Moduli Space of Genus 1 Tropical Curves

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Mentor: Yu Zhao

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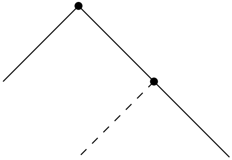
Graphs



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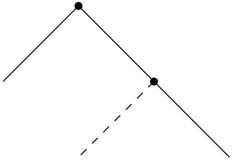

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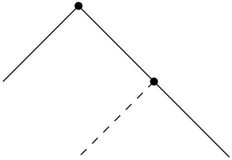
Graphs



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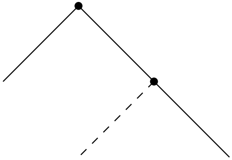
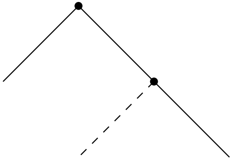


- Vertices and edges
- Edges may be unbounded, connected to only one vertex
- A *flag* consists of a vertex and an edge pointing away from the vertex

Graphs



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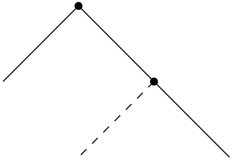

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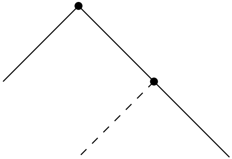

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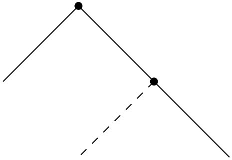

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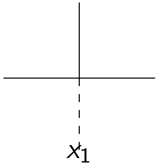
  - Ex. The genus of a tree is 0

Abstract Tropical Curves



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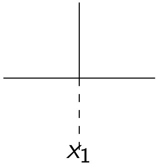

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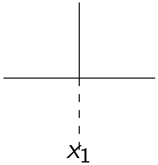

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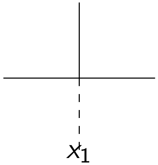

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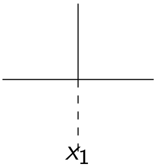
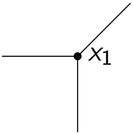
- Marked edges will have labels $x_1, x_2, \ldots, x_n$

Planar Tropical Curves





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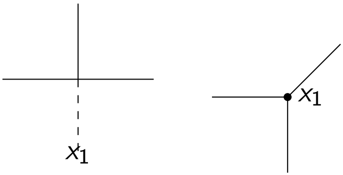

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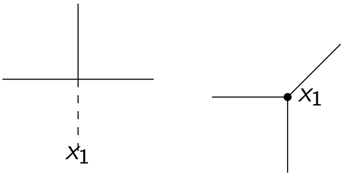

- A *planar tropical curve* consists of an underlying abstract tropical curve along with a continuous mapping h that takes Γ to $\mathbb{R}^2$
  - Each edge $e \in \Gamma$ with length $l(e)$ will be mapped to the segment $a + v \cdot l(e)$ where $a \in \mathbb{R}^2$ and v is an integral vector called the *direction vector* of F, where F is the flag at e, denoted $v(F)$

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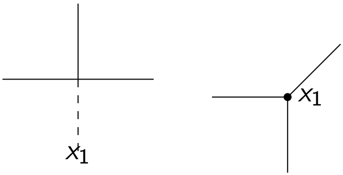

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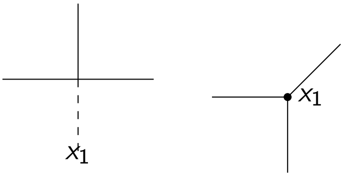

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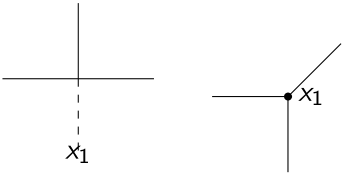

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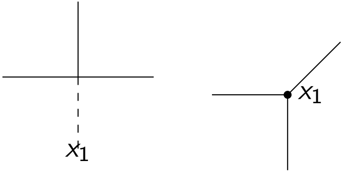

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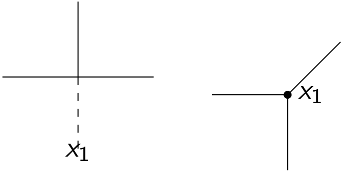

- The *degree* of a planar tropical curve is the multiset of direction vectors of unmarked unbounded edges
  - If the degree consists of the vectors $(-1, 0), (0, -1), (1, 1)$ each occurring d times, we will say the degree is d

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  - (a, b, r) where $r \geq 0$
  - Half of $\mathbb{R}^3$

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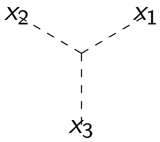
  - There is only one possible abstract tropical curve
  - The moduli space consists of one element



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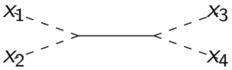
- The *combinatorial type* of an n-marked abstract tropical curve consists of all the information comprising the curve except the lengths of bounded edges
- The *combinatorial type* of an n-marked planar curve is the combinatorial type of the underlying abstract tropical curve along with the direction vectors for all flags.

Example: Moduli Space of 4-marked abstract tropical curves



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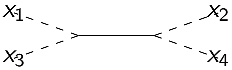

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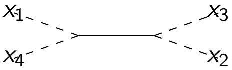

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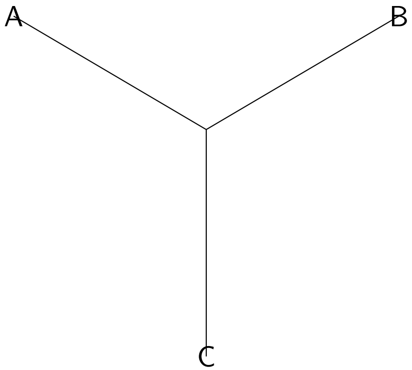
Example: Moduli Space of 4-marked abstract tropical curves



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Moduli Space of planar tropical curves

- What is the moduli space of planar tropical curves with genus ≤ 1, n marked edges, and degree d?
- Approach: Divide up all planar tropical curves into different combinatorial types, find the moduli space of each combinatorial type, and glue the individual spaces together
- To "glue" the spaces together, we can contract edges to arrive at other combinatorial types, but the sum of all the lengths must remain positive

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- $d = 1$, there are three unmarked unbounded edges with each direction $(1, 1), (-1, 0), (0, -1)$ appearing once
- To compute all combinatorial types, casework on number of vertices in the cycle
- Since we have four unbounded edges, we can have at most four vertices in cycle

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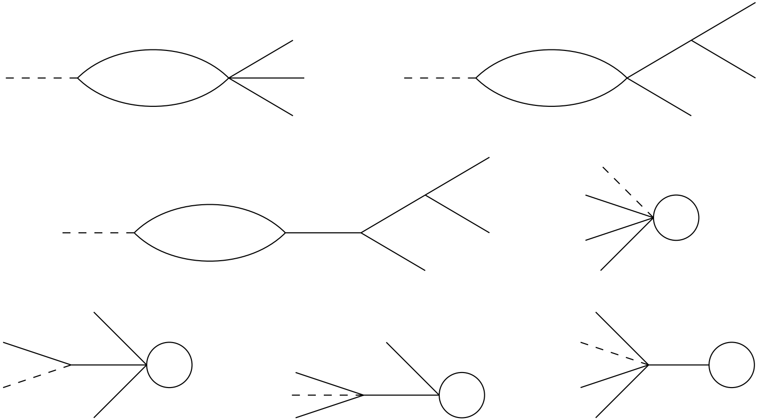

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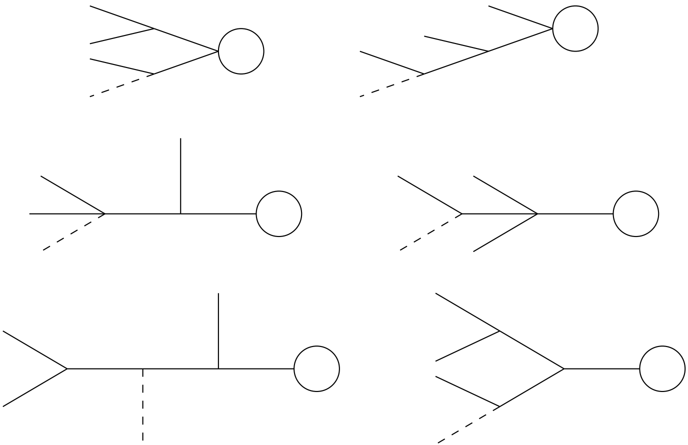

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Result: $d = 1$

The moduli space of all planar tropical curves with degree 1 and n marked edges is connected for any nonnegative integer n.

Future Work

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- Investigate other properties of the moduli space of planar curves (ie homology group)

Acknowledgements

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- my parents