Agent-based Models for Conservation Equations

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Abstract

In this research, we use agent-based models to solve conservation equations. A conservation equation is a partial differential equation that describes any conserved quantity by establishing a relationship between the density and the flux. It is used in areas such as traffic flow and fluid dynamics. Past research on numerically solving conservation equations mainly tackles the problem by establishing discrete cells in the space and approximating the densities in the cells. In this research, we use an agent-based model, in which we describe the solution through the movement of particles in the space. We propose an agent-based model for conservation equation in 1-D space. We found a change of variables that transforms the original conservation equation to the specific volume conservation equation. This transform allows us to apply results in finite volume method to the agent-based model and find a condition for the agent-based solution to converge to the exact solution of scalar conservation equations.

1 Introduction

A well-known partial differential equation is the conservation equation. In one-dimensional space, the conservation equation is

$$\partial_t \rho(x,t) + \partial_x f(x,t) = 0, \tag{1}$$

where $\rho(x,t)$ is the density and $f(x,t)$ is the flux, both of which are scalar functions of position x and time t. Let the velocity function $v(x,t) = \frac{f(x,t)}{\rho(x,t)}$, then Equation (1) becomes $\partial_t \rho + \partial_x(\rho v) = 0$. When velocity only depends on the density, Equation (1) becomes

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0. \tag{2}$$

Equation (2) has many applications, such as modelling traffic systems where the velocity of cars only depends on the density of cars.

There is much previous research on solving Equation (2). It can be solved analytically using the method of characteristics [2]. People also developed numerical methods such as the finite difference and finite volume method for obtaining approximations of the solution [3]. These methods discretize the space domain into fixed cells and track how the density in each cell changes with time. Approaching the problem differently, the agent-based model does not set up fixed cells, but rather uses the movement of agents on the space domain to track the changes in density. The agent-based model is motivated by the Lagrangian description of a flow field because it attempts to simulate the changes in density by tracking the movements of individual particles. While the agent-based model is quite intuitive in the case of modelling traffic and fluid particles, we want to explore the implementation of agent-based model to equations that may not be derived from moving particles.

Given the initial density distribution, Finite volume method finds the density in each cell at the next time step by approximating the fluxes going through the cell walls. Past research has shown that when implementing the finite volume method, choosing the correct flux approximation methods, such as the upwind method and Godunov's method, allows the numerical solution to converge to the exact solution [3]. The central question this paper investigates is how we should set up the agent-based model so that the numerical solution of the agent-based model converges to the exact solution. To tackle the problem, we introduce a change of variables that transforms the original conservation equation into the conservation of specific volume equation. We observe that the agent-based model can be understood as a finite-volume method for the specific volume conservation equation. This observation produces our main theorem, which, in informal terms, states that given an agent-based model, if its corresponding finite-volume method converges for the specific volume conservation equation, the agent-based model converges for the original conservation equation. This result gives us a condition for the convergence of the agent-based model and allow us to apply results about finite volume method to agent-based models.

The paper contains the following sections. Section 2 defines the agent-based model that we study and provides background knowledge for the finite volume method, which is necessary for understanding our main result. Section 3 introduces the change of variables and proves important properties of the change of variables. Section 4 contains the proof of our main theorem, which gives a condition for the convergence of the agent-based model. Section 5 gives an example of an agent-based model that converges. Section 6 discusses implementing the agent-based model to solve systems of conservation equation. Section 7 discusses potential topics for future research.

2 Preliminaries

2.1 Finite Volume Method

Finite volume method is a widely used numerical method for Equation (2) and is related to the agent-based model. Finite volume method calculates the change of ρ in each cell by calculating the fluxes going through the two cell walls [3]. We first discretize the space domain into cells of length Δx and the time domain into time steps Δt apart. Let ρ_j^n denote the density in the cell $[x_j, x_{j+1}]$ at the time step t_n and let F_j^n be the flux through the cell wall at x_j at time step t_n. At each time step, we calculate the flux through a cell wall using a numerical flux function $F(\rho_L, \rho_R)$ where ρ_L and ρ_R are the densities in the cells left and right of the cell wall respectively. Given a finite volume method, the function $f(\rho_L, \rho_R)$ is the same for all cell walls at all time steps. At the time step t_n, the numerical fluxes is given by $F_j^n = F(\rho_{j-1}^n, \rho_j^n)$. We then calculate the densities at the next time step with

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}(F_{j-1}^n - F_j^n). \tag{3}$$

Given ρ_j^0, the numerical density at t_0 initialized according to the initial condition $\rho(x, 0)$, we can obtain the numerical solution at any time step t_n by applying Equation (3) for n times.

Equation (3) is derived from the integral form of Equation (1). Integrating with respect to x over $[x_0, x_0 + \Delta x]$ and with respect to t over $[t_0, t_0 + \Delta t]$ of equation (1) gives us

$$\begin{aligned} \int_{x_0}^{x_0+\Delta x} \rho(x, t_0 + \Delta t) - \rho(x, t_0) dx &= \int_{t_0}^{t_0+\Delta t} f(x_0, t) - f(x_0 + \Delta x, t) dt, \\ \overline{\rho}^{(t_0+\Delta t)} &= \overline{\rho}^{t_0} + \frac{\Delta t}{\Delta x}(\overline{f}_{x_0} - \overline{f}_{x_0+\Delta t}), \end{aligned}$$

where $\overline{\rho}^{t_0}$ is the average density in $[x_0, x_0 + \Delta x]$ at t_0 and $\overline{f}_{x_0}$ is the average flux over the time period $[t_0, t_0 + \Delta t]$ at x_0.

Different finite volume methods are defined by different numerical flux function. For example, the upwind method approximates the flux using

$$F(\rho_L, \rho_R) = \frac{1}{2}(f(\rho_L) + f(\rho_R) - a(\rho_R - \rho_L))$$

where $a = f'(\rho_L)$ if $\rho_L = \rho_R$ and $a = \frac{f(\rho_L) - f(\rho_R)}{\rho_L - \rho_R}$ if $\rho_L \neq \rho_R$.

2.2 Agent-based Model

In the agent-based model, we first discretize the time domain into discrete time steps of length Δt. Let each agent carries the mass Δm. To initialize the model, we place an agent P_0 at $x = 0$. For an initial condition $\rho(x, 0)$, we place the agent P_j at $x = x_j$ whenever $\int_0^{x_j} \rho(x, 0) dx = j\Delta m$ for any integer j. As shown in Figure 1, given an distribution of the n agents, the numerical density ρ_j^n between agents P_{j+1} and P_j is calculated as $\Delta m/(x_{j+1} - x_j)$ where x_{j+1} and x_j are the location of the two agents that are adjacent to the location x. Because of the way the agent-based model is initialized, the numerical density in the initial time step equals to the average density

$$\rho_j^0 = \frac{\int_{x_j}^{x_{j+1}} \rho(x, 0) dx}{x_{j+1} - x_j}.$$

The velocities of the agents at each time step are defined by a function $V(\rho_L, \rho_R)$, where ρ_L and ρ_R are the numerical densities to the left and right of the agent. Let x_j^n denotes the position of P_j at time step t_n. Let V_j^n denote the velocity of P_j. Then $x_j^{n+1} = x_j^n + V_j^n \Delta t$.

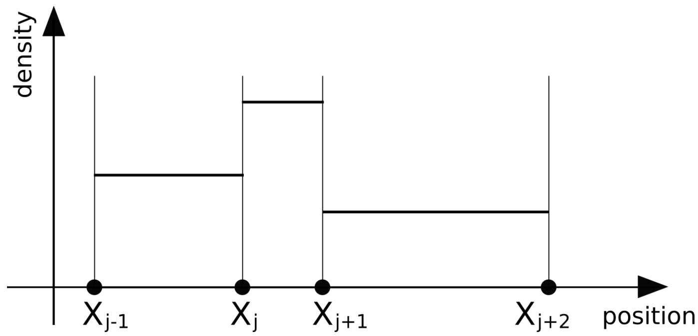

Figure 1: The Numerical Density Calculated from the Distribution of Agents

It is important to note that the agent-based model is different from a finite volume method with moving mesh, such as the one discussed in [5]. In a moving mesh finite volume method, although the cell walls are moving, the density in each cell is still approximated from the fluxes through the cell walls. On the other hand, the density in the agent-based model is approximated from the distance between the agents.

In this paper, we assume that Δt are small enough such that the agent P_j never surpasses P_{j+1} for all integer j at all time steps. This paper investigates what kind of $V(\rho_L, \rho_R)$ we should give to the agents so that the numerical solution of the agent-based model converges to the exact solution. Figure 2 and Figure 3 shows the numerical solutions obtained from a finite volume method and an agent-based model for the inviscid Burgers equation. In Figure 3, each dot represents an agent. The x-coordinate of the dot shows the position of the agent and the y-coordinate of the dot shows the numerical density on the left side of the agent. In this specific example from Figure 3, the agent-based model seems to approximate the exact solution well and suggests that it is reasonable to attempt to find a convergence condition for agent-based models.

3 Conservation of Specific Volume

This section introduces the change of variables that allows us to find the condition under which the agent-based model converges to an exact solution. This change of variable is from the observation that the distances between adjacent agents change in the same way as the densities in the cells of a finite volume method. Let d_j^n be the distance between the agents P_j and P_{j+1} at time step

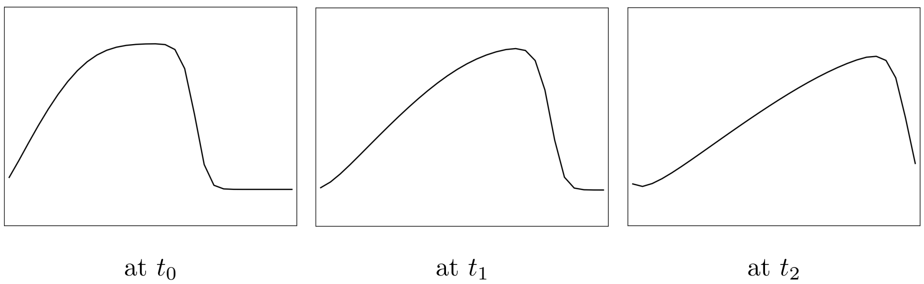

Figure 2: Numerical solution obtained from finite volume method. Horizontal axis: position. Vertical axis: density

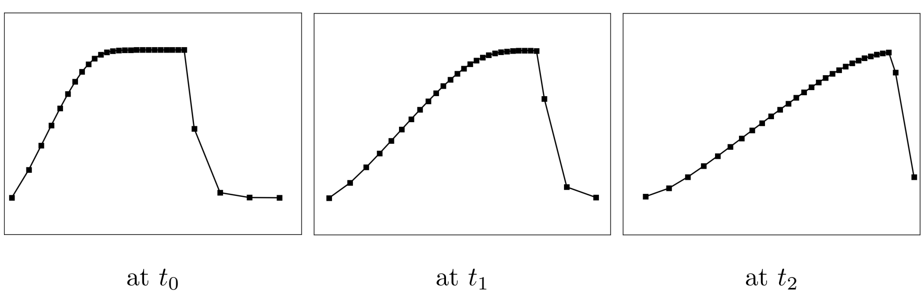

Figure 3: Numerical solution obtained from the agent-based model. Horizontal axis: position. Vertical axis: density

t_n. Then $d_j^{n+1} = d_j^n + V_{j+1}^n \Delta t - V_j^n \Delta t$. Let $\sigma_j^n = \frac{d_j^n}{\Delta m}$. Then

$$\sigma_j^{n+1} = \sigma_j^n + \frac{\Delta t}{\Delta m} \left((-V_j^n) - (-V_{j+1}^n) \right). \tag{4}$$

Note that Equation (4) is simply Equation (3) with the variables renamed. More specifically, Equation (4) is the finite volume equation for the conservation equation

$$\partial_t \sigma(m,t) - \partial_m v = 0.$$

Therefore, there is some connection between the finite volume method and the agent-based model. The change of variable introduced in this section is motivated by this observation and helps us understand the relation between the two seemingly different numerical methods.

Let $\rho : \mathbb{R} \times [0,T] \to \mathbb{R}^+$ be a strong solution to Equation (2) such that $\lim_{x\to\infty} \int_0^x \rho(x',t)dx' = \infty$ and $\lim_{x\to-\infty} \int_0^x \rho(x',t)dx' = -\infty$. We also assume that $v(\rho)$ is a continuous function. Let $m(x,0) = \int_0^x \rho dx$ and let

$$m(x,t) = m(x,0) + \int_0^t -f(x,a)da. \tag{5}$$

Taking, the derivatives of $m(x,t)$, we get the following.

$$\begin{aligned} \partial_t\, m(x,t) &= -f(x,t), \\ \partial_x m(x,t) &= \rho(x,0) - \int_0^t -\rho_t(x,a)da \\ &= \rho(x,t). \end{aligned}$$

Thus, intuitively, m can be understood as the mass because its spacial derivative is density ρ. Because $\rho(x,t) > 0$, the function $g : (x,t) \mapsto (m(x,t),t)$ is a bijection whose domain and codomain are $\mathbb{R} \times [0,T]$. Therefore, we can rewrite $\rho(x,t)$ as $p(m(x,t),t)$ where the function $p(m,t) = \rho(g^{-1}(m,t))$. Let the specific volume $\sigma(m,t)$ be defined as $\sigma(m,t) = \frac{1}{p(m,t)}$. Because g has x and t derivatives, g^{-1} has m and t derivatives, so $\sigma(m,t)$ has m and t derivatives.

Lemma 3.1. *$\sigma(m,t)$ is a strong solution for*

$$\partial_t \sigma(m,t) - \partial_m v\left(\frac{1}{\sigma(m,t)}\right) = 0. \tag{6}$$

Proof. Because $\rho(x,t)$ satisfies Equation (2),

$$\begin{aligned} 0 =& \partial_t \rho(x,t) + f'(\rho)\partial_x \rho(x,t) \\ =& \partial_t p(m(x,t),t) + f'(p)\, \partial_x p(m(x,t),t) \\ =& \frac{\partial p(m,t)}{\partial t} + \frac{\partial p(m,t)}{\partial m}\frac{\partial m(x,t)}{\partial t} + f'(p)\frac{\partial p(m,t)}{\partial m}\frac{\partial m(x,t)}{\partial x}. \end{aligned}$$

Plugging in $f(p) = pv(p)$ and the derivatives of $m(x,t)$, we have

$$0 = \frac{\partial p(m,t)}{\partial t} + p^2 v'(p) \frac{\partial p(m,t)}{\partial m}.$$

Timing the expression by $-\frac{1}{p^2}$ and plugging in $\sigma = \frac{1}{p}$, we have

$$\begin{aligned} 0 =& \frac{\partial \sigma(m,t)}{\partial t} + \frac{v'(\frac{1}{\sigma})}{\sigma^2} \frac{\partial \sigma(m,t)}{\partial m} \\ =& \partial_t \sigma(m,t) - \partial_m v(\frac{1}{\sigma(m,t)}). \end{aligned}$$

Recall that $\sigma(m,t)$ has m and t derivatives. Thus $\sigma(m,t)$ is a strong solution. □

This theorem shows that the change of variable in $g : (x,t) \mapsto (m(x,t),t)$ produces a solution for Equation (6). We now construct an expression for the inverse function g^{-1}. Essentially, we need a function that, given any m_0 and t, finds the position x_0 such that $m_0 = m(x_0,t)$, or, more intuitively, tracks the trajectory of a fixed m_0. Let $k(t)$ be a function of time such that $\partial_t m(k(t),t) = 0$. Then, applying the chain rule and plugging in the derivatives of m produces

$$\rho(k,t)\partial_t k + f(k,t) = 0.$$

Recall $f(k,t) = \rho(k,t)v(k,t)$. Thus, $\partial_t k = v(k,t)$ and the trajectory k is a stream line of the velocity field $v(x,t)$. We then deduce that the inverse with respect to x of the function $m(x,t)$ is

$$x(m,t) = x(m,0) + \int_0^t v(\frac{1}{\sigma(m,t')})dt', \tag{7}$$

where $x(m,0)$ is the inverse of $m(x,0)$. Notice that $\partial_t x(m,t) = v(\frac{1}{\sigma(m,t)})$, which shows that it is a stream line. Intuitively, Equation (7) can be understood as the original position of m and how much m has traveled until time t. Thus, the inverse of g is simply $g^{-1} : (m,t) \mapsto (x(m,t),t)$.

Finding g^{-1} allows us to transform a solution of Equation (6) into a solution of Equation (2). Let $\rho_0(x)$ be a smooth initial condition. Applying the change of variable in Equation (5) for only $t = 0$ gives us $\sigma(m,0)$. Let $\sigma(m,t)$ be a strong solution for Equation (6) with $\sigma(m,0)$ as the initial condition for $t \in [0,T]$. Let $a(x,t)$ be the function such that $a(x(m,t),t) = \sigma(m,t)$ for all $m \in \mathbb{R}$ and $t \in [0,T]$. Let $\rho(x,t) = \frac{1}{a(x,t)}$.

Lemma 3.2. *$\rho(x,t)$ is a strong solution of Equation (2) that satisfies the initial condition $\rho_0(x)$.*

Proof. The proof of this lemma relies on an observation on the symmetry of the transformation. After exchanging the names of m and x, ρ and σ, p and a, f and $-v$, Equation (2) becomes Equation (6) and Equation (5) becomes Equation (7). The arguments for Lemma 3.1 then applies to the current lemma, so $\rho(x,t)$ satisfies Equation (2). Because $x(m,0)$ is the inverse of $m(x,0)$ with respect to x, $\rho(x,0) = \rho_0(x)$. The proof is complete. □

The proof of this lemma reveals that the transformation described by Equation (5) has order 2 in the sense that if we apply this transformation two times to a solution of a conservation equation, we get the original solution. This result is confirmed by the fact that given a initial condition, there is a unique strong solution to Equation (2) [1]. Our result also implies that considering the conservation of specific volume is looking at the conservation of mass from a different perspective and solving one allows us to solve the other one.

Lemma 3.3. *For $t \in [0, T]$,*

$$\frac{\int_{m_1}^{m_2} \sigma(m,t)dm}{m_2 - m_1} = \frac{x_2 - x_1}{\int_{x_1}^{x_2} \rho(x,t)dx}, \tag{8}$$

where $x_2 > x_1$, $m_1 = m(x_1, t)$, and $m_2 = m(x_2, t)$.

Proof. Because $\partial_x m = \rho$, it comes naturally that $\int_{x_1}^{x_2} \rho(x,t)dx = m_2 - m_1$. Applying a substitution to the denominator of the expression of the left gives us

$$\begin{aligned}\int_{m_1}^{m_2} \sigma(m,t)dm &= \int_{x_1}^{x_2} \frac{1}{\rho(x,t)} \rho(x,t)dx \\ &= x_2 - x_1.\end{aligned}$$

□

4 Convergence of Maximum Norm

Let $\rho(x, 0)$ be the initial condition. Applying the transformation in Equation (5) gives us $\sigma(m, 0)$, the initial condition for the conservation of specific volume. Given a velocity rule $V(\rho_L, \rho_R)$ for the agent-based model, the finite volume method for Equation (6) is set up in the following way. Let $m_j = j\Delta m$ and we set up a cell wall at each m_j for all integer j. The initial numerical specific volume σ_j^0 between m_j and m_{j+1} is set to

$$\sigma_j^0 = \frac{\int_{m_j}^{m_{j+1}} \sigma(m,0)}{m_{j+1} - m_j}.$$

The numerical flux $-V_j^n$ going through the cell wall at m_j at time step t_n is $-V(\frac{1}{\sigma_{j-1}^n}, \frac{1}{\sigma_j^n})$ where V is the velocity rule for the agent based model. Thus, every agent-based model has a finite volume counterpart. The convergence is considered in the maximum norm. We choose to use the maximum norm because it allows us to apply our result in the domain of real numbers to periodic domains.

Let $T > 0$ and n be a positive integer such that $n\Delta t < T$.

Theorem 4.1. *Suppose the numerical flux from the finite volume method of specific volume has*

$$\max_j(|\sum_{n'=0}^{n-1} V_j^{n'}\Delta t - \int_0^{t_n} v(m_j,t)dt|) \le \gamma(\Delta t) \tag{9}$$

where $v(m,t)$ is the velocity field from an exact solution $\sigma(m,t)$ of Equation (6) and γ is a strictly increasing function such that $\lim_{\Delta t\to 0}\gamma(\Delta t) = 0$. Then, there exists $\rho(x,t)$ that is a solution for Equation (2) such that

$$\max_j(|\rho_j^n - \frac{\int_{x_j^n}^{x_{j+1}^n} \rho(x,t_n)dx}{x_{j+1}-x_j}|) \le C\gamma(\Delta t) \tag{10}$$

for all $\Delta t \le \Delta t_c$ where Δt_c and C are constants that depends only on the initial condition $\rho(x,0)$.

Proof. Let $\sigma_{\min}$ be the minimum $\sigma(m,t)$ for all $t \in [0,T]$, which equals the minimum of $\sigma(m,0)$. Let Δt_c be a solution to $\Delta m\sigma_{\min} - 2\gamma(\Delta t) > 0$. Note that Δt_c only depends on $\rho(x,0)$ because $\sigma(m,0)$ is determined by $\rho(x,0)$. Let $\Delta t < \Delta t_c$. We first show that the convergence of the numerical velocity means the convergence of the numerical specific volume. For convenience, let $\bar\sigma(m_j,t_n) = \frac{\int_{m_j}^{m_{j+1}} \sigma(m,t_n)dx}{m_{j+1}-m_j}$. Because $\sigma_j^n = \sigma_j^0 + \sum_{n'=0}^{n-1}\frac{\Delta t}{\Delta m}(V_{j+1}^{n'} - V_j^{n'})$ and $\bar\sigma(m_j,t_n) = \bar\sigma(m_j,0) + \frac{1}{\Delta m}\int_0^{t_n} v(m_{j+1},t) - v(m_j,t)dt$, we have

$$\begin{aligned}\max_j(|\sigma_j^n - \bar\sigma(m_j,t_n)|) \le & \max_j(|\frac{\Delta t}{\Delta m}\sum_{n'=0}^{n-1} V_{j+1}^{n'} - \frac{1}{\Delta m}\int_0^{t_n} v(m_{j+1},t)dt \\ & -(\frac{\Delta t}{\Delta m}\sum_{n'=0}^{n-1} V_j^{n'}\Delta t - \frac{1}{\Delta m}\int_0^{t_n} v(m_j,t)dt)|) \\ \le & \frac{2\gamma(\Delta t)}{\Delta m}.\end{aligned}$$

The finite volume method is connected to the agent-based model because the density between agents equals to the reciprocal of the specific volume between cell walls as shown below.

$$\begin{aligned}\rho_j^{n+1} =& \frac{\Delta m}{x_{j+1}^{n+1} - x_j^{n+1}} \\ =& (\frac{x_{j+1}^n - x_j^n + (V_{j+1}^n - V_j^n)\Delta t}{\Delta m})^{-1} \\ =& (\sigma_j^n + \frac{\Delta t}{\Delta m}(V_{j+1}^n - V_j^n))^{-1} \\ =& \frac{1}{\sigma_j^{n+1}}.\end{aligned}$$

Let $\rho(x,t)$ be the function constructed from $\sigma(m,t)$ using the change of variable in Equation (7). By Lemma 3.2, $\rho(x,t)$ is a solution Equation (2). Now, we need to show that the numerical solution converges to $\rho(x,t)$. For convenience, let

$$\begin{aligned}\bar{\rho}(x_j^n,t_n) =& \frac{1}{x_{j+1}^n - x_j^n}\int_{x_j^n}^{x_{j+1}^n} \rho(x,t_n)dx,\\ \bar{\rho}(x(m_j,t_n),t_n) =& \frac{1}{x(m_j,t_n)-x(m_{j+1},t_n)}\int_{x(m_j,t_n)}^{x(m_{j+1},t_n)} \rho(x,t)dx\end{aligned}$$

where $x(m,t)$ is the exact change of variable from Equation (7). Note that $\bar{\rho}(x(m_j,t_n),t_n)$ is different from $\bar{\rho}(x_j^n,t_n)$ because there is a difference between the numerical position x_j^n and the exact position $x(m_j,t_n)$. Because of Lemma 3.3, $\bar{\rho}(x(m_j,t_n),t_n) = \frac{1}{\bar{\sigma}(m_j,t_n)}$. Now we consider the error of the agent based model.

$$\begin{aligned}|\rho_j^n - \bar{\rho}(x_j^n,t_n)| =& |\frac{1}{\sigma_j^n} - \frac{1}{\bar{\sigma}(m_j,t_n)} + \frac{1}{\bar{\sigma}(m_j,t_n)} - \bar{\rho}(x_j^n,t_n)|\\ \le& |\frac{1}{\sigma_j^n} - \frac{1}{\bar{\sigma}(m_j,t_n)}| + |\bar{\rho}(x(m_j,t),t) - \bar{\rho}(x_j^n,t_n)|\end{aligned}$$

Because $\sigma_j^n \ge \sigma_{\min} - \frac{2\gamma(\Delta t_c)}{\Delta m}$,

$$\begin{aligned}|\frac{1}{\sigma_j^n} - \frac{1}{\bar{\sigma}(m_j,t_n)}| \le& \frac{|\sigma_j^n - \bar{\sigma}(m_j,t_n)|}{|\sigma_j^n\bar{\sigma}(m_j,t_n)|}\\ \le& \frac{2\gamma(\Delta t)}{\sigma_{\min}(\Delta m\sigma_{\min} - 2\gamma(\Delta t_c))}.\end{aligned}$$

Let $\rho_{\max}$ be the maximum of $\rho(x,t)$ over $t \in [0,T]$. Note that $\rho_{\max} = \frac{1}{\sigma_{\min}}$. Recall that $x_j^n = x_j^0 + \sum_{n'=0}^{n-1} V_j^{n'}\Delta t$ and $x(m_j,t_n) = x(m_j,0) + \int_0^{t_n} v(m_j,t)dt$. Because of the way the agent=based model is initialized, $x_j^0 = x(m_j,0)$. Thus, $|x_j^n - x(m_j,t_n)| = |\sum_{n'=0}^{n-1} V_j^{n'}\Delta t - \int_0^{t_n} v(m_j,t)dt|) \le \gamma(\Delta t)$. Then,

$$\bar{\rho}(x_j^n,t_n) \le \frac{(x(m_{j+1},t_n) - x(m_j,t_n))\bar{\rho}(x(m_j,t),t) + 2\rho_{\max}\gamma(\Delta t)}{x(m_{j+1},t_n) - x(m_j,t_n) - 2\gamma(\Delta t)}$$

For any $a > 0$ and $x_c \in (0,a)$, we have $\frac{a}{a-x} \le \frac{x}{a-x_c} + 1$ for all $x \in [0,x_c]$. In our case, $a = x(m_{j+1},t_n) - x(m_j,t_n)$, $x_c = 2\gamma(\Delta t_c)$, and $x = 2\gamma(\Delta t)$. Thus,

$$\begin{aligned}\bar{\rho}(x_j^n,t_n) \le& \bar{\rho}(x(m_j,t),t) + \frac{2\gamma(\Delta t)\bar{\rho}(x(m_j,t),t)}{x(m_{j+1},t_n) - x(m_j,t_n) - 2\gamma(\Delta t_c)}\\ &+ \frac{2\rho_{\max}\gamma(\Delta t)}{x(m_{j+1},t_n) - x(m_j,t_n) - 2\gamma(\Delta t)}\\ \le& \bar{\rho}(x(m_j,t),t) + \frac{4\gamma(\Delta t)\rho_{\max}}{x(m_{j+1},t_n) - x(m_j,t_n) - 2\gamma(\Delta t_c)}\end{aligned}$$

Because $x(m_{j+1},t_n) - x(m_j,t_n) = \int_{m_j}^{m_{j+1}} \sigma(m,t_n)dm \geq \Delta m \sigma_{\min}$,

$$\bar{\rho}(x_j^n, t_n) \leq \bar{\rho}(x(m_j,t),t) + \frac{4\rho_{\max}}{\Delta m \sigma_{\min} - 2\gamma(\Delta t_c)} \gamma(\Delta t)$$

Similarly, we have

$$\bar{\rho}(x_j^n, t_n) \geq \frac{(x(m_{j+1},t_n) - x(m_j,t_n))\bar{\rho}(x(m_j,t),t) - 2\rho_{\max}\gamma(\Delta t)}{x(m_{j+1},t_n) - x(m_j,t_n) + 2\gamma(\Delta t)}.$$

For any $a > 0$, $\frac{a}{a+x} > 1 - \frac{x}{a}$ when $x > 0$. Substituting $a = x(m_{j+1},t_n) - x(m_j,t_n)$ and $x = 2\rho_{\max}\gamma(\Delta t)$ produces

$$\begin{aligned}
\bar{\rho}(x_j^n, t_n) \geq & \bar{\rho}(x(m_j,t),t) - \frac{2\bar{\rho}(x(m_j,t),t)}{x(m_{j+1},t_n) - x(m_j,t_n)} \gamma(\Delta t) \\
& - \frac{2\rho_{\max}}{x(m_{j+1},t_n) - x(m_j,t_n) + 2\gamma(\Delta t)} \gamma(\Delta t), \\
\geq & \bar{\rho}(x(m_j,t),t) - \frac{4\rho_{\max}}{x(m_{j+1},t_n) - x(m_j,t_n)} \gamma(\Delta t), \\
\geq & \bar{\rho}(x(m_j,t),t) - \frac{4\rho_{\max}}{\Delta m \sigma_{\min} - 2\gamma(\Delta t_c)} \gamma(\Delta t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\bar{\rho}(x(m_j,t),t) - \bar{\rho}(x_j^n, t_n)| \leq & \frac{4\rho_{\max}}{\Delta m \sigma_{\min} - 2\gamma(\Delta t_c)} \gamma(\Delta t), \\
\leq & \frac{4}{\sigma_{\min}(\Delta m \sigma_{\min} - 2\gamma(\Delta t_c))} \gamma(\Delta t).
\end{aligned}$$

Thus,

$$\max_j |\rho_j^n - \bar{\rho}(x_j^n, t_n)| \leq \frac{6}{\sigma_{\min}(\Delta m \sigma_{\min} - 2\gamma(\Delta t_c))} \gamma(\Delta t). \tag{11}$$

Recall that $\sigma_{\min}$ only depends on $\rho(x, 0)$. The proof is complete. $\square$

This proof essentially shows that if the velocity approximation rule $V(\rho_L, \rho_R)$ gives us the correct specific volume solution when we use it in the finite volume method for specific volume, the agent-based solution converges to the exact solution. Moreover, the agent-based model converges in the same order as the finite volume method. The reason that the convergence condition is the convergence of velocity, or the specific volume flux, instead of the convergence specific volume is because $\partial_t \sigma(m,t) + \partial_m v(\frac{1}{\sigma(m,t)}) = 0$ and $\partial_t \sigma(m,t) + \partial_m \left(v(\frac{1}{\sigma(m,t)}) + k\right) = 0$ where k is a constant have the same $\sigma(m,t)$ as the solution, but the $\rho(m,t)$ obtained from $\sigma(m,t)$ will be different because the velocities in the two cases are offset by k.

5 Example: Godunov's Method

Theorem 4.1 shows that we can determine whether an agent-based model converges to the exact solution by examining whether its corresponding finite volume method converges. Therefore, if we know a finite volume method converges, we can "translate" it to an agent-based model that will also converge, allowing us to apply results in finite volume methods to the agent-based model.

Consider the conservation equation $\partial_t u(x,t) + \partial_x f(u(x,t))$. The Godunov's Method is a finite volume method that approximate the interface flux by solving the Riemann problem [3]. Suppose two adjacent cells have density u^n_{j-1} and u^n_j and their boundary is at x_j at time step t_n, we consider the Riemann problem with the condition initial condition

$$u(x,0) = \begin{cases} u^n_{j-1} & x \le x_j \\ u^n_j & x > x_j. \end{cases}$$

Let $u(x,t)$ be the solution to the Riemann problem. Suppose $u(x_j,t) = u'$ where u' is constant for $t \in (0, \Delta t]$. Then, the interface flux at the boundary x_j is $F^n_j = f(u')$. Results in solving the Riemann problem shows that the interface flux should be

$$F^n_j = \begin{cases} \min_{[u^n_{j-1}, u^n_j]} f(u) & u^n_{j-1} \le u^n_j \\ \max_{[u^n_j, u^n_{j-1}]} f(u) & u^n_{j-1} > u^n_j. \end{cases}$$

In the conservation of specific volume, $u(x,t)$ is replaced by $\sigma(m,t)$ and $f(u(x,t))$ by $-v(\frac{1}{\sigma(m,t)})$. The specific volume flux law in Godunov's method is therefore

$$-V(\frac{1}{\sigma^n_{j-1}}, \frac{1}{\sigma^n_j}) = \begin{cases} -(\max_{[\sigma^n_{j-1}, \sigma^n_j]} v(\frac{1}{\sigma})) & \sigma^n_{j-1} \le \sigma^n_j \\ -(\min_{[\sigma^n_j, \sigma^n_{j-1}]} v(\frac{1}{\sigma})) & \sigma^n_{j-1} > \sigma^n_j. \end{cases}$$

The agent-based model that correspond to this finite volume flux law therefore has the following velocity rule.

$$V(\rho^n_{j-1}, \rho^n_j) = \begin{cases} \min_{[\rho^n_{j-1}, \rho^n_j]} v(\rho) & \rho^n_{j-1} \le \rho^n_j \\ \max_{[\rho^n_j, \rho^n_{j-1}]} v(\rho) & \rho^n_{j-1} > \rho^n_j \end{cases} \tag{12}$$

The example of the inviscid Burgers equation can clarify how the agent-based model using the velocity rule in Equation (12) is implemented. The inviscid Burgers equation is $\partial_t \rho(x,t) + \partial_t(\frac{1}{2}\rho^2) = 0$, so $v(\rho) = \frac{1}{2}\rho$ is the velocity term [4]. The agent-based model derived from Godunov's method for the inviscid Burgers equation therefore has the velocity rule

$$V(\rho^n_{j-1}, \rho^n_j) = \begin{cases} \frac{1}{2}\rho^n_{j-1} & \rho^n_{j-1} \le \rho^n_j \\ \frac{1}{2}\rho^n_j & \rho^n_{j-1} > \rho^n_j \end{cases}$$

Because Godunov's method converges for scalar conservation laws, the agent-based model that uses the velocity approximation rule in Equation (12) converges to the exact solution. In fact, the agent-based model that produces

Figure 3 uses the velocity rule derived from Godunov's method. By the process described in this section, we can construct an agent-based model for any finite volume method.

6 Systems of Conservation Equations

This section discusses the possibility and the mathematical difficulty of applying the agent-based model to systems of conservation equations. Systems of conservation equation can be written as

$$\partial_t \vec{\rho}(x,t) + \partial_x \vec{f}(\vec{\rho}(x,t)) = \vec{0}, \tag{13}$$

where $\vec{\rho}$ and $\vec{f}$ are n dimensional vectors. Equation (2) studied in previous sections is the case when $n = 1$. This section only discusses the case when $n = 2$, which already shows how we can extend the agent-based model to $n > 2$ and the main difficulties in generalizing the result in $n = 1$ to $n > 1$. When $n = 2$, Equation (13) can be written as

$$\partial_t \rho_1(x,t) + \partial_x f_1(\rho_1(x,t), \rho_2(x,t)) = 0 \tag{14}$$

$$\partial_t \rho_2(x,t) + \partial_x f_2(\rho_1(x,t), \rho_2(x,t)) = 0, \tag{15}$$

where $f_1 = \rho_1 v_1(\rho_1, \rho_2)$ and $f_2 = \rho_2 v_2(\rho_1, \rho_2)$. The agent-based model is set up in the following way. There are two kinds of agents P and Q, which represents ρ_1 and ρ_2 respectively. Let each agent carries the mass Δm. To initialize the agent simulation, we place agents P_0 and Q_0 at $x = 0$. For an initial condition $\rho_1(x, 0)$, we place agent P_j at $x = x_j$ whenever $\int_0^{x_j} \rho_1(x,0)dx = j\Delta m$ for any integer j. For an initial condition $\rho_2(x, 0)$, we place agent Q_j at $x = x_j$ whenever $\int_0^{x_j} \rho_2(x,0)dx = j\Delta m$ for any integer j. Given an distribution of the n P agents, the numerical density ρ_{1j}^n between agents P_{j+1} and P_j is calculated by $\Delta m/(x_{j+1} - x_j)$ where x_{j+1} and x_j are the location of the two P agents that are adjacent to the location x. The numerical density ρ_{2j}^n is calculated in the same way using Q agents. The velocity of agent P_j at position x_j are determined by a function $V_1(\rho_{1L}, \rho_{1R}, \rho_{2L}, \rho_{2R})$ where ρ_{1L} is the numerical density ρ_1 to the left of x_j, ρ_{1R} the numerical density ρ_1 to the right of x_j, ρ_{2L} the numerical density ρ_2 to the left of x_j, and ρ_{2R} the numerical density ρ_2 to the right of x_j. When there is no Q agent at the position of P_j, $\rho_{2L} = \rho_{2R}$ because the density ρ_2 to the left and right of P_j is the same. Because it is almost impossible for the agents to completely overlap, there is often no need to consider the case of $\rho_{2L} \neq \rho_{2R}$ The velocity of agent Q_j at position x_j are determined by a function $V_2(\rho_{1L}, \rho_{1R}, \rho_{2L}, \rho_{2R})$. For the same reason, we often have $\rho_{1L} = \rho_{1R}$. Once the velocities V_1 and V_2 are calculated, the agents' positions are updated for the next time step.

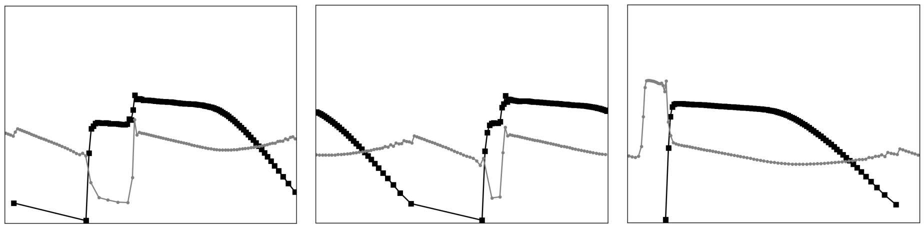

Figure 4: horizontal axis represents location; vertical axis represents density; grey dots represent ρ_1; black squares represent ρ_2

We implemented the agent-based model for the system of conservation equations that uses the velocity rule

$$v_1(\rho_1, \rho_2) = v_{m_1}\big(1 - \frac{\rho_1 + \rho_2}{\rho_m}\big),$$
$$v_2(\rho_1, \rho_2) = v_{m_2}\big(1 - (\frac{\rho_1 + \rho_2}{\rho_m})^2\big),$$

which is used for modelling traffic flow. Figure 4 shows three screen shots for the solution obtained from the agent-based model.

6.1 Change of Variables for Systems of Conservation Equations

For the scalar conservation equation, applying the change of variable in Equation (5) allows us connect the convergence of finite volume methods to the convergence of agent-based models. In this section, we apply the same change of variable to Equation (14) and Equation (15). We define the new variables $m_1(x,t)$, $m_2(x,t)$ as

$$m_1(x,t) = m_1(x,0) - \int_0^t f_1(x,t')dt', \tag{16}$$

$$m_2(x,t) = m_2(x,0) - \int_0^t f_2(x,t')dt', \tag{17}$$

where $m_1(x,0) = \int_0^x \rho_1(x',0)dx'$ and $m_2(x,0) = \int_0^x \rho_2(x',0)dx'$. Let $p_1(m_1(x,t),t) = \rho_1(x,t)$ and $p_2(m_2(x,t),t) = \rho_2(x,t)$. We then have

$$\begin{aligned}
&\partial_t \rho_1(x,t) + \partial_x(\rho_1 v_1(\rho_1,\rho_2)) \\
=&\frac{\partial p_1(m_1,t)}{\partial t} + \frac{\partial p_1(m_1,t)}{\partial m_1}\frac{\partial m_1(x,t)}{\partial t} + \frac{\partial p_1(m_1,t)}{\partial m_1}\frac{\partial m_1(x,t)}{\partial x}v_1(\rho_1,\rho_2) \\
&+ p_1(m_1,t)(\frac{\partial v_1}{\partial p_1}\frac{\partial p_1}{\partial m_1}\frac{\partial m_1}{\partial x} + \frac{\partial v_1}{\partial p_2}\frac{\partial p_2}{\partial m_2}\frac{\partial m_2}{\partial x}).
\end{aligned}$$

Because $\frac{\partial m_1(x,t)}{\partial t} = -f_1(x,t) = -\frac{\partial m_1(x,t)}{\partial x} v_1(\rho_1, \rho_2)$, we have

$$=\frac{\partial p_1(m_1,t)}{\partial t} + p_1(m_1,t)(\frac{\partial v}{\partial p_1}\frac{\partial p_1}{\partial m_1}p_1 + \frac{\partial v}{\partial p_2}\frac{\partial p_2}{\partial m_2}p_2).$$

Multiplying the equation by $-\frac{1}{p_1^2}$ produces

$$\begin{aligned}&\partial_t(\frac{1}{p_1(m_1,t)}) - \frac{\partial v}{\partial p_1}\frac{\partial p_1}{\partial m_1} - \frac{\partial v}{\partial p_2}\frac{\partial p_2}{\partial m_2}\frac{p_2}{p_1}\\ =&\partial_t\sigma_1(m_1,t) - \partial_{m1}v_1(p_1(m_1,t), p_2(m_2,t)).\end{aligned}$$

Doing the same change of variable on the equation for ρ_2, the original system of conservation equations becomes

$$\partial_t\sigma_1(m_1,t) - \partial_{m_1}v_1(p_1(m_1,t), p_2(m_2,t)) = 0, \tag{18}$$
$$\partial_t\sigma_2(m_1,t) - \partial_{m_2}v_2(p_1(m_1,t), p_2(m_2,t)) = 0. \tag{19}$$

It is important to note that the new system in Equation (18) is no longer a system of conservation equation because σ_1 is a function of m_1 and t while σ_2 is a function of m_2 and t. In fact, we need Equation (16) and Equation (17) to implicitly define the relation between m_1 and m_2 to make the system of specific volume conservation meaningful. Because the new system is not a system of conservation equations, we cannot simply treat the agent-based model as a finite volume method for specific volume, which is our the main technique for the scalar case.

7 Conclusion and Future Work

In our work, we propose an agent-based model for solving the conservation equation. By introducing a change of variables, we transform the original conservation equation to the conservation of specific volume. This transformation then allows us to prove that the agent-based solution converges if the finite volume method for the conservation of specific volume converges. This result enables us to apply past results in finite volume methods to agent-based models as each finite volume method has its agent-based version. We also extend the agent-based model for scalar conservation equations to vector conservation equations and show the difficulty of proving convergence for the agent-based model for vector equations.

One can generalize our work in the scalar conservation equation by considering convergence in different norms. For example, Theorem 4.1 likely can be extended to Lp norm. This extension would allow the application of more results in finite volume methods to agent-based models since many of the results are in Lp norms.

Another natural choice for future work is to find the criteria for the agent-based solution to converge for the vector conservation equation. This maybe difficult because the change of variable discussed in Section 6 shows that it is unlikely that we can directly connect results in finite volume methods to agent-based models as we did with the scalar case.

Lastly, it would also be meaningful to compare agent-based models with finite volume methods. Although it is likely that the two methods are the same for scalar conservation equations, the two methods have major differences for vector conservation equations. In a traditional finite volume method, different densities use the same cells, so the cell walls for different densities line up with each other. Thus, the flux approximation needs to take into account the densities on the left and the right of the wall. Agent-based models may have an advantage over traditional finite volume methods as the agents for different densities almost never line up, making velocity approximation easier.

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