On the Okounkov-Olshanski Formula for the Number of Tableaux of Skew Shapes

Daniel Zhu\textsuperscript{1}
mentor: Prof. Alejandro Morales\textsuperscript{2}

\textsuperscript{1}Montgomery Blair High School
\textsuperscript{2}UMass Amherst

May 20, 2018
MIT PRIMES Conference
Partitions

- **Partition**: way to write a nonnegative integer as a sum of positive integers, without regard to order

\[ 9 = 3 + 3 + 2 + 1 \quad \lambda = (3, 3, 2, 1) \quad |\lambda| = 9 \]

- **Young diagram**: grid of boxes representing a partition

\[
[\lambda] = \begin{array}{ccc}
1 & 2 & 3 \\
& 3 & \\
& & \\
\end{array}
\]
Partitions

- Partition: way to write a nonnegative integer as a sum of positive integers, without regard to order

\[ 9 = 3 + 3 + 2 + 1 \quad \lambda = (3, 3, 2, 1) \quad |\lambda| = 9 \]

- Young diagram: grid of boxes representing a partition

\[
\begin{array}{cccc}
\hline
& & & 3 \\
& & 3 \\
& 1 & & & 2 \\
\hline
\end{array}
\]

Daniel Zhu  The Okounkov-Olshanski Formula
Standard Young Tableaux

- Standard Young tableau (SYT): way to fill in boxes of a Young diagram with 1 through $|\lambda|$, with rows and columns strictly increasing

$$\begin{array}{ccc}
1 & 3 & 6 \\
2 & 5 & 8 \\
4 & 7 \\
9
\end{array}$$

- $f^\lambda$: number of SYT of shape $\lambda$
- $f^\lambda$ is the dimension of an irreducible representation of the symmetric group
Standard Young Tableaux

- Standard Young tableau (SYT): way to fill in boxes of a Young diagram with 1 through $|\lambda|$, with rows and columns strictly increasing

```
  1 3 6
  2 5 8
  4 7
  9
```

- $f^\lambda$: number of SYT of shape $\lambda$

- $f^\lambda$ is the dimension of an irreducible representation of the symmetric group
Standard Young Tableaux

- Standard Young tableau (SYT): way to fill in boxes of a Young diagram with 1 through $|\lambda|$, with rows and columns strictly increasing

$$
\begin{array}{ccc}
1 & 3 & 6 \\
2 & 5 & 8 \\
4 & 7 & \\
9 & \\
\end{array}
$$

- $f^\lambda$: number of SYT of shape $\lambda$
- $f^\lambda$ is the dimension of an irreducible representation of the symmetric group
Hook-Length Formula

**Theorem (Frame-Robinson-Thrall, 1954)**

\[
f^\lambda = \frac{|\lambda|!}{\prod_{u \in [\lambda]} h(u)}
\]

- \(h(u)\) is the size of a hook of a cell \(u\)

![Diagram](image)

\(h(u) = 5\)
Applications of the Hook-Length Formula

- Hook lengths of \((n, n)\) are

\[
\begin{array}{ccc}
 n+1 & n & \cdots \\
 n & n-1 & \cdots \\
\end{array}
\]

SO

\[
f^{(n,n)} = \frac{(2n)!}{\prod_{u \in [(n,n)]} h(u)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C_n
\]

- There exists a bijection between SYT of shape \((n, n)\) and Dyck paths
Applications of the Hook-Length Formula

► Hook lengths of \((n, n)\) are

\[
\begin{array}{|c|c|}
\hline
n+1 & n \\
\hline
n & n-1 \\
\hline
\end{array}
\quad \ldots \quad 
\begin{array}{|c|c|}
\hline
3 & 2 \\
\hline
2 & 1 \\
\hline
\end{array}
\]

So

\[
f^{(n,n)} = \frac{(2n)!}{\prod_{u \in [(n,n)]} h(u)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C_n
\]

► There exists a bijection between SYT of shape \((n, n)\) and Dyck paths

Daniel Zhu  The Okounkov-Olshanski Formula
Applications of the Hook-Length Formula

- Hook lengths of \((n, n)\) are

\[
\begin{array}{ccc}
 n+1 & n & \cdots \\
 n & n-1 & \cdots \\
 3 & 2 & \cdots \\
 2 & 1 & \\
\end{array}
\]

so

\[
f^{(n,n)} = \frac{(2n)!}{\prod_{u \in [(n,n)]} h(u)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C_n
\]

- There exists a bijection between SYT of shape \((n, n)\) and Dyck paths
Applications of the Hook-Length Formula

- Hook lengths of $(n, n)$ are

\[
\begin{array}{cc}
  n+1 & n \\
  n & n-1 \\
  & \vdots \\
  & \vdots \\
  3 & 2 \\
  2 & 1 \\
\end{array}
\]

so

\[
f^{(n,n)} = \frac{(2n)!}{\prod_{u \in [(n,n)]} h(u)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C_n
\]

- There exists a bijection between SYT of shape $(n, n)$ and Dyck paths
Consider $\mu$ so that $[\mu] \subseteq [\lambda]$.
Example: take $\lambda = (3,3,2,1)$, $\mu = (2,1)$

$\lambda / \mu$: number of SYT of shape $\lambda / \mu$

$\lambda / \mu$ are dimensions of irreducible representations of Hecke algebras

Is there a formula for $f^{\lambda / \mu}$?
Standard Young Tableaux of Skew Shape

Consider $\mu$ so that $[\mu] \subseteq [\lambda]$
Example: take $\lambda = (3, 3, 2, 1), \mu = (2, 1)$

$\lambda = (3, 3, 2, 1), \mu = (2, 1)$

$\mathbf{f}^{\lambda/\mu}$: number of SYT of shape $\lambda/\mu$
$\mathbf{f}^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras

Is there a formula for $\mathbf{f}^{\lambda/\mu}$?
Consider $\mu$ so that $[\mu] \subseteq [\lambda]$

Example: take $\lambda = (3, 3, 2, 1), \mu = (2, 1)$

$f^{\lambda/\mu}$: number of SYT of shape $\lambda/\mu$

$f^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras

Is there a formula for $f^{\lambda/\mu}$?
Consider $\mu$ so that $[\mu] \subseteq [\lambda]$.

Example: take $\lambda = (3,3,2,1)$, $\mu = (2,1)$

$[\lambda/\mu] = \begin{array}{cccc}
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{array}$

$\lambda = \begin{array}{cccc}
2 & 1 & 4 & \\
3 & 6 & & \\
5 & & & \\
\end{array}$

$\mu = \begin{array}{cccc}
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{array}$

- $f^{\lambda/\mu}$: number of SYT of shape $\lambda/\mu$
- $f^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras
- Is there a formula for $f^{\lambda/\mu}$?
Standard Young Tableaux of Skew Shape

Consider $\mu$ so that $[\mu] \subseteq [\lambda]$
Example: take $\lambda = (3, 3, 2, 1), \mu = (2, 1)$

$f^{\lambda/\mu}$: number of SYT of shape $\lambda/\mu$
$f^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras
Is there a formula for $f^{\lambda/\mu}$?
Consider \( \mu \) so that \([\mu] \subseteq [\lambda]\)
Example: take \( \lambda = (3, 3, 2, 1) \), \( \mu = (2, 1) \)

\[
[\lambda/\mu] = \begin{array}{cccc}
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{array}
\]

\( f^{\lambda/\mu} \): number of SYT of shape \( \lambda/\mu \)

\( f^{\lambda/\mu} \) are dimensions of irreducible representations of Hecke algebras

Is there a formula for \( f^{\lambda/\mu} \)?
Consider $\mu$ so that $[\mu] \subseteq [\lambda]$

Example: take $\lambda = (3, 3, 2, 1), \mu = (2, 1)$

$[\lambda/\mu] = \begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{array}$

$f^{\lambda/\mu}$: number of SYT of shape $\lambda/\mu$

$f^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras

Is there a formula for $f^{\lambda/\mu}$?
Consider $\mu$ so that $[\mu] \subseteq [\lambda]$
Example: take $\lambda = (3, 3, 2, 1)$, $\mu = (2, 1)$

$[\lambda/\mu] = \begin{array}{c}
\ \ \ \ \ \ \\
\ \ \ \ \ \\
\ \ \ \ \\
\ \ \ \\
\ \ \\
\ \\
\end{array}$

$\begin{array}{cccc}
2 & 1 & 4 \\
3 & 6 \\
5 & \\
\end{array}$

$\lambda/\mu$: number of SYT of shape $\lambda/\mu$

$\lambda/\mu$ are dimensions of irreducible representations of Hecke algebras

Is there a formula for $f^{\lambda/\mu}$?
Bad News: No Product Formula

- One can compute
  \[ f = 61 \]
  which is prime
- No nontrivial product formula
- In particular, \( f^{\lambda/\mu} \) does not divide \(|\lambda/\mu|!\) anymore
Bad News: No Product Formula

- One can compute

\[ f = 61 \]

which is prime

- No nontrivial product formula
- In particular, \( f^{\lambda/\mu} \) does not divide \(|\lambda/\mu|\) anymore
Bad News: No Product Formula

- One can compute
  \[ f = 61 \]
  which is prime

- No nontrivial product formula
- In particular, \( f^{\lambda/\mu} \) does not divide \( |\lambda/\mu|! \) anymore
Bad News: No Product Formula

- One can compute
  \[ f = 61 \]
  which is prime
- No nontrivial product formula
- In particular, \( f^{\lambda/\mu} \) does not divide \( |\lambda/\mu|! \) anymore
The Okounkov-Olshanski Formula

Theorem (Okounkov-Olshanski, 1996)

\[ f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u \in [\lambda]} h(u)} \sum_{T \in SSYT(\mu,d)} \prod_{(i,j) \in [\mu]} (\lambda_{d+1-T(i,j)} + i - j) \]

- SSYT(\mu, d) is semistandard tableaux of shape \( \mu \) with all entries at most \( d \)

\[
\begin{array}{ccc}
1 & 1 & 5 \\
2 & 2 & 6 \\
8 & 9 \\
9
\end{array}
\]
The Okounkov-Olshanski Formula

**Theorem (Okounkov-Olshanski, 1996)**

\[
f_{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u \in [\lambda]} h(u)} \sum_{T \in \text{SSYT}(\mu, d)} \prod_{(i, j) \in [\mu]} (\lambda_{d+1-T(i,j)} + i - j)
\]

\[
f_{\begin{array}{|c|c|}
1 & 1 \\
\hline 1 & 2 \\
\hline
\end{array}} = \frac{5!}{5 \cdot 4 \cdot 3 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \left( \frac{3 \cdot 2}{1 \cdot 1} + \frac{3 \cdot 3}{1 \cdot 2} + \frac{4 \cdot 3}{2 \cdot 2} \right)
\]

\[
= \frac{1}{3} \cdot 27 = 9
\]
Observations About the Formula

Theorem (Okounkov-Olshanski, 1996)

\[ f_{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u \in [\lambda]} h(u)} \sum_{T \in SSYT(\mu, d)} \prod_{(i, j) \in [\mu]} (\lambda_{d+1-T(i,j)} + i - j) \]

- All terms are nonnegative
- How many nonzero terms are there? (call this \( T(\lambda/\mu) \))
Observations About the Formula

Theorem (Okounkov-Olshanski, 1996)

\[ f_{\lambda/\mu} = \frac{\mid \lambda/\mu \mid !}{\prod_{u \in \lambda} h(u)} \sum_{T \in \text{SSYT} (\mu, d)} \prod_{(i,j) \in \mu} (\lambda_{d+1-T(i,j)} + i - j) \]

- All terms are nonnegative
- How many nonzero terms are there? (call this \( T(\lambda/\mu) \))
Observations About the Formula

Theorem (Okounkov-Olshanski, 1996)

\[
f_{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u \in [\lambda]} h(u)} \sum_{T \in SSYT(\mu,d)} \prod_{(i,j) \in [\mu]} (\lambda_{d+1-T(i,j)} + i - j)
\]

- All terms are nonnegative
- How many nonzero terms are there? (call this \(T(\lambda/\mu)\))
The Number of Nonzero Terms
An RPP $q$-Analogue

Our Result

Theorem (Morales-Z., 2018+)

$$T(\lambda/\mu) = \det \left[ \left( \begin{array}{c} \lambda_i - \mu_j + j - 1 \\ i-1 \end{array} \right) \right]_{i,j=1}^d$$

$T((4, 3)/(2)) = 3$ by previous example

$$\det \begin{bmatrix} \begin{array}{cc} 2 & 5 \\ 0 & 0 \end{array} & \begin{array}{cc} 1 & 4 \\ 1 & 1 \end{array} \end{bmatrix} = 1 \cdot 4 - 1 \cdot 1 = 3$$

Daniel Zhu
The Okounkov-Olshanski Formula
Our Result

Theorem (Morales-Z., 2018+)

\[ T(\lambda/\mu) = \text{det} \left[ \left( \lambda_i - \mu_j + j - 1 \right) \right]_{i,j=1}^d \]

\[ T((4, 3)/(2)) = 3 \text{ by previous example} \]

\[ \text{det} \begin{bmatrix} 2 & 5 \\ 0 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} = 1 \cdot 4 - 1 \cdot 1 = 3 \]
Theorem (Morales-Z., 2018+)

\[ T(\lambda/\mu) = \det \left[ \left( \begin{array}{c} \lambda_i - \mu_j + j - 1 \\ i - 1 \end{array} \right) \right]_{i,j=1}^d \]

\[ T((4, 3)/(2)) = 3 \text{ by previous example} \]

\[ \det \begin{bmatrix} (2, 0) & (5, 0) \\ (1, 1) & (4, 1) \end{bmatrix} = 1 \cdot 4 - 1 \cdot 1 = 3 \]
Proof Idea

Daniel Zhu

The Okounkov-Olshanski Formula
Proof Idea
Proof Idea
Proof Idea
The Number of Nonzero Terms
An RPP $q$-Analogue

Proof Idea

[Diagram with blue lines]

Daniel Zhu
The Okounkov-Olshanski Formula
Proof Idea
Reverse Plane Partitions

- Reverse plane partition (RPP): nonnegative integers, rows and columns weakly increasing

\[
\begin{array}{ccc}
0 & 0 & 3 \\
1 & 1 & 3 \\
8 & 8 & \leq \\
8
\end{array}
\]

- \text{RPP}(\lambda/\mu): RPP of shape \lambda/\mu
An RPP Generating Function

Definition

For $T \in \text{RPP}(\lambda/\mu)$, let $|T|$ be the sum of entries in $T$. Then let

$$\text{rpp}_{\lambda/\mu}(q) = \sum_{T \in \text{RPP}(\lambda/\mu)} q^{|T|}.$$ 

Theorem (Stanley, 1971)

$$\lim_{q \to 1} \text{rpp}_{\lambda/\mu}(q) \cdot (1 - q)^{|\lambda/\mu|} = \frac{f_{\lambda/\mu}}{|\lambda/\mu|!}$$
An RPP Generating Function

**Definition**

For $T \in \text{RPP}(\lambda / \mu)$, let $|T|$ be the sum of entries in $T$. Then let

$$
rpp_{\lambda / \mu}(q) = \sum_{T \in \text{RPP}(\lambda / \mu)} q^{|T|}.
$$

**Theorem (Stanley, 1971)**

$$
\lim_{q \to 1} rpp_{\lambda / \mu}(q) \cdot (1 - q)^{|\lambda / \mu|} = \frac{f_{\lambda / \mu}}{|\lambda / \mu|!}
$$
Our Result

Theorem (Morales-Z., 2018+)

\[
\frac{\text{rpp}_{\lambda/\mu}(q)}{\text{rpp}_{\lambda}(q)} = \sum_{T \in \text{SSYT}(\mu, d)} q^{p(T)} \prod_{u \in \mu} (1 - q^{w(u, T(u))})
\]

where

- \( w(u, t) = \lambda_{d+1-t} + i - j \) where \( u = (i, j) \)
- \( p(T) = \sum_{u \in \mu, m_T(u) \leq t < T(u)} w(u, t) \)
- \( m_T(u) \) is the minimum positive integer \( t \) such that replacing the entry of \( u \) in \( T \) with \( t \) still yields a semistandard tableau
Our Result

Theorem (Morales-Z., 2018+)

\[
\frac{\text{rpp}_{\lambda/\mu}(q)}{\text{rpp}_{\lambda}(q)} = \sum_{T \in \text{SSYT}(\mu,d)} q^{p(T)} \prod_{u \in [\mu]} \left(1 - q^{w(u,T(u))}\right)
\]

where

- \( w(u, t) = \lambda_{d+1-t} + i - j \) where \( u = (i, j) \)
- \( p(T) \) is a sum of \( w(u, t) \) for some \( u, t \)

Chen and Stanley have a similar \( q \)-analogue for semistandard tableaux
Other results:

- Two other determinant formulas for $T(\lambda/\mu)$
- Equivalence of formulas for $f^{\lambda/\mu}$ of Okounkov-Olshanski and Knutson-Tao

Future work:

- Prove $q$-analogue without equivariant $K$-theory, possibly combinatorially
- Relate Okounkov-Olshanski to other formulas
Other Work

Other results:

- Two other determinant formulas for $T(\lambda/\mu)$
- Equivalence of formulas for $f^{\lambda/\mu}$ of Okounkov-Olshanski and Knutson-Tao

Future work:

- Prove $q$-analogue without equivariant $K$-theory, possibly combinatorially
- Relate Okounkov-Olshanski to other formulas
Other Work

Other results:

- Two other determinant formulas for $T(\lambda/\mu)$
- Equivalence of formulas for $f^{\lambda/\mu}$ of Okounkov-Olshanski and Knutson-Tao

Future work:

- Prove $q$-analogue without equivariant $K$-theory, possibly combinatorially
- Relate Okounkov-Olshanski to other formulas
Other Work

Other results:

▶ Two other determinant formulas for $T(\lambda/\mu)$
▶ Equivalence of formulas for $f^{\lambda/\mu}$ of Okounkov-Olshanski and Knutson-Tao

Future work:

▶ Prove $q$-analogue without equivariant $K$-theory, possibly combinatorially
▶ Relate Okounkov-Olshanski to other formulas
Acknowledgements

- My mentor, Prof. Alejandro Morales
- MIT-PRIMES Program
- Dr. Tanya Khovanova
- MIT Math Department
- My family
Summary

Theorems (Morales-Z., 2018+)

\[ T(\lambda/\mu) = \det \left[ \left( \lambda_i - \mu_j + j - 1 \right) \right]_{i,j=1}^d \]

\[ \frac{\text{rpp}_{\lambda/\mu}(q)}{\text{rpp}_{\lambda}(q)} = \sum_{T \in \text{SSYT}(\mu,d)} q^{p(T)} \prod_{u \in [\mu]} \left( 1 - q^{w(u,T(u))} \right) \]