On Quasirandom Permutations

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An ordering of the elements of a set
Permutations

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- Elements of the symmetric group $S_n$
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- Elements of the symmetric group $S_n$
- Denoted $(4, 2, 3, 1)$, or $4231$ for short.
Randomness:

- Cryptography
- Unbiased ordering of products
- Selection of election districts
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What kinds of properties do random permutations have?
Permutation density helps define quasirandomness.

- A sequence of distinct integers $a_1, a_2, \ldots, a_k$ is order-isomorphic to a permutation $\pi \in S_k$ if they are ordered the same.

- Example: The sequences 295 and 396 are both order-isomorphic to the permutation 132, but 123 and 483 are not.

This helps study patterns of subsequences within permutations.
Permutation Density

**Definition**

The density of a pattern permutation $\pi$ in a permutation $\tau$, denoted by $t(\pi, \tau)$, is the probability that the restriction of $\tau$ to a random $|\pi|$-point set is order-isomorphic to $\pi$. 

**Example**

The density $t(12, 132) = \frac{2}{3}$, while $t(21, 132) = \frac{1}{3}$. In general, the density $t(21, \tau)$ equals the number of inversions in $\tau$ divided by $|\tau|^2$. 

**Example**

For a random permutation $\tau \in S_n$ and any fixed permutation $\pi$ (of which there are $|\pi|$ of a given length), $\mathbb{E}t(\pi, \tau) = \frac{1}{|\pi|!}$.
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Example
For a random permutation $\tau \in S_n$ and any fixed permutation $\pi$ (of which there are $|\pi|!$ of a given length),

$$\mathbb{E} t(\pi, \tau) = \frac{1}{|\pi|!}.$$
Sequences of permutations \( \{\tau_j\} \) are called convergent if as \( j \to \infty \),

- Lengths \( |\tau_j| \to \infty \)
- Sequences of densities \( t(\pi, \tau_j) \) converge, for any permutation \( \pi \)

Advantage: we can ignore higher-order terms, e.g. \( \binom{n}{2}/n^2 = 1/2 + o(1) \).
Quasirandomness

Behavior of random permutations with respect to subpermutation densities:

**Definition**

A convergent sequence of permutations \( \{\tau_j\} \) is called *quasirandom* if for every permutation \( \pi \),

\[
\lim_{j \to \infty} t(\pi, \tau_j) = \frac{1}{|\pi|!}.
\]
Convergent sequences of permutations can be characterized by corresponding limit objects known as *permutons*.

**Definition**

A *permuton* is a probability measure $\mu$ on the unit square $[0, 1]^2$ with *uniform marginals*, meaning the individual distributions of the two coordinates are uniform.

The definition of density in permutations, $t(\pi, \tau)$, can be extended to density in permutons, $t(\pi, \mu)$.
Permutation Limits

Convergent sequences of permutations can be characterized by corresponding limit objects known as \textit{permutons}.

\section*{Definition}

A \textit{permuton} is a probability measure $\mu$ on the unit square $[0, 1]^2$ with \textit{uniform marginals}, meaning the individual distributions of the two coordinates are uniform.

The definition of density in permutations, $t(\pi, \tau)$, can be extended to density in permutons, $t(\pi, \mu)$.

\section*{Theorem}

\textit{For every convergent sequence of permutations $\{\tau_j\}$, there exists a corresponding permuton $\mu$ with the same densities of pattern permutations.}
**Symmetry**

**Definition**
A permuton $\mu$ is called $k$-symmetric if sampling permutations of length $k$ from $\mu$ is uniformly random, i.e. the densities are all $1/k!$.

**Example**
The following permuton is 2-symmetric:
Inflation

Are there non-uniform three-symmetric permutons?
Inflation

Are there non-uniform three-symmetric permutons? Yes.

Definition

A permutation $\tau$ of length $n$ is called $k$-inflatable if the permuton $\mu$ corresponding to mass uniformly distributed along the graph of the permutation on an $n \times n$ grid is $k$-symmetric.

Example

The inflation of $3421$ is the following permuton:
Densities in Inflations

Definition

Let $B(\pi)$ be the set of all pairs $(b, \sigma)$ corresponding to ways of dividing $\pi$ into consecutive blocks of size $b_1, b_2, \ldots, b_k$, with relative ordering $\sigma$.

Theorem

The density of $\pi$ in the inflation of $\tau$ is

$$t(\pi, \text{inflated}(\tau)) = \frac{|\pi|!}{|\tau||\pi|} \sum_{(b, \sigma) \in B(\pi)} \left[ \binom{|\tau|}{|\sigma|} t(\sigma, \tau) \cdot \prod_{x \in b} \frac{1}{x!^2} \right].$$
3-Inflatable Permutations

Theorem

A permutation \( \tau \in S_n \) is 3-inflatable if and only if \( t(12, \tau) = \frac{1}{2} \) and

\[
t(123, \tau) = t(321, \tau) = \frac{2n - 7}{12(n - 2)},
\]

\[
t(132, \tau) = t(213, \tau) = t(231, \tau) = t(312, \tau) = \frac{4n - 5}{24(n - 2)}.
\]

Corollary

\[ n \equiv 0, 1, 17, 64, 80, 81 \pmod{144}. \]

There are 750 rotationally symmetric permutations of size 17 that are 3-inflatable, e.g. g54abc319hf678ed2.
3-Inflatable Example

g54abc319hf678ed2
Quasirandomness is only dependent on densities of four-point permutations.

**Theorem (Kral and Pikhurko, 2013)**

Any four-symmetric permuton $\mu$ is the uniform probability measure.
Theorem

Let $S = S_4 \setminus D$ for some equi-dense $D \subseteq S_4$. If a convergent sequence \( \{\tau_j\} \) of permutations satisfies $t(\pi, \tau_j) = 1/4! + o(1)$ for every $\pi \in S$, then it is quasirandom.

Equi-dense subset of size 8 $\Longrightarrow$ better condition for quasirandomness, only requiring densities of 16/24 four-point permutations.
\[ \int F(X,Y)^2 \, dV = \int F(X,Y)XY \, dV = \int F(x,y)^2 \, dv = \frac{1}{9}. \]
We call a group of permutations equi-dense if each element of the group has the same coefficient in the expression of each of these integrals as a linear combination of densities of permutations in $S_4$.

\[
\begin{align*}
\int F(X, Y)^2 \, dV &= \int F(X, Y)XY \, dV = \int F(x, y)^2 \, dv = \frac{1}{9}.
\end{align*}
\]

<table>
<thead>
<tr>
<th>Permutation</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234, 2134</td>
<td>1/3</td>
<td>1/4</td>
<td>1/6</td>
</tr>
<tr>
<td>1243, 2143</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>1324, 2314, 3124, 3214</td>
<td>1/4</td>
<td>1/4</td>
<td>1/6</td>
</tr>
<tr>
<td>1342, 1423, 2341, 2413, 3142, 3241, 4123, 4213</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>1432, 2431, 4132, 4231</td>
<td>0/1</td>
<td>1/24</td>
<td>1/12</td>
</tr>
<tr>
<td>3412, 3421, 4312, 4321</td>
<td>0/1</td>
<td>0/1</td>
<td>1/12</td>
</tr>
</tbody>
</table>
We call a group of permutations *equi-dense* if each element of the group has the same coefficient in the expression of each of these integrals as a linear combination of densities of permutations in $S_4$. 

\[ \int_{A} F(X, Y)^2 \, dV = \int_{B} F(X, Y)XY \, dV = \int_{C} F(x, y)^2 \, dv = \frac{1}{9}. \]
Future Work

- Find a complete list of minimal subsets of permutations for which having density $1/24$ is a sufficient condition for quasirandomness.
- Better understand inflatable permutations, including examples with inflated density $1/24$ of some $\pi \in S_4$.
- Use the technique of flag algebras to generate bounds on densities given those of a certain subset.
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