Anti-Ramsey Type Problems

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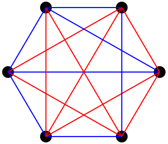

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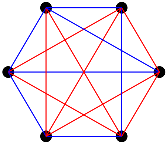

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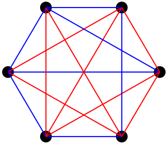

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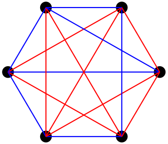

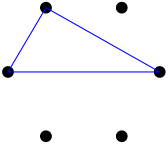

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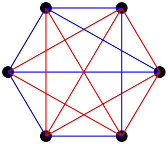

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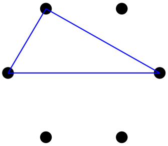
- For example, $R(3, 3) = 6$, so we can always find a monochromatic triangle in a K_6.
- $r_k(p)$ is the smallest n such that coloring the edges of K_n with k colors will always produce a monochromatic copy of K_p.



Generalized Ramsey Numbers

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- $f(n, p, q)$ is the minimal number of colors of a (p, q) coloring of K_n
- Finding an asymptotic estimate for $f(n, p, 2)$ is equivalent to finding an asymptotic estimate for $r_k(p)$ (difficult).

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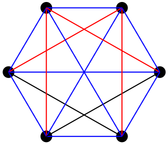
- But there does exist a $(3,2)$ coloring using 3 colors, so $f(6,3,2) = 3$:



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- A $(3, 3)$ coloring is equivalent to a proper edge-coloring (one in which no two adjacent edges have the same color), so $f(n, 3, 3)$ equals n for n odd and $n - 1$ for n even.

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- $f(n, 4, 4)$ is known to be $n^{1/2+o(1)}$ (also due to Mubayi)

More general bounds

Theorem (Erdős and Gyárfás, 1997)

For some c depending on p and q, $f(n,p,q) \leq cn^{\frac{p-2}{\binom{p}{2}-q+1}}$

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- However, Conlon et al. showed that $f(n,p,p-1)$ is subpolynomial in n
- Their coloring is a generalization of Mubayi's optimal coloring for $f(n,4,3)$

Our (4,3)-Coloring

Partition $\{1, 2, \cdots, n\}$ into $t = \lceil 2^{\sqrt{\log n}} \rceil$ equally sized sets and label them $1 - t$. Do this for $k = \lceil 2\sqrt{\log n} \rceil$ partitions so that every edge crosses between two sets in some partition.

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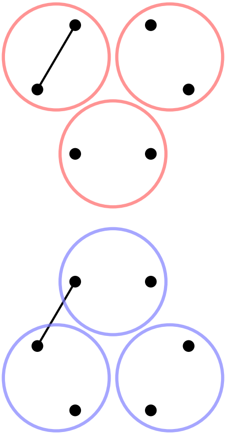

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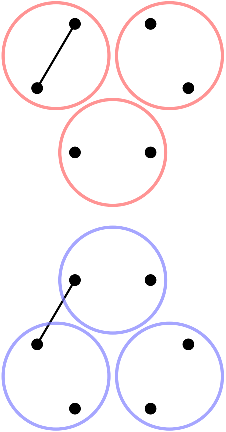

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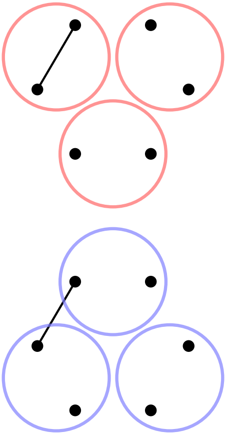

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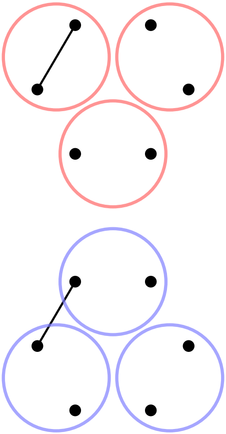

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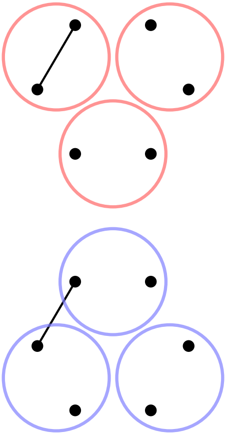
- The triple $(c_1(e), c_2(e), c_3(e))$ is the color of e.



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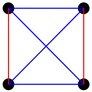

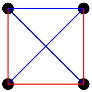

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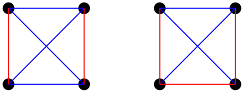

- In total we used $t^2 2^k$ colors, which is $2^{O(\sqrt{\log n})}$ since $k = \lceil 2\sqrt{\log n} \rceil$ and $t = \lceil 2^{\sqrt{\log n}} \rceil$.

Future work

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- Work on the lower bound

Acknowledgements

I would like to thank:

- My mentor, Dr. Asaf Ferber
- The MIT PRIMES-USA program
- The MIT math department
- My parents