 - 3q_1 q_2 + 6q_1^3 q_2^3 - 3q_1^3 q_2^2 - 3q_1^3 q_2^4) z_1^2 z_2 + 6q_1^3 q_2 z_1^2 z_2 + 6q_1^3 q_2 z_1^2 z_2 + 6q_1^3 q_2 z_1^2 z_2$$

$$+ (-3q_1^2 q_2 - 3q_1^3 q_2 + 6q_1^3 q_2^3 - 3q_1^3 q_2^4) z_1^2 z_2 z_3 + \cdots$$

Frank Wang
Montgomery High School

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