Counting Bimonotone Subdivisions

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Subdivisions

- **Subdivision**: Of a point configuration $A$ in $\mathbb{R}^2$, a subdivision is a collection of convex polygons such that:
  - The union of the polygons is $\text{conv}(A)$
  - Each pair of polygons does not intersect or intersects at a common vertex or side
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Bimonotone

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Tent Functions

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- $f$ induces a subdivision of projected polygons on the plane of $A$
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Supermodularity

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  \[ f(x) + f(y) \leq f(\min(x, y)) + f(\max(x, y)) \text{ for all } x, y \]
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• Example: An IQ test with $n$ questions
  
  • The joint distribution of $n$ scores takes $f(x)$
  • The score for each question has a density
  • Scores on separate questions are positively correlated
Bimonotone and Supermodularity

• For a tent function $f$, the subdivision is bimonotone if and only if $f$ is supermodular
Bimonotone and Supermodularity

- For a tent function $f$, the subdivision is bimonotone if and only if $f$ is supermodular.
- The goal of this project is to count the number of bimonotone subdivisions and compare this to the total number of subdivisions.
Our Work: $2 \times n$ Grids

- First consider subdivisions of a $2 \times n$ lattice grid
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- First consider subdivisions of a $2 \times n$ lattice grid
- To use a recursion, we extend this to grids with $m$ points at the top and $n$ at the bottom
Recursion

- Using inclusion-exclusion for the unconnectedness of the top right and bottom right vertices, the number of bimonotone subdivisions is

\[ A_{m,n} = \begin{cases} 2A_{m,n-1} + 2A_{m-1,n} - 2A_{m-1,n-1}, & m > n \\ 2A_{m,n-1}, & m = n \\ 0, & m < n \end{cases} \]
Recursion

• Similarly, for the total number of subdivisions,

\[ B_{m,n} = 2A_{m,n-1} + 2A_{m-1,n} - 2A_{m-1,n-1} \]
Theorem

For a lattice grid with \( m \) points at the top and \( n \) points at the bottom:

- The number of bimonotone subdivisions is given by
  \[ A_{m,n} = \frac{2^{m-2}}{(n-1)!} P_n(m), \]
  where \( P_n(m) \) is some monic polynomial with degree \( n - 1 \).

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• Proof by induction
• We repeatedly substitute smaller terms into the recursion, giving for $A_{m,n}$:

$$\frac{2^{m-2}}{(n-2)!} (P_{n-1}(m) + (P_{n-1}(m) + P_{n-1}(m-1) + \cdots + P_{n-1}(n)))$$
Proof Idea

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\frac{2^{m-2}}{(n-2)!} \left( P_{n-1}(m) + (P_{n-1}(m) + P_{n-1}(m-1) + \cdots + P_{n-1}(n)) \right)
$$

- We find the highest degree term using Faulhaber’s formula for the sum of the $p$th powers of the first $m$ positive integers:

$$
\sum_{k=1}^{m} k^p = \frac{m^{p+1}}{p+1} + \frac{1}{2} m^p + \sum_{k=2}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!} m^{p-k+1}
$$

where the $B_k$ are the Bernoulli numbers
Future Research

• Prove these conjectures:
  • The number of bimonotone subdivisions of a $2 \times n$ lattice grid is $2^{n-1}$ times the $n$th large Schröder number
  • The total number of subdivisions of a $2 \times n$ lattice grid is $2^{n-1}$ times the $n$th Delannoy number
Future Research

- Prove these conjectures:
  - The number of bimonotone subdivisions of a $2 \times n$ lattice grid is $2^{n-1}$ times the $n$th large Schröder number
  - The total number of subdivisions of a $2 \times n$ lattice grid is $2^{n-1}$ times the $n$th Delannoy number
- Find recursive formulas for $3 \times n$ and larger lattice grids
- Find closed form expressions for the number of bimonotone/total subdivisions
- Extend formulas into higher dimensions
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