Extractable Tree-Statistics from the Quasisymmetric Bernardi Polynomial

Lucy Cai, Espen Slettnes, Jeremy Zhou

Mentor: Duncan Levear
Advisor: Professor Olivier Bernardi

Ninth Annual MIT PRIMES Conference

May 18-19, 2019
Introduction & Definitions

Findings

Consequences & Open Questions
Introduction & Definitions

Let \( D = (V, A) \) be a directed graph.

**Notation**

Let \( f : V \to \mathbb{N} \) be a coloring of the vertices.

- **Ascents** are the elements of \( f^>_A := \{(u, v) \in A | f(v) > f(u)\} \).
- **Descents** are the elements of \( f^<_A := \{(u, v) \in A | f(v) < f(u)\} \).

The **Quasisymmetric Bernardi polynomial (QSBP)** is the formal power series

\[
B_D(x; y, z) = \sum_{f : V \to \mathbb{N}} \left( \prod_{v \in V} x_{f(v)} \right) y^{|f^>_A|} z^{|f^<_A|}
\]

for indeterminates \((x_i)_{i \in \mathbb{N}}\).

- The QSBP is the generating function over all colorings counted by the number of ascents \((y)\), descents \((z)\), and uses of each color \((x_i)\), respectively.
- Motivated by Richard Stanley’s Tutte symmetric function.
\[ B_D(x; y, z) = B((x_1, x_2, \ldots); y, z) = \sum_{f: V \to \mathbb{N}} \left( \prod_{v \in V} x_{f(v)} \right) y^{|f^>_A|} z^{|f^<_A|} \]

\[ = \ldots + x_1 x_3^2 x_4 x_5^2 y^8 z^2 + \ldots \]
## Introduction & Definitions

### Open Question (Stanley, 1995)

Does the Tutte symmetric function distinguish all non-isomorphic trees?

### Open Question (Awan & Bernardi, 2018): Analogue for Digraphs

Does the QSBP distinguish all non-isomorphic rooted trees?

- We want to find information about rooted trees from their QSBP.

### Definition

- A *tree-statistic* is a function on the set of rooted trees.
- Tree-statistic $S$ is *extractable* if for all rooted trees $T_1, T_2$ where $B_{T_1}(x; y, z) = B_{T_2}(x; y, z)$, it follows that $S(T_1) = S(T_2)$.

- We want extractable tree-statistics $S$ because if $S(T_1) \neq S(T_2)$, $B_{T_1}(x; y, z) \neq B_{T_2}(x; y, z)$. 
Introduction & Definitions

Layer 1, Co-height 0
Layer 2, Co-height 1
Layer 3, Co-height 2
Layer 4, Co-height 3
Layer 5, Co-height 4
Layer 6, Co-height 5

\[ h_v = 2 \]

\[ w_v = |S_v| = 8 \]
Introduction & Definitions

Definition

- Rooted tree $T = (V, A)$, vertex $v \in V$, given $(a_u)_{u \in V}$ a vertex-statistic, we define $P^a_v$ to denote the multiset

$$\{a_u \mid u \in S_v\}$$

- $P^a_T$ means $P^a_{v_T}$, where $v_T$ is the root of $T$.

Definition

- Co-height profile of a tree $T$ is $P^h_T$,
- Weight profile is $P^w_T$.

Definition

We say e.g., the co-height profile profile of a tree $T$ is $P^{ph}_T$.
$h_v = 2$

$P_v^h = \{2, 3, 3, 3, 4, 4, 4, 5\}$

$P_T^h = \{0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5\}$

$P_T^{P_h} = \{2, 3, 3, 3, 4, 4, 4, 5\},$
\{3, 4\}, \{3\}, \{3, 4, 4, 5\},$
\{4\}, \{4, 5\}, \{4\},$
\{5\}\}$

$P_T^{P_v} = \{0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5\},$
\{1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5\}, \{1\}, \{1, 2, 3, 3, 4\},$
\{2\}, \{2, 3, 3, 3, 4, 4, 4, 5\}, \{2\}, \{2, 3, 3, 4\},$
\{3, 4\}, \{3\}, \{3, 4, 4, 5\}, \{3, 4\}, \{3\},$
\{4\}, \{4, 5\}, \{4\}, \{4\}$
\{5\}\}$
Agenda

1. Introduction & Definitions
2. Findings
3. Consequences & Open Questions
Findings

Theorem

For a rooted tree $T$, the coheight profile $P^h_T$ is extractable.

$$x_1^1 x_2^3 x_3^4 x_4^1 y^1 |A| z^0 \rightarrow P^h_T = \{0, 1, 1, 1, 2, 2, 2, 2, 3\}$$
Findings

Theorem

The coheight profile profile $P^h_T$ is extractable.

Coheight profile of $v$: $P^h_v = \{1, 2, 2, 2, 3\}$

Coheight profile profile: $P^h_T = \{0, 1, 1, 1, 2, 2, 2, 2, 3\}, \{1, 2\}, \{1\}, \{1, 2, 2, 2, 3\}, \{2\}, \{2\}, \{2\}, \{2, 3\}, \{3\}$
Findings

$T_1$

$P^h_T = \{0, 1, 1, 2, 2\}$

$T_2$

$P^h_T = \{0, 1, 1, 2, 2\}$
Findings

$T_1$

$P^h_T = \{0, 1, 1, 2, 2\}$

$P^h_T = \{1, 2, 2\}$

$P^h_T = \{2\}$

$P^h_T = \{2\}$

$T_2$

$P^h_T = \{0, 1, 1, 2, 2\}$

$P^h_T = \{1\}$

$P^h_T = \{1, 2\}$

$P^h_T = \{2\}$

$P^h_T = \{2\}$
Agenda

1. Introduction & Definitions
2. Findings
3. Consequences & Open Questions
Consequences

Corollary

We can extract:

1. the number of leaves in each layer.
2. the outdegree distribution of each layer.
3. the *height profile* of each layer.
4. the *weight profile* of each layer.
Consequences: 2-Caterpillars

Definition
An $n$-caterpillar tree is a rooted directed tree in which all vertices are at most distance $n$ from a central path.

Corollary
We can distinguish between all 2-caterpillar trees.
The previous theorem cannot distinguish these two trees:

$T_1$

$T_2$

By computer evidence, they do not have the same QSBP, but they have the same coheight profile profiles.
Open Questions

Main Question
Does the QSBP distinguish between all rooted directed trees?

1. Does the QSBP distinguish between all rooted directed trees with 4 layers?
2. For what $n > 2$ can the QSBP distinguish between all $n$-caterpillars?
3. Under what conditions will two trees $T_1, T_2$ share the same coheight profile?
Acknowledgements

We wish to thank

- Our mentor Duncan Levear of Brandeis University, and his advisor Professor Olivier Bernardi, for valuable guidance.
- Professor Pavel Etingof, Dr. Tanya Khovanova, and Dr. Slava Gerovitch for organizing MIT PRIMES.
- The MIT Math Department.