Agent-based Models for Conservation Equations

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Conservation Equations

\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial f(x, t)}{\partial x} = 0
\]

\[\rho_t + f_x = 0\]

\(\rho\) : density

\(f\) : flux
Possible Usage

- Cars
- Blood
- Electric Charge

Figure: Red Blood Cells

Figure: Traffic Flow
Density

$\rho(x, t)$: Density defined as mass per unit length.
Example: $\rho = \frac{\text{cars}}{\text{length}}$
Flux

$f(x, t)$: Flux defined as the amount of mass passing through $x$ per unit time.

\[
f = \frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta x} \frac{\Delta x}{\Delta t} = \rho v
\]
Derivation of the Conservation Equation

\[ \frac{d}{dt} \int_{x_0}^{x_0+\Delta x} \rho(x, t) \, dx = f(x_0, t) - f(x_0 + \Delta x, t) \]

\[ \rho_t = -f_x \]
Constitutive Laws

We need to relate flux with density.

▶ Greenshield’s Law

\[ v = v_m \left(1 - \frac{\rho}{\rho_m}\right) \]
\[ f = v_m \left(1 - \frac{\rho}{\rho_m}\right) \rho \]
\[ \rho_t + (v_m \left(1 - \frac{\rho}{\rho_m}\right) \rho)_x = 0 \]

▶ Burger’s Equation

\[ v(\rho) = \frac{1}{2} \rho \]
\[ f(\rho) = \frac{1}{2} \rho^2 \]
\[ \rho_t + (\frac{1}{2} \rho^2)_x = 0 \]
Finite Volume Method

Let flux $f(x, t)$ at $x = x_0$ and $t = t_0$ be written as $f_{x_0}^{t_0}$

$$\frac{d}{dt} \int_{x_0}^{x_0 + \Delta x} \rho(x, t) dx = f(x_0, t) - f(x_0 + \Delta x, t)$$

$$\frac{\Delta x}{\Delta t} (\rho_{x}^{t+1} - \rho_{x}^{t}) = f_{x-\frac{1}{2}}^{t} - f_{x+\frac{1}{2}}^{t}$$

$$\rho_{x}^{t+1} = \rho_{x}^{t} + \frac{\Delta t}{\Delta x} (f_{x-\frac{1}{2}}^{t} - f_{x+\frac{1}{2}}^{t})$$
Example: Upwind Method

\[ f(\bar{\rho}_x, \bar{\rho}_{x+1}) = \frac{1}{2} \left( f(\bar{\rho}_x) + f(\bar{\rho}_{x+1}) - a(\bar{\rho}_{x+1} - \bar{\rho}_x) \right) \]

Where \( a = \left| \frac{f(\bar{\rho}_x) - f(\bar{\rho}_{x+1})}{\bar{\rho}_x - \bar{\rho}_{x+1}} \right| \)

Figure: Screenshots of the numerical solution. Horizontal axis: position. Vertical axis: density
Agent-based Models

1. \[ x^{i+1} = x^i + v(\rho_L, \rho_R)\Delta t \]

2. The density \( \rho(x, t) \) is approximated as \( \frac{\Delta m}{x_{j+1} - x_j} \)

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Figure: Agent Model
Example: Greenshields Law

\[
\begin{align*}
\rho_t + (v_m(1 - \frac{\rho}{\rho_m})\rho)_x &= 0 \\
v &= v_m(1 - \frac{\rho}{\rho_m}) \\
v(u_L, u_R) &= \begin{cases} 
v_{\text{max}} & u_L \geq u_R \\
v_{\text{min}} & u_L \leq u_R \end{cases}
\end{align*}
\]

**Figure:** Each agent correspond to a point. X axis: location. Y axis: density
Finite Volume Method for Specific Volume

- $\sigma$: the specific volume, defined as $\frac{\Delta x}{\Delta m}$, equals to $\frac{1}{\rho}$.
- Consider the amount of distance between two particles:

$$\Delta x^{i+1} = \Delta x^i + v_R \Delta t - v_L \Delta t$$

$$\frac{\Delta x^{i+1}}{\Delta m} = \frac{\Delta x^i}{\Delta m} + \frac{\Delta t}{\Delta m} \left(( -v_L ) - ( -v_R ) \right)$$

$$\sigma^{i+1} = \sigma^i + \frac{\Delta t}{\Delta m} \left(( -v_L ) - ( -v_R ) \right)$$

- Equals to the finite volume formula ($\bar{\rho}_x^{t+1} = \bar{\rho}_x^t + \frac{\Delta t}{\Delta x} (f_{x-\frac{1}{2}} - f_{x+\frac{1}{2}})$) for specific volume $\sigma$ where the flux is $-\nu$. 
Mass Function and Its Inverse

Mass function: \( m(x, t) = m(x, 0) + \int_0^t -f(x, a) da \) where \( m(x, 0) \) is the initial mass value at \( x \)

\[ \frac{\partial m}{\partial t} = -f(x, t) \]

\[ \frac{\partial m}{\partial x} = \rho(x, t) \]

\( x(m, t) \) is defined as \( m^{-1}(m, t) \)

\[ \frac{\partial x(m, t)}{\partial m} = \frac{1}{\rho} \]

\[ \frac{\partial m^{-1}(m(x, t), t)}{\partial t} \bigg|_x = \frac{\partial m^{-1}(m, t)}{\partial t} \bigg|_m + \frac{\partial m^{-1}(m, t)}{\partial m} \frac{\partial m}{\partial t} \]

\[ \frac{\partial x(m, t)}{\partial t} = \frac{f}{\rho} = v \]
Conservation of Specific Volume

\[ \sigma_t|_m + (-\nu)_m = 0 \]

- The conservation equation's finite volume formula is
  \[ \overline{\sigma}^{i+1} = \overline{\sigma}^i + \frac{\Delta t}{\Delta m} ((-\nu_L) - (-\nu_R)) \]

- Agent-based model can be viewed as a finite volume method for the specific volume where the total distance that passes each cell wall is recorded.

- Any finite volume method has its agent-based version.

- If a finite volume method converges to a solution for the specific volume conservation equation, its agent-based model converges to a solution for the original conservation equation.
Vector Conservation Equation

\[ \vec{\rho}_t + \vec{f}_t = 0 \]

Example:

\[ \begin{align*}
    v_1 &= v_{m_1} \left( 1 - \frac{\rho_1 + \rho_2}{\rho_m} \right) \\
    v_2 &= v_{m_2} \left( 1 - \left( \frac{\rho_1 + \rho_2}{\rho_m} \right)^2 \right)
\end{align*} \]

Figure: Red Dots: \( \rho_1 \) agents. Blue Dots: \( \rho_2 \) agents.
Future Goals

- Comparing agent-based solver with other solvers
- Vector conservation equations
- Conservation equation in 2-D space
- Adapt to source and sink terms.
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References