Sampling over tilings of the plane: A computational approach against political gerrymandering

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Motivation

What is political gerrymandering?

https://www.fairvote.org/new_poll_everybody_hates_gerrymandering
Motivation

What is political gerrymandering?

Gerrymandering is the practice of drawing boundaries of electoral districts in a way that gives one party an unfair advantage over the others.

https://www.fairvote.org/new_poll_everybody_hates_gerrymandering
Motivation

Figure 1: Map of Virginia’s electoral districts, 2011

Uniform Sampling

We can think of electoral redistricting as tiling a grid of cells.

How can we determine if a tiling is fair?
- Sampling repeatedly for statistics — Law of Large Numbers

How do we sample tilings of the plane uniformly at random?
- Distribution is very complicated
- Potential solution: use Markov chains
Markov chains

Definition

A sequence of random variables \((X_0, X_1, \ldots)\) is a Markov chain with state space \(\Omega\) and transition matrix \(P\) if for all states \(x, y \in \Omega\), we have

\[
Pr\{X_{t+1} = y \mid X_t = x\} = P(x, y).
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Example (continued)

We can repeatedly multiply by $P$ to get

$$
\mu_t = \mu_0 P^t = \begin{bmatrix} \pi_A & \pi_B \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}^t
$$
Our Proposed Markov chain

State space: tiling $n \times n$ with tiles of area $n$

- Simple case: rectangular tilings
Our Proposed Markov chain

Transitions: merge and re-split

What is this Markov chain doing?
Stationary Distributions

Definition

A distribution $\pi$ is called a *stationary distribution* of the Markov chain if it satisfies the following property:

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Stationary Distributions

**Definition**

A distribution $\pi$ is called a *stationary distribution* of the Markov chain if it satisfies the following property:

$$ \pi = \pi P. $$

- Fair tilings should be natural — “compact”
- Ideally, stationary distribution of our chain favors “compact” tiling

Irreducibility

**Definition**

A chain $P$ is called *irreducible* if for any two states $x, y \in \Omega$, there exists an integer $t$ (possibly depending on $x$ and $y$) such that

$$P^t(x, y) > 0.$$
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Definition

Let $T(x) := \{ t \geq 1: P^t(x, x) > 0 \}$. The chain $P$ is aperiodic if for all states $x \in \Omega$, the $gcd(T(x))$ is 1.
Aperiodicity

Definition

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Convergence Theorem

**Theorem (Convergence Theorem)**

Suppose that $P$ is irreducible and aperiodic, with stationary distribution $\pi$. Then $P^t\pi_0 \xrightarrow{t \to \infty} \pi$ for any $\pi_0$.

Why is this important?

- Want to use Markov Chain for sampling
- Running Markov chain algorithmically — want to draw from stationary distribution
  - If we can show irreducible and aperiodic, then this is possible
- Allows us to compute statistics like averages
Mixing Time of a Markov chain

How long should we run this Markov chain?

- e.g. the larger the diameter, the longer this takes
- Vertices = states, edges = nonzero probabilities

https://en.wikipedia.org/wiki/Betweenness_centrality
Diameter of our Markov chain

Conjecture: diameter = $O(n \log n)$

- Diameter over rectangular tilings explains mixing time
- Count tilings to understand how large graph on state space is
Future Work

- Prove irreducibility
- Calculate diameter/bottleneck/mixing time
- Understand stationary distribution of current Markov chain — need to run experiments
- Does our Markov chain favor “compact” tilings?
  - If not, can weigh transitions in Markov chain
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