Hilbert Series of Quasi-invariant Polynomials

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Let $s_{ij}$ be the operator interchanging $x_i$ and $x_j$ in a function $f(x_1, x_2, ..., x_n)$.
m-Quasiinvariance

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- Ex. $s_{13}(x_1^2x_2 + x_3^5) = (x_3^2x_2 + x_1^5)$
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**Definition**

Let \( m \) be a non-negative integer and \( k \) be a field. A polynomial \( F \in k[x_1, x_2, ..., x_n] \) is \( m \)-quasiinvariant if for all \( 1 \leq i < j \leq n \)

\[
(1 - s_{ij})F(x_1, x_2, ..., x_n)
\]

is divisible by \((x_i - x_j)^{2m+1}\).
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- $Q_m$ is the space of $m$-quasiinvariant polynomials
Examples

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Examples for $n = 2$:

$(k = \mathbb{C})$

- $F(x, y) = 2x^3 + 6xy^2 \in Q_1$ since $F(x, y) - F(y, x) = 2(x - y)^3$
- $F(x, y) = x^5 - 5x^3y^2 \in Q_1$ since $F(x, y) - F(y, x) = (x - y)^3(x^2 + 3xy + y^2)$
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$(k = \mathbb{F}_2)$

- $F(x, y) = x^8 \in Q_3$ since $F(x, y) - F(y, x) = x^8 - y^8 = (x - y)^8$
Want to measure "size" of space of quasi-invariant polynomials
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$Q_{m,d} = \text{vector space of homogeneous } m\text{-quasiinvariants of degree } d$
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- $Q_{m,d}$ = vector space of homogeneous $m$-quasiinvariants of degree $d$
- $Q_m$ can be decomposed into

$$\bigoplus_{d \geq 0} Q_{m,d} = Q_{m,0} \oplus Q_{m,1} \oplus \ldots$$
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**Definition**

The Hilbert series of the space of \( m \)-quasiinvariants to be

\[
HS_m(t) = \sum_{d \geq 0} t^d \dim(Q_{m,d})
\]
Module Structure

- Generalization of vector space over a field
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- $Q_m$ is a module over the ring of symmetric polynomials
- Closed under addition
- Closed under multiplication by ring elements (symmetric polynomials)
- Satisfies distributive property
$Q_m$ is a finitely generated module over the ring of symmetric polynomials.
More Properties

- $Q_m$ is a finitely generated module over the ring of symmetric polynomials
- Thus, $HS_m(t)$ can be written as

$$\frac{P(t)}{\prod_{i=1}^{n}(1 - t^i)}$$

where $P(t)$ is a polynomial with integer coefficients.
Theorem (Felder and Veselov)

Hilbert series of $m$-quasiinvariants in $\mathbb{C}$ is

$$HS_m(t) = n! t^{m\binom{n}{2}} \sum_{\text{Young diagrams}} \prod_{i=1}^{n} t^{m(l_i-a_i)+l_i} \frac{1-t^i}{h_i(1-t^{h_i})}$$

For example, when $n=4$ and $m=5$ the Hilbert series is

$$1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 + ...$$

Young diagrams are objects useful in representation theory

Want to generalize in $\mathbb{F}_p$
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- Young diagrams are objects useful in representation theory
- Want to generalize in $\mathbb{F}_p$
Let $g$ be a generic homogeneous polynomial of degree $d$. What can we say about the Hilbert series of the space of quasiinvariants divisible by $g$?

Work with $n=2$.

Ex. If $g = x^2 + 5y^2$, the Hilbert series for the space of $2$-quasiinvariants divisible by $g$ is $t^5 + t^4(1-t^2)(1-t^2)$.
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Work with $n=2$

Ex. If $g = x^2 + 5y^2$, the Hilbert series for the space of 2-quasiinvariants divisible by $g$ is

$$
\frac{t^5 + t^4}{(1 - t)(1 - t^2)}
$$
Theorem

If \( g = (ax^k + by^k) \) and \( a^2 \neq b^2 \)
then the Hilbert series divisible by \( g \) is

\[
t^k \left( t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+\min(i,k)} - \sum_{i=1}^{m} t^{2(m-i)+\min(i,k)+2} \right) \frac{(1 - t)(1 - t^2)}{(1 - t)(1 - t^2)}
\]
Determine when the Hilbert Series for $Q_m$ in $n$ variables is greater in $\mathbb{F}_p$ than in $\mathbb{C}$
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**Theorem**

*If there exists integers $a \geq 1$, $k \geq 0$, and $b \geq 0$ such that*

$$p^a(nk + 1) + 2b \binom{n}{2} \leq mn$$

$$p^a(2k + 1) + 2b \geq 2m + 1,$$

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then the Hilbert series of $Q_m$ in $n$ variables is greater in $\mathbb{F}_p$ than in $\mathbb{C}$

- If $a = 1$, $k = 0$, $b = 0$, then the Hilbert series is greater for $2m + 1 \leq p \leq mn$
Conjecture

The previous conditions are necessary for the Hilbert series to be greater in $F_p$ than in $C$. 
The previous conditions are necessary for the Hilbert series to be greater in \( \mathbb{F}_p \) than in \( \mathbb{C} \).

Furthermore, the minimal non-symmetric polynomial in \( \mathbb{F}_p \) is of the form

\[
G = P_k^{p^a} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2b}
\]

where \( P_k \) is a generator of degree \( kn + 1 \) in \( \mathbb{C} \).
### Status of Project

**Figure: n=4**

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Archer Wang (Mentor: Dr. Xiaomeng Xu)
Future Studies

- Generalize results for first problem for generic $g$
- Compute Hilbert series for finite fields using the representation theory of the Cherednik algebra
Acknowledgements

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