Approximating the Hurwitz Zeta Function

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Presentation
The Riemann Zeta function \( \zeta(s) \) is defined for complex inputs \( s \) as

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]
The Hurwitz Zeta Function

The Hurwitz Zeta function generalizes the Riemann Zeta.

Definition

The Hurwitz Zeta function $\zeta(s, a)$ is defined for complex inputs $s$ and $a$ with $\text{Re}(a) > 0$ and $\text{Re}(s) > 1$ as follows:

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$ 

Definition

Compare this to the definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

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**Definition**

Compare this to the definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$
The Hurwitz zeta function can be analytically continued to almost all complex arguments Re(s) ≤ 1 as follows:

\[ \zeta(s, q) = \frac{\Gamma(1-s)}{2\pi i} \int_C z^{s-1} e^{qz} \frac{1}{1-e^z} dz. \]
The Hurwitz zeta function can be analytically continued to almost all complex arguments $\Re(s) \leq 1$ as follows:

Theorem

$$\zeta(s, q) = \Gamma(1 - s) \frac{1}{2\pi i} \int_C \frac{z^{s-1}e^{qz}}{1 - e^z} \, dz.$$
The Hurwitz Zeta Function

**Figure:** A graph of the Hurwitz zeta function as a function of $a$ with $s = 3 + 4i$. 
Recall the definition of the Hurwitz Zeta function:

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Recall the definition of the Hurwitz Zeta function:

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**Query**

How can we approximate the Hurwitz Zeta function for any satisfactory inputs $s$ and $a$ to arbitrary precision?
Definition

The **Lerch Transcendent** \( \Phi(s, a, z) \) is defined for complex inputs \( s, a, z \) as

\[
\Phi(s, a, z) = \sum_{n=1}^{\infty} \frac{z^n}{(n + a)^s}.
\]
The **Lerch Transcendent** $\Phi(s, a, z)$ is defined for complex inputs $s, a, z$ as

$$\Phi(s, a, z) = \sum_{n=1}^{\infty} \frac{z^n}{(n + a)^s}.$$

Once again, recall that the Hurwitz Zeta function $\zeta(s, a)$ is defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.$$
The **gamma function** $\Gamma(s)$ is defined by the integral

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

2. The **upper incomplete gamma function** $\Gamma(s, z)$ is defined by the integral

$$\Gamma(s, z) = \int_z^\infty t^{s-1} e^{-t} dt$$

The incomplete gamma function generalizes the gamma function.
Theorem (Bailey–Borwein, 2015)

Let \( \lambda \) be a parameter with \( 0 < \lambda < 2\pi \). Define \( \sigma(x) \) to be the sign function. Then for real \( a \) and complex \( s \) with \( 0 < a < 1 \) and \( \text{Re}(s) > 1 \), we have

\[
\zeta(s, a) = \frac{\sqrt{\pi} \lambda^{s-1/2}}{(s-1)\Gamma\left(\frac{s}{2}\right)} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{|n+a|^s} \left( \frac{\Gamma\left(\frac{s}{2}, \lambda(n+a)^2\right)}{\Gamma\left(\frac{s}{2}\right)} \sigma(n+a) \frac{\Gamma\left(\frac{s+1}{2}, \lambda(n+a)^2\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right)
\]

\[
+ \pi^{s-1/2} \sum_{m=1}^{\infty} \frac{1}{m^{1-s}} \left( \frac{\Gamma\left(\frac{1-s}{2}, \frac{m^2\pi^2}{\lambda}\right)}{\Gamma\left(\frac{s}{2}\right)} \cos(2\pi ma) + \frac{\Gamma\left(1-\frac{s}{2}, \frac{m^2\pi^2}{\lambda}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \sin(2\pi ma) \right).
\]
Our Research

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Approximating the Hurwitz Zeta Function
Difficulties in Approximating the Hurwitz Zeta Function

Large Imaginary Parts

Set \( a = e^{m+pi} \), where \( 0 \leq p < 2\pi \), for brevity. Consider just the first term in our summation, \( \frac{1}{a^s} \).

\[
\left| \frac{1}{a^s} \right| = \frac{e^{p \cdot \text{Im}(s)}}{|a|^\text{Re}(s)}.
\]

Thus, when \( |a|^\text{Re}(s) \ll 1 \) or \( p \cdot \text{Im}(s) \gg 1 \), \( \frac{1}{a^s} \) may grow very large in magnitude.
Analyzing Convergence

Theorem

When $a > 0$ and $N$ some (presumably large) positive integer,

$$\left| \sum_{n=N+1}^{\infty} (n + a)^{-s} \right| < \frac{N^{1 - \text{Re}(s)}}{\text{Re}(s) - 1}.$$
Analyzing Convergence

**Theorem**

When $a > 0$ and $N$ some (presumably large) positive integer,

$$\left| \sum_{n=N+1}^{\infty} \frac{(n + a)^{-s}}{n^s} \right| < \frac{N^{1-\text{Re}(s)}}{\text{Re}(s) - 1}.$$

**Idea**

For real $s$, the series $\sum_{n=N}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

**Corollary**

To achieve $k$ digits of precision in $\zeta(s, a)$, we need $O\left(10^{\frac{k}{\text{Re}(s) - 1}}\right)$ terms as $k \to \infty$. 
Theorem

For real $s$ and $a$ with $s > 1$ and $a > 0$, and integer $n$ so that $|n + a| \geq \frac{s}{2}$ and $|n + a| \geq 10$,

$$\Gamma \left( \frac{s}{2}, \pi(n + a)^2 \right) < 10^{-(n+a)^2}.$$
Analyzing Convergence

Theorem

For real \( s \) and \( a \) with \( s > 1 \) and \( a > 0 \), and integer \( n \) so that \( |n + a| \geq \frac{s}{2} \) and \( |n + a| \geq 10 \),

\[
\Gamma \left( \frac{s}{2}, \pi(n + a)^2 \right) < 10^{-(n+a)^2}.
\]

Corollary

For any given ordered pair \((s, a)\), we need \( O(\sqrt{k}) \) terms to obtain \( k \) digits of precision in \( \zeta(s, a) \).
Future Research

- Analyze the performance of other series
- Expand the scope of our analyses to complex $s$ and/or $a$.
- Optimize Implementation
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