**Problem:** A fish population starts out at 50 fish and grows 4-fold each year with 100 fish dying each year

**Mathematical Formalism**
- Population at time $t$ is $p_t$
- Recurrence: $p_t = 4 \cdot p_{t-1} - 100$
- Base case: $p_0 = 50$

**Natural question:** What is $p_t$ for any $t$?
Recurrence and Base Case:  \( p_t = 4 \cdot p_{t-1} - 100 \), with \( p_0 = 50 \)

Iterative Calculations

- \( p_0 = 50 \)
- \( p_1 = 100 \)
- \( p_2 = 300 \)
- \( p_3 = 1100 \)
- \( p_4 = 4300 \)

We want a closed form!
A generating function takes a sequence of real numbers and makes it the coefficients of a formal power series.

Generating Function

Let \( \{f_n\}_{n \geq 0} \) be a sequence of real numbers. Then the formal power series

\[
F(x) = \sum_{n \geq 0} f_n x^n
\]

is called the **ordinary generating function** of the sequence \( \{f_n\}_{n \geq 0} \).
When using generating functions we will look at power series \textit{formally}, meaning we \textit{ignore convergence}.

**Convergence**

Consider the power series expansion

\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots.
\]

When \(|x| < 1\), you can plug in \(x\) and the RHS = LHS. For example, when \(x = \frac{1}{2}\):

\[
\frac{1}{1 - \frac{1}{2}} = 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.
\]
Example Cont.

\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots.
\]

When \(|x| > 1\), plugging in \(x\) does not yield meaningful equalities. Consider \(x = 2\):

\[
\frac{1}{1-2} = -\frac{1}{2} \neq 1 + 2 + 4 + 8 + \ldots = \infty.
\]

**Formal power series:** Do not plug in values for \(x\), because it is meaningless! We only care about the coefficients of the series.
Define the generating function:

\[ G(x) = \sum_{n \geq 0} p_n x^n. \]

First few terms: \( G(x) = 50 + 100x + 300x^2 + \ldots \)

Express Recurrence: \( p_{t+1} = 4 \cdot p_t - 100 \)

\[ \sum_{n \geq 0} p_{n+1} \cdot x^{n+1} = \sum_{n \geq 0} (4 \cdot p_n - 100) \cdot x^{n+1} \]

\[ = \sum_{n \geq 0} 4 \cdot p_n \cdot x^{n+1} - \sum_{n \geq 0} 100 \cdot x^{n+1} \]
Generating Function equality:

\[ \sum_{n \geq 0} p_{n+1} \cdot x^{n+1} = \sum_{n \geq 0} 4 \cdot p_n \cdot x^{n+1} - \sum_{n \geq 0} 100 \cdot x^{n+1} \]

- Left hand side: \( G(x) - p_0 \), since it’s missing the first term of the sequence \( \{p_n\}_{n \geq 0} \)
- Right hand side term 1: \( 4x \cdot G(x) \)
- Right hand side term 2: \( - \frac{100x}{1-x} \), since \( \frac{1}{1-x} = 1 + x + x^2 + \ldots \)

Recurrence in terms of \( G(x) \):

\[ G(x) - p_0 = 4x \cdot G(x) - \frac{100x}{1-x} \]
Want to solve following equation for closed form for $p_t$:

$$G(x) - p_0 = 4x \cdot G(x) - \frac{100x}{1 - x}$$

After rearranging,

$$G(x) = \frac{p_0}{1 - 4x} - \frac{100x}{(1 - x)(1 - 4x)}.$$

We have obtained an explicit formula for the $G(x)$, the generating function of the sequence $\{p_n\}$. 
Want closed form for coefficient of \( x^n \) in \( G(x) \) because this is \( p_n \).

\[
G(x) = \frac{p_0}{1 - 4x} - \frac{100x}{(1 - x)(1 - 4x)}.
\]

First term’s contribution is easy to calculate:

\[
\frac{p_0}{1 - 4x} = 50 \sum_{n \geq 0} (4x)^n = 50 \sum_{n \geq 0} 4^n x^n
\]
Expanding 2nd term yields confusion:

\[
\frac{100x}{(1 - x)(1 - 4x)} = 100x \sum_{n \geq 0} x^n \cdot \sum_{n \geq 0} 4^n x^n.
\]

Another approach: partial fraction decomposition

We want to find constants \(A\) and \(B\) such that

\[
\frac{100x}{(1 - x)(1 - 4x)} = \frac{A}{1 - x} + \frac{B}{1 - 4x}.
\]

With \(A = \frac{100}{3}\) and \(B = -\frac{100}{3}\),

\[
\frac{100x}{(1 - x)(1 - 4x)} = \frac{100}{3} \cdot \frac{1}{1 - 4x} - \frac{100}{3} \cdot \frac{1}{1 - x}.
\]
Using Partial Fractions

\[
\frac{100x}{(1-x)(1-4x)} = \frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x}.
\]

Expanding using power series yields:

\[
\frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x} = \frac{100}{3} \left( \sum_{n\geq0} 4^n x^n - \sum_{n\geq0} x^n \right).
\]

Thus 2nd term’s contribution to coefficient of \(x^n\) is:

\[
\frac{100}{3} (4^n - 1).
\]
An explicit formula for $p_n$

Recall

$$G(x) = \frac{p_0}{1 - 4x} - \frac{100x}{(1 - x)(1 - 4x)}.$$

First term’s contribution:

$$50 \cdot 4^n.$$

Second term’s contribution:

$$\frac{100}{3} (4^n - 1).$$

Combining contributions, closed-form formula for $p_n$ is:

$$p_n = 50 \cdot 4^n - 100 \cdot \frac{4^n - 1}{3}.$$
Exponential generating functions are every similar to ordinary generating functions.

**Exponential Generating Function**

Let \( \{f_n\}_{n \geq 0} \) be a sequence of real numbers. Then the formal power series

\[
F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!},
\]

is called the *exponential generating function* of the sequence \( \{f_n\}_{n \geq 0} \).

*Intuition:* Dividing by \( n! \) allows for \( f_n \) to grow faster.
Motivating Example

Recurrence Relation: Solve for $a_n$ if $a_0 = 1$, and $a_n$ satisfies the following recurrence

$$a_{n+1} = (n + 1)(a_n - n + 1).$$

First few terms

- $a_0 = 1$
- $a_1 = 2$
- $a_2 = 4$
- $a_3 = 9$
- $a_4 = 28$
- $a_5 = 125$

This series grows too fast for an ordinary generating function. Therefore an exponential generating function is used.
Solving recurrence with exponential generating functions

Defining generating function:

\[ A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \]

is the exponential generating function of the sequence \( \{a_n\}_{n \geq 0} \).

Expressing recurrence \( a_{n+1} = (n + 1)(a_n - n + 1) \):

\[
\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n + 1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n - 1) \frac{x^{n+1}}{n!}.
\]
Solving recurrence cont.

\[ \sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!}. \]

- LHS $= A(x) - 1$
- RHS first term: $xA(x)$
- RHS second term: $-x^2 e^x + xe^x = (x - x^2)e^x$

Plugging in above:

\[ A(x) - 1 = xA(x) - x^2 e^x + xe^x. \]

Rearranging yields,

\[ A(x) = \frac{1}{1 - x} + xe^x. \]

Thus coefficient $a_n$ for $\frac{x^n}{n!}$ is $a_n = n! + n$. 

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Generating Functions

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The End