On Quasirandom Permutations

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Permutations

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- Elements of the symmetric group S_n

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- Elements of the symmetric group S_n
- Denoted $(4, 2, 3, 1)$, or 4231 for short.

Introduction

Randomness:

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What kinds of properties do random permutations have?

Patterns

Permutation density helps define quasirandomness.

- A sequence of distinct integers $a_1, a_2, \ldots, a_k$ is *order-isomorphic* to a permutation $\pi \in S_k$ if they are ordered the same.
- Example: The sequences 295 and 396 are both order-isomorphic to the permutation 132, but 123 and 483 are not.

This helps study patterns of subsequences within permutations.

Permutation Density

Definition

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Example

For a random permutation $\tau \in S_n$ and any fixed permutation π (of which there are $|\pi|!$ of a given length),

$$\mathbb{E}\ t(\pi, \tau) = \frac{1}{|\pi|!}.$$

Permutation Sequences

Sequences of permutations $\{\tau_j\}$ are called convergent if as $j \to \infty$,

- Lengths $|\tau_j| \to \infty$
- Sequences of densities $t(\pi, \tau_j)$ converge, for any permutation π

Advantage: we can ignore higher-order terms, e.g. $\binom{n}{2}/n^2 = 1/2 + o(1)$.

Quasirandomness

Behavior of random permutations with respect to subpermutation densities:

Definition

A convergent sequence of permutations $\{\tau_j\}$ is called *quasirandom* if for every permutation π,

$$\lim_{j\to\infty} t(\pi, \tau_j) = \frac{1}{|\pi|!}.$$

Permutation Limits

Convergent sequences of permutations can be characterized by corresponding limit objects known as *permutons*.

Definition

A *permuton* is a probability measure μ on the unit square $[0, 1]^2$ with *uniform marginals*, meaning the individual distributions of the two coordinates are uniform.

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Theorem

For every convergent sequence of permutations $\{\tau_j\}$, there exists a corresponding permuton μ with the same densities of pattern permutations.

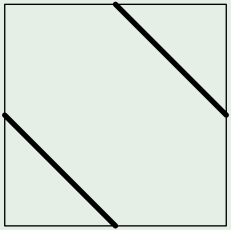
Symmetry

Definition

A permuton μ is called *k-symmetric* if sampling permutations of length k from μ is uniformly random, i.e. the densities are all $1/k!$.

Example

The following permuton is 2-symmetric:



Inflation

Are there non-uniform three-symmetric permutons?

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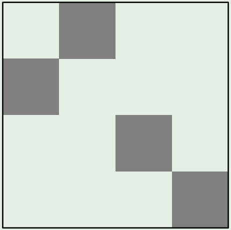
Are there non-uniform three-symmetric permutons? **Yes.**

Definition

A permutation τ of length n is called k-inflatable if the permuton μ corresponding to mass uniformly distributed along the graph of the permutation on an $n \times n$ grid is k-symmetric.

Example

The inflation of 3421 is the following permuton:



Densities in Inflations

Definition

Let $B(\pi)$ be the set of all pairs (b, σ) corresponding to ways of dividing π into consecutive blocks of size $b_1, b_2, \ldots, b_k$, with relative ordering σ.

Theorem

The density of π in the inflation of τ is

$$t(\pi, \textit{inflated}(\tau)) = \frac{|\pi|!}{|\tau|^{|\pi|}} \sum_{(b,\sigma)\in B(\pi)} \left[\binom{|\tau|}{|\sigma|} t(\sigma, \tau) \cdot \prod_{x \in b} \frac{1}{x!^2} \right].$$

3-Inflatable Permutations

Theorem

A permutation $\tau \in S_n$ *is 3-inflatable if and only if* $t(12,\tau) = \frac{1}{2}$ *and*

$$t(123,\tau) = t(321,\tau) = \frac{2n-7}{12(n-2)},$$

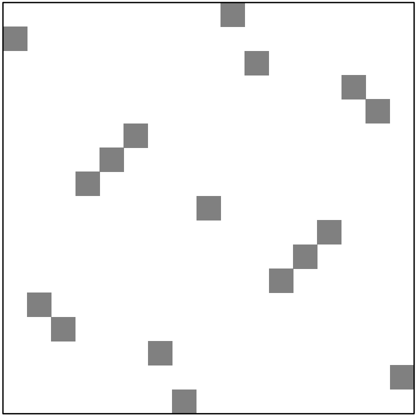
$$t(132,\tau) = t(213,\tau) = t(231,\tau) = t(312,\tau) = \frac{4n-5}{24(n-2)}.$$

Corollary

$$n \equiv 0, 1, 17, 64, 80, 81 \pmod{144}.$$

There are 750 rotationally symmetric permutations of size 17 that are 3-inflatable, e.g. g54abc319hf678ed2.

3-Inflatable Example



g54abc319hf678ed2

Four-Symmetry

Quasirandomness is only dependent on densities of four-point permutations.

Theorem (Kral and Pikhurko, 2013)

Any four-symmetric permuton μ is the uniform probability measure.

Better Condition for Quasirandomness

Theorem

Let $S = S_4 \setminus D$ for some *equi-dense* $D \subseteq S_4$. If a convergent sequence $\{\tau_j\}$ of permutations satisfies $t(\pi, \tau_j) = 1/4! + o(1)$ for every $\pi \in S$, then it is quasirandom.

Equi-dense subset of size 8 $\implies$ better condition for quasirandomness, only requiring densities of 16/24 four-point permutations.

Equi-Dense

$$\int F(X,Y)^2 \, \mathsf{d}V = \int F(X,Y)XY \, \mathsf{d}V = \int F(x,y)^2 \, \mathsf{d}v = \frac{1}{9}.$$

Equi-Dense

$$\underbrace{\int F(X,Y)^2 \, \mathrm{d}V}_{A} = \underbrace{\int F(X,Y)XY \, \mathrm{d}V}_{B} = \underbrace{\int F(x,y)^2 \, \mathrm{d}v}_{C} = \frac{1}{9}.$$

Permutation	A	B	C
1234, 2134	1/3	1/4	1/6
1243, 2143	1/6	1/6	1/6
1324, 2314, 3124, 3214	1/4	1/4	1/6
1342, 1423, 2341, 2413, 3142, 3241, 4123, 4213	1/12	1/12	1/12
1432, 2431, 4132, 4231	0/1	1/24	1/12
3412, 3421, 4312, 4321	0/1	0/1	1/12

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1324, 2314, 3124, 3214	1/4	1/4	1/6
1342, 1423, 2341, 2413, 3142, 3241, 4123, 4213	1/12	1/12	1/12
1432, 2431, 4132, 4231	0/1	1/24	1/12
3412, 3421, 4312, 4321	0/1	0/1	1/12

We call a group of permutations *equi-dense* if each element of the group has the same coefficient in the expression of each of these integrals as a linear combination of densities of permutations in S_4.

Future Work

- Find a complete list of minimal subsets of permutations for which having density $1/24$ is a sufficient condition for quasirandomness.
- Better understand inflatable permutations, including examples with inflated density $1/24$ of some $\pi \in S_4$.
- Use the technique of flag algebras to generate bounds on densities given those of a certain subset.

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References

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