Partially Ordered Sets

Sasha Shashkov

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A poset is a set $P$ equipped with a binary relation denoted $\leq$. Similar to totally ordered set such as $\mathbb{N}$ or $\mathbb{R}$ except two elements may be incomparable.

Posets satisfy the following axioms:

1. **Reflexivity**: $\forall t \in P$, $t \leq t$ and $t \geq t$.
2. **Anti-Symmetry**: If $s \geq t$ and $s \leq t$, then $s = t$.
3. **Transitivity**: If $s \geq t$ and $t \geq u$, then $s \geq u$. 
Definition

For $s, t \in P$, we say that $s$ covers $t$ if $s > t$ and there is not $u \in P$ such that $s > u > t$.

Definition

A Hasse Diagram is an undirected graph visualizing cover relations, with each element as a vertex and an edge between two elements if one covers the other. The greater element is ”above” the lesser one.
Examples

$B_3$

$D_{12}$
**Definition**

The product $P \times Q$ of two posets $P$ and $Q$ is the poset of all pairs $(s, t)$ with $s \in P$ and $t \in Q$ with $(s, t) \leq (s', t')$ iff $s \leq s'$ and $t \leq t'$.

**Example**

The Hasse Diagram for the poset $2 \times 2$ is shown below. It is isomorphic to $B_2$. In fact, $2^n \cong B_n$.

![Hasse Diagram](image-url)
Subsets

Definition

A *chain* of length $\ell$ is a sequence of elements $t_0 < t_1 < t_2 \ldots < t_\ell$ in $P$, and a *multichain* of length $\ell$ is a sequence of elements $t_0 \leq t_1 \leq t_2 \ldots \leq t_\ell$.

Definition

An *order ideal* of an element of $P$ is a set of elements $I$ such that if $t \in I$ and $s \leq t$, then $s \in I$.

Definition

An *interval* $[s, t]$ of $P$ is the set of all elements $u$ such that $s \leq u \leq t$. $Int(P)$ is the set of all intervals in $P$. 
The Incidence Algebra of a poset $P$, denoted $I_P$ or just $I$, is the algebra of all functions $f : \text{Int}(P) \rightarrow K$, where $K$ is any field.

Multiplication or convolution is defined as:

$$fg(s, t) = \sum_{s \leq u \leq t} f(s, u)g(u, t)$$

The identity $\delta$ (or sometimes 1) is:

$$\delta(s, t) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$$
The Zeta Function

The zeta function $\zeta$ is defined by $\zeta(s, t) = 1$ for all $s \leq t \in P$.

We can see that

$$\zeta^2(s, t) = \sum_{s \leq u \leq t} \zeta(s, u)\zeta(u, t) = \sum_{s \leq u \leq t} 1 = \#[s, t]$$

or more generally

$$\zeta^n(s, t) = \sum_{s = t_0 \leq t_1 \ldots \leq t_n = t} 1 = \# \text{ of multichains of length } n \text{ from } s \text{ to } t$$

We can also show

$$(2 - \zeta)^{-1}(s, t) = \text{total number of chains from } s \text{ to } t$$
The Möbius Function

Definition

The Möbius function $\mu$ (or $\mu_P$ for a specific poset) is the inverse of $\zeta$, so $\mu(\zeta(s, t)) = \zeta(\mu(s, t)) = \delta(s, t)$. It is defined explicitly as

$$
\mu(s, t) = \begin{cases} 
1 & s = t \\
-\sum_{s \leq u < t} \mu(s, u) & s < t 
\end{cases}
$$

Theorem

Let $P$ and $Q$ be finite posets and let $P \times Q$ be their product poset. Let $s, s' \in P$ and $t, t' \in Q$ such that $(s, t) \leq (s', t')$ in $P \times Q$. Then

$$
\mu_{P \times Q}((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t')
$$
Möbius Inversion

Theorem

If $P$ is a poset such that every order ideal is finite, and $f, g$ are two functions from $P \rightarrow K$, where $K$ is any field, then

$$f(t) = \sum_{s \leq t} g(s)$$

if and only if:

$$g(t) = \sum_{s \leq t} f(s) \mu(s, t)$$

for all $t \in P$.

We also have the dual version:

$$f(t) = \sum_{s \geq t} g(s)$$

if and only if:

$$g(t) = \sum_{s \geq t} f(s) \mu(s, t)$$
We can quickly calculate that $\mu_2(0, 1) = -1$. Since $2^n \cong B_n$, we calculate that

$$\mu_{B_n}(s, t) = \prod_i \mu_2(s_i, t_i) = (-1)^{#(s \neq t)}$$

where $s_i = 1$ if $i \in s$ and 0 if not ($t_i$ is defined analogously)

As sets this means $\mu_{B_n}(S, T) = (-1)^{#(S - T)}$. Dual Möbius inversion then gives us:

$$f(T) = \sum_{S \supseteq T} g(S) \text{ if and only if: } g(T) = \sum_{S \supseteq T} f(S)(-1)^{#(S - T)}$$

This statement is equivalent to the Principle of Inclusion-Exclusion.
If \( n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \), then

\[ D_n \cong (a_1 + 1) \times (a_2 + 1) \times \ldots \times (a_k + 1) \]

Easy computation gives us

\[ \mu(s, n) = \mu \left( \frac{n}{s} \right) = \begin{cases} (-1)^p & \text{if } \frac{n}{s} \text{ is square-free} \\ 0 & \text{otherwise} \end{cases} \]

Where \( p \) is the number of prime factors of \( \frac{n}{s} \). Möbius inversion gives

\[ f(n) = \sum_{s \mid n} g(s) \text{ if and only if: } g(n) = \sum_{s \mid n} f(s) \mu \left( \frac{n}{s} \right) \]

This formula is known as the Möbius transform.