Anti-Ramsey Type Problems

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Motivation: Ramsey Numbers

- Color each edge of the complete graph $K_n$ red or blue

Ramsey's Theorem:

There is always a blue copy of $K_r$ or a red copy of $K_s$ if $n$ is sufficiently large. The smallest such $n$ is denoted $R(r, s)$. For example, $R(3, 3) = 6$, so we can always find a monochromatic triangle in a $K_6$.

$r_k(p)$ is the smallest $n$ such that coloring the edges of $K_n$ with $k$ colors will always produce a monochromatic copy of $K_p$. 

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**Definition**

For positive integers $p$ and $q$ with $p \geq 3$ and $2 \leq q \leq \binom{p}{2}$, a $(p, q)$-coloring is an edge-coloring of $K_n$ where every copy of $K_p$ has at least $q$ distinct colors.
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- $f(n, p, q)$ is the minimal number of colors of a $(p, q)$ coloring of $K_n$
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- $f(n, p, q)$ is the minimal number of colors of a $(p, q)$ coloring of $K_n$
- Finding an asymptotic estimate for $f(n, p, 2)$ is equivalent to finding an asymptotic estimate for $r_k(p)$ (difficult).
Example: $f(6,3,2)$

- Since $R(3,3) = 6$, no coloring of $K_6$ with 2 colors can be a $(3,2)$-coloring. So $f(6,3,2) > 2$. 

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But there does exist a $(3,2)$ coloring using 3 colors, so $f(6,3,2) = 3$: 
A (3, 3) coloring is equivalent to a proper edge-coloring (one in which no two adjacent edges have the same color), so $f(n, 3, 3)$ equals $n$ for $n$ odd and $n – 1$ for $n$ even.
Small Cases

- A $(3,3)$ coloring is equivalent to a proper edge-coloring (one in which no two adjacent edges have the same color), so $f(n,3,3)$ equals $n$ for $n$ odd and $n−1$ for $n$ even.
- For $f(n,4,3)$, the best known lower bound is $\Omega(\log n)$ and best known upper bound is $2^{O(\sqrt{\log n})}$ (from a coloring constructed by Mubayi)
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$f(n, 4, 4)$ is known to be $n^{1/2 + o(1)}$ (also due to Mubayi)
More general bounds

**Theorem (Erdős and Gyárfás, 1997)**

For some $c$ depending on $p$ and $q$, $f(n, p, q) \leq cn^{\frac{p-2}{2} - q + 1}$

Their proof is nonconstructive (uses probabilistic method).

They also showed that $f(n, p, p)$ has to be polynomial in $n$.

However, Conlon et al. showed that $f(n, p, p - 1)$ is subpolynomial in $n$.

Their coloring is a generalization of Mubayi's optimal coloring for $f(n, 4, 3)$. 

Anti-Ramsey Type Problems
More general bounds

Theorem (Erdős and Gyárfás, 1997)

For some $c$ depending on $p$ and $q$, $f(n, p, q) \leq cn\left(\frac{p-2}{\binom{p}{2}}-q+1\right)$

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Our (4,3)-Coloring

Partition \{1, 2, \ldots, n\} into \( t = \lceil 2^{\log n} \rceil \) equally sized sets and label them 1 – \( t \). Do this for \( k = \lceil 2^{\sqrt{\log n}} \rceil \) partitions so that every edge crosses between two sets in some partition.

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For \( e = \{a, b\} \), let \( c_1(e) \) be the smallest \( i \) for which \( e \) is crossing in the \( i \)th partition. In the picture, \( c_1(e) = 2 \).
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- Let \(c_3(e)\) be a binary string of length \(k\) where the \(i\)th entry is 1 iff \(e\) is crossing in the \(i\)th partition. Here \(c_3(e) = (0, 1)\).
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- The triple \( (c_1(e), c_2(e), c_3(e)) \) is the color of \( e \).
Our (4,3)-Coloring.

Why does this work?

No monochromatic triangles
This leaves only the following bad $K_4$s:

In total we used $t^2k$, which is $2O(\sqrt{\log n})$ since $k = \lceil 2\sqrt{\log n} \rceil$ and $t = \lceil 2\sqrt{\log n} \rceil$. 

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Our (4,3)-Coloring.

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![Graphs](image-url)
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- In total we used $t^2 2^k$ colors, which is $2^{O(\sqrt{\log n})}$ since $k = \lceil 2\sqrt{\log n} \rceil$ and $t = \lceil 2\sqrt{\log n} \rceil$. 

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Modify the above coloring by choosing a coloring on $K_t$ and using this to determine $c_2(e)$. 

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Future work

- Modify the above coloring by choosing a coloring on $K_t$ and using this to determine $c_2(e)$.
- Work on the lower bound
I would like to thank:

- My mentor, Dr. Asaf Ferber
- The MIT PRIMES-USA program
- The MIT math department
- My parents