Sequence of finite sets $S_0, S_1, S_2, ...$
Want to determine or describe $f(i) = \#S_i$
- Closed form formulas
- Open form formulas
- Recurrences
- Estimates
- Generating functions
Generating Functions

\[ f(i) = \# S_i \]

**Definition**

An ordinary generating function is a formal power series with complex coefficients

\[ \sum_{n \geq 0} f(n) x^n = \sum_{n \geq 0} \sum_{a \in S_n} x^n \quad \text{or} \quad = \sum_{n \geq 0} \sum_{a \in S_n} w(a) x^n \]

Generating functions of the form \( \sum_{n \geq 0} f(n) \frac{x^n}{B(n)} \) may be used.
Examples

Example
Say $\# S_i = 1$ for all $i$. Then the generating function is

$$1 + x + x^2 + \ldots = \frac{1}{1 - x}$$

In the ring of formal power series over $\mathbb{C}$, there is only one power series that is the inverse of $(1 - x)$.

Example
A partition of $n$ is a weakly decreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_k = n$. Let $p(n)$ be the number of partitions of $n$. Then

$$\sum_{n\geq 0} p(n)x^n = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots) \ldots = \prod_{n\geq 1} \frac{1}{1 - x^n}$$
Rational Generating Functions

Definition

A rational generating function has the form

\[ \sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)} \]

where \( P(x) \) and \( Q(x) \) are polynomials with complex coefficients.

Example

The generating function from the previous slide

\[ 1 + x + x^2 + \ldots = \frac{1}{1 - x} \]

is a rational generating function.
Recurrences

When do we have a rational generating function?

**Theorem**

The following conditions are equivalent:

- \( \sum_{n \geq 0} f(n)x^n \) is a rational generating function \( \frac{P(x)}{Q(x)} \),
  \[ Q(x) = 1 + b_1x + \ldots + b_dx^d. \]

- For fixed \( b_i \) and \( d \) and sufficiently large \( n \),
  \[ f(n) + b_1f(n - 1) + \ldots + b_d f(n - d) = 0. \]

**Proof.**

\[
Q(x) \sum_{n \geq 0} f(n)x^n = \sum_{n \geq 0} (f(n) + b_1f(n - 1) + \ldots + b_d f(n - d))x^n = P(x)
\]
Fibonacci Numbers

Theorem

- \(\sum_{n\geq 0} f(n)x^n\) is a rational generating function \(P(x)/Q(x)\),
  \[Q(x) = 1 + b_1x + \ldots + b_dx^d.\]
- For fixed \(b_i\) and \(d\) and sufficiently large \(n\),
  \(f(n) + b_1f(n-1) + \ldots + b_df(n-d) = 0\).

Example

- \(f(0) = 0,\ f(1) = 1,\ f(n) - f(n-1) - f(n-2) = 0\) for \(n \geq 2\)
- \(\sum_{n\geq 0} f(n)x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6\ldots\)
- \(Q(x) = 1 - x - x^2\) and \(Q(x) \sum_{n\geq 0} f(n)x^n = x\)
- \(\sum_{n\geq 0} f(n)x^n = \frac{x}{1 - x - x^2}\)
Lemma

\[(1 + x)^j = \sum_{n \geq 0} \frac{j(j - 1) \ldots (j - n + 1)}{n!} x^n\]

for all \(j \in \mathbb{C}\).

Lemma

\[\frac{\beta}{(1 - \gamma x)^j} = \sum_{n \geq 0} \left( \frac{\beta(n + j - 1)(n + j - 2) \ldots (n + 1)}{(j - 1)!} \gamma^n \right) x^n\]

for all positive integers \(j\).
Theorem

Let $S(x)$ and $Q(x)$ be polynomials over $\mathbb{C}$. Then

$$\frac{S(x)}{Q(x)} = R(x) + \frac{P(x)}{Q(x)}$$

where the degree of $P(x)$ be less than the degree of $Q(x)$.

Theorem

Let the degree of $P(x)$ be less than the degree of $Q(x)$. Then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \frac{\beta_{ij}}{(1 - \gamma_i x)^j}$$

where $Q(x) = \prod_{i=1}^{k} (1 - \gamma_i x)^{d_i}$.
Fundamental Property

$$Q(x) = \prod_{i=1}^{k} (1 - \gamma_i x)^{d_i}$$ has degree $n$

**Proof.**

Letting $P(x)$ vary, we can consider the vector spaces over $\mathbb{C}$

$$V_1 = \left\{ f : \mathbb{N} \to \mathbb{C} \text{ such that } \sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)}, \ deg P(x) < n \right\}$$

$$V_2 = \left\{ f : \mathbb{N} \to \mathbb{C} \text{ such that } \sum_{n \geq 0} f(n)x^n = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \frac{\beta_{ij}}{(1 - \gamma_i x)^j} \right\}$$

1. $V_2 \subseteq V_1$
2. $\{x^i/Q(x) \mid 0 \leq i < n\}$ spans $V_1$
3. $\{1/(1 - \gamma_i x)^j \mid 1 \leq i \leq k, 1 \leq j \leq d_i\}$ is linearly independent in $V_2$
Fibonacci Numbers

Example

\[ f(0) = 0, \ f(1) = 1, \ f(n) - f(n-1) - f(n-2) = 0 \text{ for } n \geq 2 \]

\[
\sum_{n \geq 0} f(n)x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 \ldots = \frac{x}{1 - x - x^2}
\]

\[ = \frac{x}{(1 - \left(\frac{1+\sqrt{5}}{2}\right)x)(1 - \left(\frac{1-\sqrt{5}}{2}\right)x)} = \frac{1/\sqrt{5}}{1 - \left(\frac{1+\sqrt{5}}{2}\right)x} + \frac{-1/\sqrt{5}}{1 - \left(\frac{1-\sqrt{5}}{2}\right)x} \]

Thus,

\[ f(n) = \frac{1}{\sqrt{5}} \left( \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right) \]
Thank You

I would like to thank

- My mentor, Aleksandra Utiralova
- The PRIMES program
- Sasha Shashkov
- My parents