The Hilbert Polynomial of the Irreducible Representation of the Rational Cherednik Algebra of Type $A_n$ in Characteristic $p \nmid n$

Merrick Cai  
Mentor: Daniil Kalinov, MIT

Kings Park High School

May 19-20, 2018  
MIT PRIMES Conference
A vector space defined over a field \( k \), is a collection of vectors, which may be multiplied by scalars \( \lambda \in k \), and added together.
A vector space defined over a field $\mathbb{k}$, is a collection of vectors, which may be multiplied by scalars $\lambda \in \mathbb{k}$, and added together.

- $\mathbb{k}^n$
- $\mathbb{k}[x_1, x_2, \ldots, x_n]$
- $\mathbb{k}[[x_1, x_2, \ldots, x_n]]$
- $\mathbb{k}[\partial_x, x]$
- $\text{Mat}_n(\mathbb{k})$
An algebra is a vector space $V$ equipped with a bilinear product; i.e., the vectors can be multiplied while preserving linearity.
An algebra is a vector space \( V \) equipped with a bilinear product; i.e., the vectors can be multiplied while preserving linearity.

- \( \mathbb{k}[x_1, \ldots, x_n] \)
- \( \mathbb{k}[[x_1, \ldots, x_n]] \)
- \( \mathbb{k}[x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}] \)
- \( \text{Mat}_n(\mathbb{k}) \)
An algebra $A$ is graded if $A = \bigoplus_{n \geq 0} A_n$ for subspaces $A_n$ and $A_iA_j \subset A_{i+j}$.
An algebra $A$ is graded if $A = \bigoplus_{n \geq 0} A_n$ for subspaces $A_n$ and $A_i A_j \subset A_{i+j}$.

**Example**

The algebra $A = \mathbb{k}[x_1, x_2, \ldots, x_n]$ has a grading given by $A_i$; the subspace of homogeneous degree $i$ polynomials.
The Hilbert series of a graded algebra $A$ is given by

$$h(z) = \sum_{n \geq 0} \dim(A_n)z^n.$$
The Hilbert series of a graded algebra $A$ is given by

$$h(z) = \sum_{n \geq 0} \dim(A_n) z^n.$$  

Example

The algebra $A = \mathbb{k}[x_1, \ldots, x_n]$ has the usual grading by degree. Then $\dim(A_i) = \binom{n+i-1}{i}$, so $h_A(z) = \sum_{i \geq 0} \binom{n+i-1}{i} z^i = \frac{1}{(1-z)^n}$. 

Merrick Cai; Mentor: Daniil Kalinov
A representation of an algebra $A$ is a vector space $V$ equipped with a homomorphism $\rho : A \to \text{End}(V)$.
A representation of an algebra $A$ is a vector space $V$ equipped with a homomorphism $\rho : A \rightarrow \text{End}(V)$.

**Example**

Take $V = \mathbb{C}^n$ and $G = S_n$. Then the group algebra $\mathbb{k}[S_n]$ acts on $v \in V$ by permuting the indices; e.g., $[(123)](x, y, z) = (z, x, y)$. 
A representation of an algebra $A$ is a vector space $V$ equipped with a homomorphism $\rho : A \to \text{End}(V)$.

**Example**

Take $V = \mathbb{C}^n$ and $G = S_n$. Then the group algebra $\mathbb{k}[S_n]$ acts on $v \in V$ by permuting the indices; e.g., $[(123)](x, y, z) = (z, x, y)$.

A subrepresentation is a subspace $W \subset V$ which remains closed under the action of $\rho(A)$. 
A representation \((A, V)\) is irreducible if there does not exist any (proper) subspace \(W \subset V\) which is closed under the action of \(A\).
A representation \((A, V)\) is irreducible if there does not exist any (proper) subspace \(W \subset V\) which is closed under the action of \(A\).

**Example**

Let \(A = \mathbb{k}[S_n]\) be the group algebra of \(S_n\) and \(V = \mathbb{C}^n\) be a vector space where \(S_n\) acts by permutations. Then \(\text{Span}\{(1, 1, 1, \ldots, 1)\}\) is an irreducible subrepresentation.
Let $V = \mathbb{k}[x]$. The differential operator acts by $\partial_x x^k = kx^{k-1}$. In characteristic 0 we can define the algebra of differential operators as a subalgebra in $End(\mathbb{k}[x])$ generated by $x$ and $\partial_x$. 
Let $V = \mathbb{k}[x]$. The differential operator acts by $\partial_x x^k = kx^{k-1}$. In characteristic 0 we can define the algebra of differential operators as a subalgebra in $End(k[x])$ generated by $x$ and $\partial_x$.

But in characteristic $p$, $\partial_x^p$ acts by 0, this is problematic. So to define $k[x, \partial_x]$, use the fact that $[\partial_x, x] = 1$. So $k[x, \partial_x] = \mathbb{k}\langle x, y \rangle / ([y, x] = 1)$. 
The Cherednik algebra \( H_{t,c}(n) \) in characteristic 0 is generated by the following in \( \text{End}(\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)) \):

- Polynomials in \( \mathbb{k}[x_1, \ldots, x_n] \) Acts by multiplication
- Elements of \( S_n \) Acts by permuting the \( x_i \)'s
- The Dunkl operators \( D_y^i \) An extension of the partial derivative

\[
D_y^i = t \frac{\partial}{\partial x_i} - c \sum_{k \neq i} \sigma_{ik} x_i - x_k
\]

The relevant cases are \( t = 1 \) and \( t = 0 \). We will work with \( t = 0 \).

We need more abstract definition for characteristic \( p \) as for differential operators.
The Cherednik algebra $H_{t,c}(n)$ in characteristic 0 is generated by the following in $\text{End}(\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n))$:

- Polynomials in $\mathbb{k}[x_1, \ldots, x_n]$
  - Acts by multiplication
The Cherednik algebra $H_{t,c}(n)$ in characteristic 0 is generated by the following in $\text{End}(\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n))$:

- Polynomials in $\mathbb{k}[x_1, \ldots, x_n]$:
  - Acts by multiplication
- Elements of $S_n$:
  - Acts by permuting the $x_i$'s
The Cherednik algebra $H_{t,c}(n)$ in characteristic 0 is generated by the following in $\text{End}(\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n))$:

- Polynomials in $\mathbb{k}[x_1, \ldots, x_n]$
  - Acts by multiplication
- Elements of $S_n$
  - Acts by permuting the $x_i$’s
- The Dunkl operators $D_{y_i}$
  - An extension of the partial derivative
  - $D_{y_i} = t\partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k}$
  - $[D_{y_i}, D_{y_j}] = 0$
The Cherednik algebra $H_{t,c}(n)$ in characteristic 0 is generated by the following in $\text{End}(\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n))$:

- Polynomials in $\mathbb{k}[x_1, \ldots, x_n]
  - Acts by multiplication
- Elements of $S_n
  - Acts by permuting the $x_i$’s
- The Dunkl operators $D_{y_i}$
  - An extension of the partial derivative
  - $D_{y_i} = t\partial_{x_i} - c \sum_{k \neq i} \frac{1-\sigma_{ik}}{x_i-x_k}$
  - $[D_{y_i}, D_{y_j}] = 0$

The relevant cases are $t = 1$ and $t = 0$. We will work with $t = 0$. We need more abstract definition for characteristic $p$ as for differential operators.
The Dunkl operator can be described by

\[ D_{y_i} = t \partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} \].

For \( t = 1 \), an example of

\[ D_{y_1}(x_1 x_2 x_2 x_3 x_3) \]:

\[ 1 \partial_{x_1}(x_1 x_2 x_2 x_3 x_3) = x_2 x_2 x_3 x_3 (x_1 x_2 x_2 x_3 x_3) = x_3 (x_1 x_2 x_2 x_3 x_3) = -x_1 x_2 x_3 x_3 \]

For \( k > 3 \), then

\[ D_{y_1}(x_1 x_2 x_2 x_3 x_3) = \partial_{x_1}(x_1 x_2 x_2 x_3 x_3) - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1 x_2 x_2 x_3 x_3) = x_2 x_2 x_3 x_3 + c (x_1 x_2 x_2 x_3 x_3 + x_2 x_1 x_2 x_3 x_3 + x_1 x_2 x_2 x_3 x_3 - (n - 3) x_2 x_2 x_3 x_3) \].

The singular polynomials are those which are in the kernel of all Dunkl operators \( D_{y_i} - y_j \) for all \( i, j \).
The Dunkl operator can be described by \( D_y = t \partial x_i - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} \). For \( t = 1 \), an example of \( D_{y_1} (x_1 x_2^2 x_3^3) \):
The Dunkl operator can be described by \( D_{y_i} = t \partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} \).

For \( t = 1 \), an example of \( D_{y_1} (x_1 x_2^2 x_3^3) \):

- \( 1 \partial_{x_1} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 \)
The Dunkl operator can be described by $D_{y_i} = t \partial_{x_i} - c \sum_{k \neq i} \frac{1-\sigma_{ik}}{x_i-x_k}$.

For $t = 1$, an example of $D_{y_1}(x_1 x_2^2 x_3^3)$:

- $1 \partial_{x_1} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3$
- $\frac{1-\sigma_{12}}{x_1-x_2} (x_1 x_2^2 x_3^3) = x_3^3 \left( \frac{x_1 x_2^2 - x_1^2 x_2}{x_1-x_2} \right) = -x_1 x_2 x_3^3$
The Dunkl operator can be described by \( D_{y_i} = t \partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} \).

For \( t = 1 \), an example of \( D_{y_1} (x_1^2 x_2^2 x_3^3) \):

1. \( 1 \partial_{x_1} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 \)
2. \( \frac{1 - \sigma_{12}}{x_1 - x_2} (x_1 x_2^2 x_3^3) = x_3^3 \left( \frac{x_1 x_2^2 - x_1^2 x_2}{x_1 - x_2} \right) = -x_1 x_2 x_3^3 \)
3. \( \frac{1 - \sigma_{13}}{x_1 - x_3} (x_1 x_2^2 x_3^3) = x_2^2 \left( \frac{x_1 x_3^3 - x_1^3 x_3}{x_1 - x_3} \right) = -x_1^2 x_2^2 x_3 - x_1 x_2^2 x_3^2 \)
The Dunkl operator can be described by $D_{y_i} = t\partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k}$.

For $t = 1$, an example of $D_{y_1} (x_1x_2^2x_3^3)$:

- $1\partial_{x_1} (x_1x_2^2x_3^3) = x_2^2x_3^3$
- $\frac{1 - \sigma_{12}}{x_1 - x_2} (x_1x_2^2x_3^3) = x_3^3 \left( \frac{x_1x_2^2 - x_1^2x_2}{x_1 - x_2} \right) = -x_1x_2x_3^3$
- $\frac{1 - \sigma_{13}}{x_1 - x_3} (x_1x_2^2x_3^3) = x_2^2 \left( \frac{x_1x_3^3 - x_1^3x_3}{x_1 - x_3} \right) = -x_1^2x_2x_3 - x_1x_2^2x_3^2$
- For $k > 3$, then $\frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1x_2^2x_3^3) = x_2^2x_3^3$
The Dunkl operator can be described by

\[ D_{y_i} = t \partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k}. \]

For \( t = 1 \), an example of \( D_{y_1}(x_1 x_2^2 x_3^3) \):

- \( 1 \partial_{x_1} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 \)
- \( \frac{1 - \sigma_{12}}{x_1 - x_2} (x_1 x_2^2 x_3^3) = x_3^3 \left( \frac{x_1 x_2^2 - x_1^2 x_2}{x_1 - x_2} \right) = -x_1 x_2 x_3^3 \)
- \( \frac{1 - \sigma_{13}}{x_1 - x_3} (x_1 x_2^2 x_3^3) = x_2^2 \left( \frac{x_1 x_3^3 - x_1^3 x_3}{x_1 - x_3} \right) = -x_1^2 x_2 x_3 - x_1 x_2^2 x_3^2 \)
- For \( k > 3 \), then \( \frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 \)
- \( D_{y_1}(x_1 x_2^2 x_3^3) = \partial_{x_1} (x_1 x_2^2 x_3^3) - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 + c \left( x_1 x_2 x_3^3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3^2 - (n - 3) x_2 x_3^3 \right) \)
The Dunkl operator can be described by $D_{y_i} = t\partial_{x_i} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k}$.

For $t = 1$, an example of $D_{y_1} (x_1 x_2^2 x_3^3)$:

- $1\partial_{x_1} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3$
- $\frac{1 - \sigma_{12}}{x_1 - x_2} (x_1 x_2^2 x_3^3) = x_3^3 \left( \frac{x_1 x_2^2 - x_1^2 x_2}{x_1 - x_2} \right) = -x_1 x_2 x_3^3$
- $\frac{1 - \sigma_{13}}{x_1 - x_3} (x_1 x_2^2 x_3^3) = x_2^2 \left( \frac{x_1 x_3^3 - x_1^3 x_3}{x_1 - x_3} \right) = -x_1^2 x_2^2 x_3 - x_1 x_2^2 x_3^2$
- For $k > 3$, then $\frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3$
- $D_{y_1} (x_1 x_2^2 x_3^3) = \partial_{x_1} (x_1 x_2^2 x_3^3) - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1 x_2^2 x_3^3) = x_2^2 x_3^3 + c (x_1 x_2 x_3^3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3^2 - (n - 3) x_2 x_3^3)$

The singular polynomials are those which are in the kernel of all Dunkl operators $D_{y_i - y_j}$ for all $i, j$. 
By $M_{t,c}$ denote the Verma module $\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$ with a standard structure of $H_{t,c}(n)$ representation.
By $M_{t,c}$ denote the Verma module $\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$ with a standard structure of $H_{t,c}(n)$ representation.

Ideal of symmetric polynomials is a subrepresentation.
By $M_{t,c}$ denote the Verma module
\[ \mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n) \]
with a standard structure of $H_{t,c}(n)$ representation.

Ideal of symmetric polynomials is a subrepresentation.

Denote by $N_{t,c}$ the quotient by this subrepresentation, which is the baby Verma module.
The contravariant form $B : S\mathfrak{h} \otimes S\mathfrak{h}^* \to \mathbb{k}$ is defined by $B(1, 1) = 1$ and for $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$, $g \in S\mathfrak{h}$, $f \in S\mathfrak{h}^*$, then $B(yg, f) = B(g, Dy(f))$ and $B(g, xf) = B(Dx(g), f)$.

The kernel of $B$ is given by $x \in S\mathfrak{h}^*$ such that for all $y \in S\mathfrak{h}$, then $B(y, x) = 0$. 
The contravariant form $B : S\mathfrak{h} \otimes S\mathfrak{h}^* \to \mathbb{k}$ is defined by $B(1, 1) = 1$ and for $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$, $g \in S\mathfrak{h}$, $f \in S\mathfrak{h}^*$, then $B(yg, f) = B(g, D_y(f))$ and $B(g, xf) = B(D_x(g), f)$.

The kernel of $B$ is given by $x \in S\mathfrak{h}^*$ such that for all $y \in S\mathfrak{h}$, then $B(y, x) = 0$.

- The kernel is a subrepresentation
- Define $L_{t,c} = M_{t,c}/\ker B$
- $L_{t,c} = N_{t,c}/\ker B$
- $L$ is an irreducible representation of $H_{t,c}$
To find the Hilbert polynomial of the irreducible quotient $L_{t,c}$ in the polynomial representation of the rational Cherednik algebra of type $A_n$, when the characteristic $p \nmid n$.

The singular polynomials generate a subrepresentation so we would like to find them and remove them.
To find the smallest $d$ such that degree $d$ polynomials in the simultaneous kernel of the Dunkl operators $D_{y_i - y_j}$ exist, and find the dimension of this kernel.
Balagovic and Chen showed that if no singular polynomials existed, the Hilbert polynomial of $N_{t,c}(\tau)$ is as follows:
Balagovic and Chen showed that if no singular polynomials existed, the Hilbert polynomial of $N_{t,c}(\tau)$ is as follows:

$$t = 1 \implies h_{N_{1,c}(\tau)}(z) = \frac{(1 - z^{2p})(1 - z^{3p}) \cdots (1 - z^{np})}{(1 - z)^{n-1}},$$

They showed that $h_{L_{t,c}(\tau)}(z) = (1 - z^{p_1}) \cdots (1 - z^n)$ for some polynomial $h$ with integer coefficients. They also proved that $\ker B$ is a maximal proper submodule of the Verma module $M_{t,c}(\tau)$, and that $L_{t,c}(\tau)$ is irreducible.
Balagovic and Chen showed that if no singular polynomials existed, the Hilbert polynomial of $N_{t,c}(\tau)$ is as follows:

\[
t = 1 \implies h_{N_{1,c}(\tau)}(z) = \frac{(1 - z^{2p})(1 - z^{3p}) \cdots (1 - z^{np})}{(1 - z)^{n-1}},
\]

\[
t = 0 \implies h_{N_{0,c}(\tau)}(z) = \frac{(1 - z^2)(1 - z^3) \cdots (1 - z^n)}{(1 - z)^{n-1}}.
\]
Balagovic and Chen showed that if no singular polynomials existed, the Hilbert polynomial of $N_{t,c}(\tau)$ is as follows:

\[ t = 1 \implies h_{N_{1,c}(\tau)}(z) = \frac{(1 - z^{2p}) (1 - z^{3p}) \cdots (1 - z^{np})}{(1 - z)^{n-1}}, \]

\[ t = 0 \implies h_{N_{0,c}(\tau)}(z) = \frac{(1 - z^2) (1 - z^3) \cdots (1 - z^n)}{(1 - z)^{n-1}}. \]

They showed that

\[ h_{L_{t,c}(\tau)}(z) = \left(\frac{1 - z^p}{1 - z}\right)^{n-1} h(z^p) \]

for some polynomial $h$ with integer coefficients.
Balagovic and Chen showed that if no singular polynomials existed, the Hilbert polynomial of $N_{t,c}(\tau)$ is as follows:

\[
t = 1 \implies h_{N_{1,c}(\tau)}(z) = \frac{(1 - z^{2p})(1 - z^{3p}) \cdots (1 - z^{np})}{(1 - z)^{n-1}},
\]

\[
t = 0 \implies h_{N_{0,c}(\tau)}(z) = \frac{(1 - z^2)(1 - z^3) \cdots (1 - z^n)}{(1 - z)^{n-1}}.
\]

They showed that

\[
h_{L_{t,c}(\tau)}(z) = \left(\frac{1 - z^p}{1 - z}\right)^{n-1} h(z^p)
\]

for some polynomial $h$ with integer coefficients.

They also proved that $\ker B$ is a maximal proper submodule of the Verma module $M_{t,c}(\tau)$, and that $L_{t,c}(\tau)$ is irreducible.
Methods

- We wrote a program in Sage to compute the dimensions of the subspaces for various $p, n$ in $L_{t,c}$
We wrote a program in Sage to compute the dimensions of the subspaces for various $p, n$ in $L_{t,c}$

We compared the dimension to those predicted by Balagovic/Chen for $N_{t,c}$ to find existence of singular polynomials
We wrote a program in Sage to compute the dimensions of the subspaces for various $p, n$ in $L_{t,c}$

We compared the dimension to those predicted by Balagovic/Chen for $N_{t,c}$ to find existence of singular polynomials

We computed these singular polynomials
Methods

- We wrote a program in Sage to compute the dimensions of the subspaces for various $p, n$ in $L_{t,c}$
- We compared the dimension to those predicted by Balagovic/Chen for $N_{t,c}$ to find existence of singular polynomials
- We computed these singular polynomials
- We conjectured a pattern and looked to prove it
For $p \mid n$:
- The singular polynomials are $x_i$ for $i = 1, 2, \ldots, n$
- The Hilbert polynomial is 1
For $p|n$:
- The singular polynomials are $x_i$ for $i = 1, 2, \ldots, n$
- The Hilbert polynomial is 1
The case $t = 1$ and $p|n$ was done by Devadas and Sun.
Progress for $p = 2$ and $t = 0$

For $p = 2$, the following polynomials are singular for distinct $i, j, k$:

- $x_i^2 + x_i x_j + x_j^2$
- $x_i x_j + x_j x_k + x_k x_i$
Progress for $p|n − 1$ and $t = 0$

For $p$ odd and distinct $i, j, k, l$, the following polynomials are singular:

- $(x_j + x_k)(x_i - x_j - x_k)$
- $(x_i - x_j)(x_k - x_l)$
The Hilbert polynomial for $p = 2$ and $t = 0$ is
$$h_{L_0,c}(z) = 1 + (n - 1)z + (n - 1)z^2 + z^3$$

Etingof conjectures that for $n = kp + r$, then
$$h_{L_0,c}(z) = [r]_z ![p]_z Q_r(n, z) \text{ and }$$
$$h_{L_1,c}(z) = [r]_{z^p} ![p]_{z^p} [p]_z^{n-1} Q_r(n, z^p),$$
for
$$Q_r(n, z) = \binom{n-1}{r-1} z^{r+1} + \sum_{i=0}^{r} \binom{n-r-2+i}{i} z^i,$$
$$[k]_z! = [k]_z [k-1]_z \cdots [1]_z, \text{ and } [w]_z = \frac{1-z^w}{1-z}.$$
In the future, we would like to find the Hilbert polynomials for $L_{t,c}$, and the singular polynomials for various $p, n$. We would like to study more cases in $t = 0$ and prove irreducibility, then consider the connection between $t = 0$ and $t = 1$. 
I would like to thank:

- My parents
- My mentor, Daniil Kalinov
- Dr. Pavel Etingof
- Dr. Tanya Khovanova
- The MIT Math Department
- The MIT PRIMES program
- Sheela Devadas, Yi Sun, Martina Balagovic, Harrison Chen