Width and Trunk of Satellite Knots

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What is a knot?

Definition

A **knot** is a smooth embedding from S^1 to $\mathbb{R}^3$, where S^1 is the unit circle in $\mathbb{R}^2$.

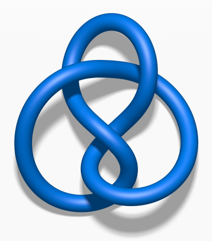


Figure: Figure 8 Knot

Knot Classes

Definition

Two knots are in the same **knot class** if we can deform one into the other without any self intersection during the deformation.

By convention, a knot class is denoted K while a single knot is denoted k.

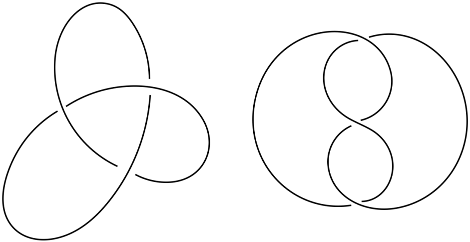
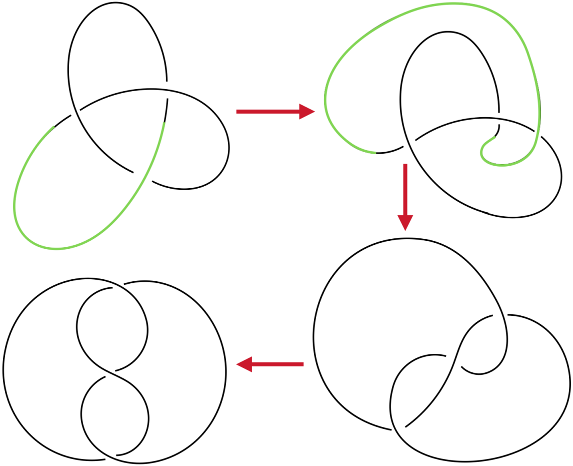


Figure: Knots in the Trefoil Knot Class

Transformation



Tori and Meridian Disks

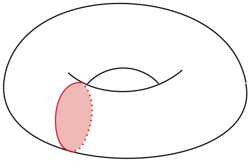


Figure: Solid torus with meridian disk

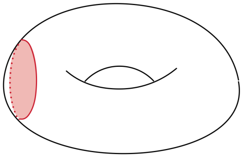


Figure: Not a meridian disk

Definition

A **meridian disk** is a disk properly embedded in the solid torus in the way depicted in the first figure.

Local Minima and Maxima of Knots

Height Function

Define $h : \mathbb{R}^3 \to \mathbb{R}$ to be the standard height function: $h(x, y, z) = z$.

For any r, the pre-image of r under h, $h^{-1}(r)$ is a horizontal plane.

Critical Points

Under h, knots have local minima and local maxima, known as **critical points**.

Critical and Regular Levels

Levels

Critical levels are denoted $c_1, c_2, \ldots, c_n$. Regular levels are located between each critical level: $c_1 < r_1 < c_2 < \ldots < c_{n-1} < r_{n-1} < c_n$.

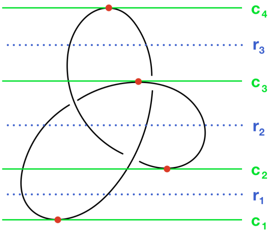


Figure: Critical and Regular levels of the trefoil knot

Width and Trunk

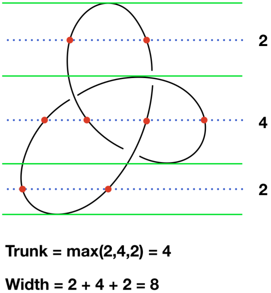


Figure: Width and Trunk of the trefoil knot

- Let ω_i be the number of intersections of each regular level with k.
- Width: $\omega(k) = \sum_{i=1}^{n-1} \omega_i$.
- Trunk: $\mathrm{tr}(k) = \max_{1 \leq i \leq n-1} \omega_i$.
- Knot class width: $\omega(K) = \min_{k \in K} \omega(k)$.
- Knot class trunk: $\mathrm{tr}(K) = \min_{k \in K} \mathrm{tr}(k)$.

Defining Satellite Knots

Knots Inside Solid Torus

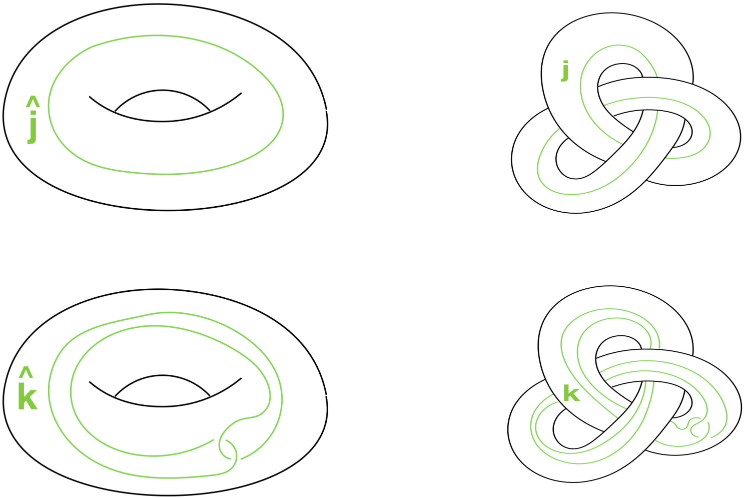
- Let V be a solid torus.
- Define $\hat{j}$ as the core of V.
- Let $\hat{k}$ be a knot inside V.

Let f be a smooth embedding from V to $\mathbb{R}^3$, and let $j = f(\hat{j})$ and $k = f(\hat{k})$.

Definition

The knot k is the **satellite knot** with companion j.

Images of Satellite Knots



Winding Number of a Satellite Knot

Definition

The **winding number** n of a knot is the absolute value of the sum of the signed intersections based on orientation of any meridian disk with the knot.

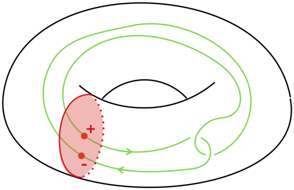


Figure: Winding Number $n = 0$

Wrapping Number of a Satellite Knot

Definition

The **wrapping number** m of a knot is the minimum number of intersections any meridian disk has with the knot.

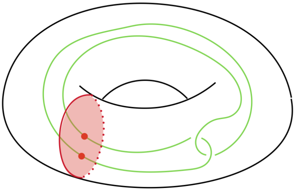


Figure: Wrapping Number $m = 2$

We always have $m \geq n$.

Motivation

Recall: $\omega(k)$ is the width of k, $\text{tr}(k)$ is the trunk of k, n is the winding number and m is the wrapping number.

Theorem (Guo, Li)

$$\omega(K) \geq n^2\omega(J).$$

Theorem (Kavi)

$$\text{tr}(K) \geq n \cdot \text{tr}(J).$$

As there are already results for winding number, what about the wrapping number?

Conjecture

$$\text{tr}(K) \geq \lambda \cdot m \cdot \text{tr}(J) \text{ for some } 0 < \lambda \leq 1.$$

How to bound the trunk of k?

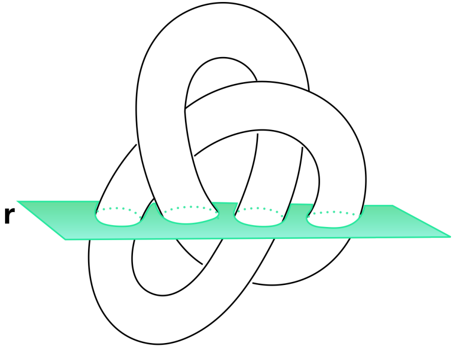


Figure: Intersection of a regular level with the solid torus

- Trunk number of a knot is the maximum number of intersections any regular level has with the knot.
- Suppose a regular level intersects the solid torus in t pieces $P_1, P_2, \ldots, P_t$.
- Recall if P_i is a meridian disk then $|P_i \cap k| \geq m$.
- How many P_i are meridian disks?

Arrangement of P_i on a plane

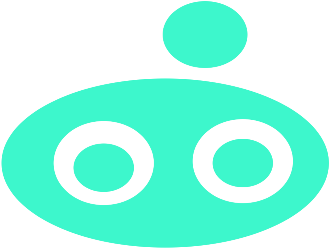
- Each P_i must have an odd number of boundaries.
- The innermost piece must be a meridian disk and there must be a meridian disk outside as well.

Figure: Two Examples of Invalid Arrangements

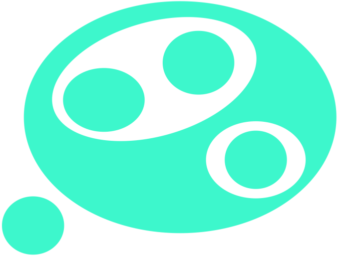
Examples of valid Arrangements

Definition

Define $A(t)$ to be an arrangement of t pieces in a plane, and let $\lambda(A(t))$ be the number of meridian disks in such an arrangement.



(a) $\lambda(A(4)) = 3$



(b) $\lambda(A(5)) = 4$

Experimental Results

Conjecture

$$\lambda(A(t)) \geq \lfloor \frac{t+3}{2} \rfloor.$$

```
Number of pieces: 2; Innermost Circles: 2
Number of pieces: 3; Innermost Circles: 3
Number of pieces: 4; Innermost Circles: 3
Number of pieces: 5; Innermost Circles: 4
Number of pieces: 6; Innermost Circles: 4
Number of pieces: 7; Innermost Circles: 5
Number of pieces: 8; Innermost Circles: 5
Number of pieces: 9; Innermost Circles: 6
Number of pieces: 10; Innermost Circles: 6
Number of pieces: 11; Innermost Circles: 7
Number of pieces: 12; Innermost Circles: 7
Number of pieces: 13; Innermost Circles: 8
Number of pieces: 14; Innermost Circles: 8
Number of pieces: 15; Innermost Circles: 9
Number of pieces: 16; Innermost Circles: 9
Number of pieces: 17; Innermost Circles: 10
Number of pieces: 18; Innermost Circles: 10
Number of pieces: 19; Innermost Circles: 11
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Figure: Program results

Main Result

- $\lambda(A(t)) \geq \lfloor \frac{t+3}{2} \rfloor$ proved.
- Note: $\frac{\lambda(A(t))}{t} > \frac{1}{2}$ for all t.

Theorem

If a knot K is a satellite knot with companion knot J and m denotes the wrapping number of k, then $\text{tr}(K) > \frac{1}{2} m \cdot \text{tr}(J)$.

Further Research

Next, we will study the relation between width and wrapping number.

Conjecture

$$\omega(K) > \frac{1}{4} m^2 \omega(J).$$

Also, we will observe specific satellite knots and determine for which ones we can find a value of λ higher than $\frac{1}{2}$.

Acknowledgements

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Bibliography

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- Derek Davies and Alexander Zupan. *Natural Properties of the Trunk of a Knot.* https://arxiv.org/pdf/1608.00019.pdf