DERIVATIVE ASYMPTOTICS OF UNIFORM GELFAND-TSETLIN PATTERNS

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Abstract. Bufetov and Gorin introduced the idea of applying differential operators which are diagonalized by the Schur functions to Schur generating functions, a generalization of probability generating functions to particle systems. This technology allowed the authors to access asymptotics of a variety of particle systems. We use this technique to analyze uniformly distributed Gelfand-Tsetlin patterns where the top row is fixed. In particular, we obtain limiting moments for the difference of empirical measure for two adjacent rows in uniformly random Gelfand-Tsetlin patterns.

1. Introduction

Consider the following setup. Let T be a triangular array of nonnegative integers

$$\begin{array}{ccccc}
\lambda_1^{(M)} & \lambda_2^{(M)} & \lambda_3^{(M)} & \cdots & \lambda_M^{(M)} \\
 & \ddots & \vdots & \vdots & \vdots \\
 & & \lambda_1^{(3)} & \lambda_2^{(3)} & \lambda_3^{(3)} \\
 & & & \lambda_1^{(2)} & \lambda_2^{(2)} \\
 & & & & \lambda_1^{(1)}
\end{array}$$

such that

$$\lambda_i^{(j)} \geq \lambda_i^{(j-1)} \geq \lambda_{i+1}^{(j)}.$$

Such an array is called a *Gelfand-Tsetlin (GT) pattern*. If $(\lambda_1^{(M)}, \ldots, \lambda_M^{(M)}) = \lambda$, we say that T has *top row* λ; denote the set of all T with top row λ by $\mathcal{T}_\lambda^{(M)}$. This article is concerned with understanding uniform distributions on $\mathcal{T}_\lambda^{(M)}$ as $M \to \infty$, under suitable limit conditions for λ. We note that the uniform distributions on $\mathcal{T}_\lambda^{(M)}$ can be interpreted as the uniform distribution of lozenge tilings of certain domains, see [No].

Although there are many methods in studying uniform distributions on $\mathcal{T}_\lambda^{(M)}$, such as via correlation kernels [Pe] or log-partition functions [No], we follow the approach of considering the moments: For each k and $T \in \mathcal{T}_\lambda^{(M)}$ define the random variable

$$\mathfrak{m}_k^{(N)}(T) = \frac{1}{M} \sum_{i=1}^{N} \left(\frac{\lambda_i^{(N)} + N - i}{M} \right)^k.$$

In [BuG1] and [BuG2], it was shown that if $\lambda = \lambda(M)$ forms a regular sequence (see Definition 2.6) and $N/M \to \eta \in (0,1)$ as $M \to \infty$, then the random moments for uniform $T \in \mathcal{T}_\lambda^{(M)}$ converge

$$\mathfrak{m}_k^{(N)}(T) \to \mathfrak{m}_k(\eta) \tag{1}$$

almost surely where $\mathfrak{m}_k(\eta) \in \mathbb{R}$ as $M \to \infty$ for each $k = 0, 1, 2, \ldots$.

The main goal of this article is to describe the asymptotics of the finite difference of moments between adjacent levels in the array for regular sequences λ. Mathematically speaking, we study the $M \to \infty$ limit of

$$\mathfrak{d}_k^{(N)}(T) = \mathfrak{m}_k^{(N)}(T) - \mathfrak{m}_k^{(N-1)}(T).$$

Our main result can be stated as follows. If λ is regular and $N/M \to \eta \in (0,1)$ as $M \to \infty$, then for uniform $T \in \mathcal{T}_\lambda^{(M)}$

$$\mathfrak{d}_k^{(N)}(T) \to \mathfrak{d}_k(\eta) \in \mathbb{R} \tag{2}$$

almost surely as $M \to \infty$ and we provide an explicit formula for $\mathfrak{d}_k(\eta)$ for each $k = 1, 2, \ldots$ (see Theorem 3.2 for a precise statement). Along the way, we also give a proof of (1) which is similar to that of [BuG1] but with methods optimized for our purposes (see Theorem 3.1 for a precise statement).

This problem is motivated by a discretization of a similar problem in random matrix theory. We briefly describe this problem, then explain its significance. Suppose X_M is an $M \times M$ Hermitian random matrix with entries in $\mathbb{C}$ such that its distribution is invariant under conjugation by unitary matrices. Let X_M' denote an $(M-1) \times (M-1)$ principal submatrix of X_M. Suppose

$$\lambda_1 \geq \ldots \geq \lambda_M \qquad \text{and} \qquad \mu_1 \geq \ldots \geq \mu_{M-1}$$

are the eigenvalues of X_M and X_M' respectively. The analogous statement to (1) is the convergence

$$\frac{1}{M}\sum_{i=1}^{M} \lambda_i^k \to m_k \qquad k = 0, 1, 2 \ldots \tag{3}$$

and the analog to (2) is the convergence

$$\frac{1}{M}\sum_{i=1}^{M} \lambda_i^k - \frac{1}{M}\sum_{i=1}^{M-1} \mu_i^k \to d_k \qquad k = 1, 2 \ldots, \tag{4}$$

both under suitable limit conditions on X_M. The first limit has been studied ubiquitously in random matrix theory. In fact, the study of random matrix theory may be argued to have begun by the consideration of this limit by Wigner for certain Gaussian matrices [AGZ]. The second limit has been considered more recently, and has been studied for several random matrix ensembles including the Jacobi ensembles [GZ] and Hermite ensembles ([Bu], [ErSc], [GA]).

One motivation for considering the limit (4) is that it is considering a discrete derivative of (3) along the direction of matrix dimension.

However, there is another perhaps deeper connection between (4) and (3). This connection can be understood by the Markov-Krein correspondence which we now state.

Theorem 1.1 (Markov-Krein Correspondence)**.** *For every probability measure μ on $\mathbb{R}$ with compact support, absolutely continuous density with respect to the Lebesgue measure,* *and density bounded by* 1*, there exists a probability measure* $\nu[\mu]$ *with compact support on* $\mathbb{R}$ *such that*

$$\sum_{n=0}^{\infty} m_n z^n = \exp\left(\sum_{n=1}^{\infty} \frac{d_n}{n} z^n\right) \tag{5}$$

where $m_n = \int x^n \mu(dx)$ *and* $d_n = \int x^n \nu(dx)$*.*

It was shown in [Bu] that for the Gaussian unitary ensemble, the limits m_k, d_k in (3) and (4) satisfy (5) as $M \to \infty$. Although unavailable in the literature, it is a folk theorem that (5) continues to hold true for X_M belonging to more general classes of distribution. This suggests an explicit procedure for constructing such a measure $\nu[\mu]$.

Returning to the discrete problems (1) and (2), our main motivation was to find a discrete analogue of the Markov-Krein correspondence. Such a correspondence has not been studied in the literature, as far as the authors are aware.

Our method relies on analyzing the Schur generating function (SGF), a certain generalization of the characteristic function, for uniform distributions on $\mathcal{T}_\lambda^{(M)}$. More specifically, we use a family of difference operators, which are diagonalized by the Schur symmetric functions, to access the moments of the measures $\mathfrak{m}_k^{(N)}(T)$ and $\mathfrak{d}_k^{(N)}(T)$. For uniform distributions on $\mathcal{T}_\lambda^{(M)}$, the SGF has a simple description in terms of Schur symmetric functions. We utilize this special form for the SGF and access the limits (1) and (2) by understanding the action of these difference operators on the SGF and the asymptotics of this action.

Finally, we note that the methods used in this article can be generalized to models broader than uniform distributions on $\mathcal{T}_\lambda^{(M)}$. The generality of the methods encompass a variety of models in asymptotic representation and two dimensional statistical mechanics.

The remainder of this article is organized as follows. In Section 2 we provide a more detailed explanation of the model and the associated objects. In Section 3 we state and prove the main results, (1) and (2) given above.

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2. Setup

2.1. **Definitions.** A partition λ of length L is a sequence of positive integers

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_L.$$

Given a partition λ' of length at most $L-1$, we say that $\lambda' \prec \lambda$ (pronounced "λ and λ' interlace") if

$$\lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \cdots \leq \lambda'_{L-1} \leq \lambda_L.$$

Throughout, ρ generally refers to a probability distribution on partitions, so the probability of picking some partition λ is $\rho(\lambda)$.

We will extensively use Schur functions (also known as Schur polynomials) in our proofs, and many details and basic results about them can be found in [BG]. For completeness, we review the basic definitions here. Suppose λ is partition of length at most N. Then the *Schur function* $s_\lambda(x_1, \ldots, x_N)$ is a symmetric polynomial given by

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[x_i^{\lambda_j + N - i} \right]_{i,j=1}^{N}}{\prod_{i<j}(x_i - x_j)}.$$

The Schur functions form a linear basis for all symmetric polynomials in N variables.

We will also use the tool of Schur generating functions.

Definition 2.1. *Given a probability distribution ρ on partitions, its Schur generating function (abbreviated SGF) is given by*

$$\mathcal{S}_\rho(x_1, \ldots, x_N) = \sum_\lambda \rho(\lambda) \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1, \ldots, 1)}.$$

The reason we use these are twofold - the SGF for the particular model we are interested in is relatively simple (see Theorem 2.2), and the probabilistic statistics we are interested in can be related to the SGF by Theorem 2.5.

2.2. **Uniform Distribution on GT Patterns.** Let λ be some deterministic partition of length M, and consider all sequences of partitions

$$\lambda^{(1)} \prec \cdots \prec \lambda^{(M)} = \lambda.$$

This is known as a *GT pattern with top row* λ. Given the uniform distribution on GT patterns with top row λ, let $\rho^{(N)}$ be the resulting marginal distribution for $\lambda^{(N)}$ where $N < M$. It turns out that the SGF of $\rho^{(N)}$ has a really nice form.

Proposition 2.2. *The SGF for $\rho^{(N)}$ is given by*

$$\mathcal{S}_{\rho^{(N)}}(x_1, \ldots, x_N) = \frac{s_\lambda(x_1, \ldots, x_N, 1^{M-N})}{s_\lambda(1^M)}.$$

Proof. This is a consequence of the so called branching rule, which states that

$$s_\lambda(x_1, \ldots, x_N, 1^{M-N}) = \sum_{\mu : \mu = \lambda^{(N)} \prec \cdots \prec \lambda^{(M)} = \lambda} s_\mu(x_1, \ldots, x_N). \tag{6}$$

This implies that

$$s_\lambda(1^M) = \sum_{\mu : \mu = \lambda^{(N)} \prec \cdots \prec \lambda^{(M)} = \lambda} s_\mu(1^N).$$

In particular if $N = 1$, we see that $s_\lambda(1^M)$ is the total number of GT patterns with top row λ. The probability of $\lambda^{(N)} = \mu$ is

$$\rho^{(N)}(\mu) = \frac{s_\mu(1^N)}{s_\lambda(1^M)} \cdot \#\{\mu = \lambda^{(N)} \prec \cdots \prec \lambda^{(M)} = \lambda\} \tag{7}$$

because the number of ways to get $\lambda^{(N)} = \mu$ is equal to the number of ways to get μ from λ times the number of configurations with μ in the top row divided by the total number of configurations (because things above μ and below are independent conditioned on μ). To finish, note that

$$\begin{aligned}\frac{s_\lambda(x_1,\dots,x_N,1^{M-N})}{s_\lambda(1^M)} &= \sum_{\mu:\mu=\lambda^{(N)}\prec\cdots\prec\lambda^{(M)}=\lambda} \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1^N)}\cdot\frac{s_\mu(1^N)}{s_\lambda(1^M)} \\ &= \sum_\mu \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1^N)}\cdot \rho^{(N)}(\mu) \\ &= \mathcal{S}_{\rho^{(N)}}(x_1,\dots,x_N)\end{aligned}$$

where the first equality follows from (6) and the second follows from (7). □

We will be random measures known as *counting measures* which we define below.

Definition 2.3. *Suppose ρ is a probability distribution on partitions of length N, and suppose M is some positive integer. Then, a counting measure for ρ is given by*

$$m[\rho] = \frac{1}{M}\sum_{i=1}^N \delta\left(\frac{\lambda_i+N-i}{M}\right),$$

where we are treating λ_i as a random variable for the ith part of the partition.

Remark 1. This is one of the few times we will be referring explicitly to the random variables λ_i. The only other place is in the proof of Theorem 2.5.

Our objects of study are

$$m^{(N)} := m[\rho^{(N)}],$$

and its discrete derivative given by

$$\mu^{(N)} := M(m^{(N)} - m^{(N-1)}).$$

Remark 2. To make the derivative interpretation explicit (as we will do later as well), let $\eta = N/M$. In the limit of large M, we have

$$\text{“}\frac{dm^{(N)}}{d\eta}\text{”} \approx \frac{m^{(N)}-m^{(N-1)}}{1/M} = \mu^{(N)}.$$

In particular, we will be studying the *moments* of these measures, and their asymptotics as $M\to\infty$.

2.3. **Moments and Operators.** The point of this section is that the moments of $m^{(N)}$ and $\mu^{(N)}$ can be found by applying certain differential operators on the SGF of $\rho^{(N)}$. The following makes this explicit.

Definition 2.4. *Define a differential operator on functions of $(x_1,\dots,x_N)$ by*

$$\mathcal{D}_{N,k} := \frac{1}{\prod_{1\le i<j\le N}(x_i-x_j)}\left(\sum_{i=1}^N (x_i\partial_i)^k\right)\prod_{1\le i<j\le n}(x_i-x_j)$$

where $\partial_i := \frac{\partial}{\partial x_i}$.

The next theorem shows the connection of this operator to moments of counting measures.

Theorem 2.5 ([BuG1, Proposition 4.6]). *Let ρ be a probability distribution on partitions of length N, and let M be some positive integer, and let $m[\rho]$ be its counting measure. The moments of this measure are given by*

$$\mathbb{E}\left(\int_{\mathbb{R}} x^k m[\rho](dx)\right)^n = \frac{1}{M^{n(k+1)}}(\mathcal{D}_{N,k})^n \mathcal{S}_\rho(x_1,\dots,x_N)\bigg|_{x_1=\cdots=x_N=1}.$$

Proof. We'll sketch the proof of this result here. The key claim is that Schur functions s_λ are eigenfunctions of the operator $\mathcal{D}_{N,k}$ (and of course this is the reason for introducing it in the first place). In particular, we have that

$$\mathcal{D}_{N,k} s_\lambda(x_1,\dots,x_N) = \left(\sum_{i=1}^N (\lambda_i + N - i)^k\right) s_\lambda(x_1,\dots,x_N).$$

This is a straightforward computation with the determinant definition of the Schur functions. By the definition of $m[\rho]$, we have that

$$\mathbb{E}\left(\int_{\mathbb{R}} x^k m[\rho](dx)\right)^n = \sum_\lambda \rho(\lambda)\left(\frac{1}{M}\sum_{i=1}^N \left(\frac{\lambda_i + N - i}{M}\right)^k\right)^n.$$

By the above observation, this can be written as

$$\mathbb{E}\left(\int_{\mathbb{R}} x^k m[\rho](dx)\right)^n = \frac{1}{M^{n(k+1)}}\sum_\lambda \rho(\lambda)\frac{\mathcal{D}_{N,k}^n s_\lambda(x_1,\dots,x_N)}{s_\lambda(x_1,\dots,x_N)}.$$

Evaluating the right side at $x_1 = \cdots = x_N = 1$, the result is immediate by definition of $\mathcal{S}_\rho$. ☐

2.4. **Convergence.** Our goal is to look at asymptotics as $M \to \infty$. To do this, we need to first define the notion of convergence of partitions.

Definition 2.6 ([BuG1, Definition 2.5]). *A sequence $\lambda(M)$ of partitions is called regular if there is a piecewise continuous function $f(t)$ and a constant C such that*

$$\lim_{M\to\infty} \sum_{j=1}^M \left|\frac{\lambda_j(M)}{M} - f(j/M)\right| = 0$$

and

$$\left|\frac{\lambda_j(M)}{M} - f(j/M)\right| < C$$

for all $j = 1,\dots,M$ and $M = 1,2,\dots$.

Remark 3. Essentially what this means is that the partitions converge to some shape $f(t)$. The second condition is a technical one to guarantee that the measures that we will associate to the partitions are uniquely determined by their moments.

The key result we want is about the asymptotics of Schur functions for a regular sequence. In particular, the Schur functions become asymptotically multiplicative. This came up originally in [GP], and later appeared in [BuG1]. The formulation in [BuG1] is most useful for our purposes.

Theorem 2.7 ([BuG1, Theorem 4.2])**.** *Suppose $\lambda(M)$ is a regular sequence of partitions. There exists an analytic function $H(x)$ defined in a neighborhood U of 1 such that for any $k \geq 1$, we have*

$$\lim_{M\to\infty} \frac{1}{M} \log\left(\frac{s_{\lambda(M)}(x_1,\ldots,x_k,1^{M-k})}{s_{\lambda(M)}(1^M)}\right) = H(x_1)+\cdots+H(x_k)$$

where the convergence is uniform over $x_1,\ldots,x_k$ in compact subsets $K \subset U$.

3. Main Results

Theorem 3.1 ([BuG1, Theorem 5.1])**.** *Let $\lambda(M)$ be a regular sequence, and consider the uniform GT-pattern with top row $\lambda(M)$. Fix some constant $0<\eta<1$, and let $N = \lfloor \eta M \rfloor$. Then, the random measures $m^{(N)}$ converge to a deterministic measure m in the limit $M \to \infty$ with moments*

$$\int_{\mathbb{R}} x^k m(dx) = \sum_{\ell=0}^{k} \eta^{\ell+1}\frac{1}{(\ell+1)!}\binom{k}{\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^k H'(z)^{k-\ell}\right),$$

where $H(x)$ is the function of Theorem 2.7.

We also have the "derivative" of the above result.

Theorem 3.2. *Let $\lambda(M)$ be a regular sequence, and consider the uniform GT-pattern with top row $\lambda(M)$. Fix some constant $0<\eta<1$, and let $N=\lfloor \eta M\rfloor$. Then, the random measures $\mu^{(N)}$ converge to a deterministic measure μ in the limit $M\to\infty$ with moments*

$$\int_{\mathbb{R}} x^k \mu(dx) = \sum_{\ell=0}^{k} \eta^{\ell}\frac{1}{\ell!}\binom{k}{\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^k H'(z)^{k-\ell}\right),$$

where $H(x)$ is the function from Theorem 2.7.

As mentioned in the introduction, Theorem 3.1 is a direct consequence of [BuG1, Theorem 5.1], and Theorem 3.2, which is our new result, is an extension of it. We present a proof of Theorem 3.1 similar in spirit to that given in [BuG1], but with methods optimized so that we can change just a few steps to get Theorem 3.2.

Remark 4. Observe that the moments of μ are simply the derivatives of the moments of m with respect to η. This can be viewed as the finite difference of measures becoming a derivative of measures in the limit, see Remark 2.

3.1. **Two Lemmas.** We introduce two lemmas that will be useful in our analysis of the moments of the counting measure. In particular, they help to reduce the number of variables in the calculation. Before proceeding, we introduce the following notation.

Given a function $f(z_1,\ldots,z_n)$, define

$$\sum_{\text{cyc}} f(z_1,\ldots,z_n) := f(z_1,\ldots,z_n) + f(z_2,\ldots,z_n,z_1)+\cdots+f(z_n,z_1,\ldots,z_{n-1}).$$

Lemma 3.3 ([BuG1, Lemma 5.5]). *Let $g(z)$ be a function analytic in some neighborhood of $z = 1$. Then*

$$\lim_{z_i \to 1} \sum_{\text{cyc}} \frac{g(z_1)}{(z_1 - z_2)(z_1 - z_3) \cdots (z_1 - z_n)} = \frac{1}{(n-1)!} \frac{\partial^{n-1} g(z)}{\partial z^{n-1}} \bigg|_{z=1}.$$

Lemma 3.4. *For positive integers k, we have that*

$$\mathcal{D}_{N,k} = \sum_{m=1}^{k} \left\{ {k \atop m} \right\} \sum_{\ell=0}^{m} \binom{m}{\ell} \ell! \sum_{\{i_0, \ldots, i_\ell\} \subseteq [N]} \sum_{\text{cyc}} \frac{x_{i_0}^m \partial_{i_0}^{m-\ell}}{(x_{i_0} - x_{i_1}) \cdots (x_{i_0} - x_{i_\ell})},$$

where $\left\{ {k \atop m} \right\}$ are Stirling numbers of the second kind.

Proof. Let $\Delta(x) = \prod_{i<j}(x_i - x_j)$ denote the Vandermonde determinant. We first compute $\partial_p^m \Delta(x)$. By the Leibniz rule, we have

$$\partial_p^m \Delta(x) = \sum_{\sum_{i<j} k_{i,j} = m} \binom{m}{k_{1,2}, \ldots, k_{n-1,n}} \prod_{i<j} \partial_p^{k_{i,j}} (x_i - x_j).$$

With some work, this reduces to

$$\partial_p^m \Delta(x) = m! \prod_{i<j} (x_i - x_j) \sum_{\substack{S \subseteq [N] \setminus \{p\} \\ |S| = m}} \frac{1}{\prod_{i \in S} (x_i - x_p)}.$$

It is well known that

$$\sum_{i=1}^{N} (x_i \partial_i)^k = \sum_{i=1}^{N} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} x_i^m \partial_i^m,$$

so in fact

$$\begin{aligned} \mathcal{D}_{N,k} &= \Delta(x)^{-1} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} \sum_{i=1}^{N} x_i^m \sum_{\ell=0}^{m} \binom{m}{\ell} (\partial_i^\ell \Delta(x)) \partial_i^{m-\ell} \\ &= \sum_{m=1}^{k} \left\{ {k \atop m} \right\} \sum_{\ell=0}^{m} \binom{m}{\ell} \ell! \sum_{i=1}^{N} x_i^m \sum_{\substack{S \subseteq [N] \setminus \{i\} \\ |S| = \ell}} \left(\prod_{j \in S} \frac{1}{x_j - x_i} \right) \partial_i^{m-\ell}. \end{aligned}$$

But we have that

$$\sum_{i=1}^{N} x_i^m \sum_{\substack{S \subseteq [N] \setminus \{i\} \\ |S| = \ell}} \left(\prod_{j \in S} \frac{1}{x_j - x_i} \right) \partial_i^{m-\ell} = \sum_{\{i_0, \ldots, i_\ell\} \subseteq [N]} \sum_{\text{cyc}} \frac{x_{i_0}^m \partial_{i_0}^{m-\ell}}{(x_{i_0} - x_{i_1}) \cdots (x_{i_0} - x_{i_\ell})},$$

which completes the proof. ∎

3.2. **Proof of Theorem 3.1.** We have to show that

$$\lim_{M\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} x^k m^{(N)}(dx)\right) = \sum_{\ell=0}^{k} \eta^{\ell+1} \frac{1}{(\ell+1)!}\binom{k}{\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^k H'(z)^{k-\ell}\right) \tag{8}$$

and

$$\lim_{M\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} x^k m^{(N)}(dx)\right)^2 = \lim_{M\to\infty}\left(\mathbb{E}\left(\int_{\mathbb{R}} x^k m^{(N)}(dx)\right)\right)^2. \tag{9}$$

We will first show (8). Combining Theorem 2.5 and Proposition 2.2, we have that

$$\mathbb{E}\left(\int_{\mathbb{R}} x^k m^{(N)}(dx)\right) = \frac{1}{M^{k+1}} \lim_{x_i\to 1} \mathcal{D}_{N,k}\left(\frac{s_\lambda(x_1,\dots,x_N,1^{M-N})}{s_\lambda(1^M)}\right). \tag{10}$$

However, by Theorem 2.7, we can write

$$S(x_1,\dots,x_N) := \frac{s_\lambda(x_1,\dots,x_N,1^{M-N})}{s_\lambda(1^M)} = \exp\left(\sum_{i=1}^{N} MH(x_i)\right) T_N(x_1,\dots,x_N), \tag{11}$$

where T_N is analytic in some open neighborhood of (1^N). We see that $T_N(1^N)=1$ and

$$\lim_{M\to\infty} \frac{1}{M}\log T_N(x_1,\dots,x_k,1^{N-k}) = 0$$

for any fixed k and uniformly in some open neighborhood of (1^k). We can differentiate the above result to get

$$\lim_{M\to\infty} \frac{1}{M}\frac{\partial_1^{a_1}\cdots\partial_k^{a_k} T_N(x_1,\dots,x_k,1^{N-k})}{T_N(x_1,\dots,x_k,1^{N-k})} = 0 \tag{12}$$

for any $(a_1,\dots,a_k)\in\mathbb{Z}_{\geq 0}^k$. By Lemma 3.4, we have that

$$\mathcal{D}_{N,k}S = \sum_{m=1}^{k}\left\{ {k \atop m} \right\}\sum_{\ell=0}^{m}\binom{m}{\ell}\ell! \sum_{\{i_0,\dots,i_\ell\}\subseteq[N]}\sum_{\text{cyc}} \frac{x_{i_0}^m \partial_{i_0}^{m-\ell} S}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})}. \tag{13}$$

We wish to take the $\lim_{x_i\to 1}$ of both sides. To do this we apply the following proposition.

Proposition 3.5. *The leading order term of*

$$L := \lim_{x_i\to 1}\sum_{\text{cyc}} \frac{x_{i_0}^m \partial_{i_0}^{m-\ell} S}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})}$$

is

$$\frac{1}{\ell!} M^{m-\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^m H'(z)^{m-\ell}\right)\bigg|_{z=1}.$$

Proof. Since all functions are symmetric, we may assume $\{i_0,\dots,i_\ell\}=\{1,\dots,\ell+1\}$. We see that we can set $x_r=1$ for $r\geq \ell+2$, so we simply have

$$L = \lim_{x_1,\dots,x_{\ell+1}\to 1}\sum_{\text{cyc}} \frac{x_1^m\partial_1^{m-\ell} S(x_1,\dots,x_{\ell+1},1^{N-\ell-1})}{(x_1-x_2)\cdots(x_1-x_{\ell+1})}.$$

From (11) and (12), we have that the leading order term of

$$\partial_1^{m-\ell} S(x_1,\dots,x_{\ell+1},1^{N-\ell-1})$$

is $M^{m-\ell}H'(x_1)^{m-\ell}S(x_1,\dots,x_{\ell+1},1^{N-\ell-1})$. Thus, the leading order term of L is

$$M^{m-\ell}\lim_{x_1,\dots,x_{\ell+1}\to 1}\sum_{\text{cyc}}\frac{x_1^m H'(x_1)^{m-\ell}}{(x_1-x_2)\cdots(x_1-x_{\ell+1})}S(x_1,\dots,x_{\ell+1},1^{N-\ell-1}).$$

Applying Lemma 3.3 and noting that $S(1^N)=1$ yields the desired result. □

Therefore, the leading order term of the right side of (13) under the limit $x_i\to 1$ is

$$\sum_{m=1}^{k}\left\{ {k \atop m} \right\}\sum_{\ell=0}^{m}\binom{m}{\ell}\binom{N}{\ell+1}M^{m-\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^m H'(z)^{m-\ell}\right)\bigg|_{z=1}. \tag{14}$$

The order of the summand is $M^{\ell+1}M^{m-\ell}=M^{m+1}$, so the only contribution in the limit comes from $m=k$, so the leading order term is

$$M^{k+1}\sum_{\ell=0}^{M}\frac{\eta^{\ell+1}}{(\ell+1)!}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^m H'(z)^{m-\ell}\right)\bigg|_{z=1}.$$

Thus, by (10), we have that

$$\lim_{M\to\infty}\mathbb{E}\left(\int_{\mathbb{R}}x^k m^{(N)}(dx)\right)=\sum_{\ell=0}^{k}\eta^{\ell+1}\frac{1}{(\ell+1)!}\binom{k}{\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}\left(z^k H'(z)^{k-\ell}\right),$$

as desired.

We'll now show (9). We just want to show that the leading order terms of $(\mathcal{D}_{N,k}S)^2$ and $(\mathcal{D}_{N,k})^2S$ match up when we take $x_i\to 1$. We see that

$$(\mathcal{D}_{N,k}S)^2=\sum_{m=1}^{k}\sum_{m'=1}^{k}\left\{ {k \atop m} \right\}\left\{ {k \atop m'} \right\}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{m}\binom{m}{\ell}\ell!\binom{m'}{\ell}\ell'!$$
$$\sum_{\{i_0,\dots,i_\ell\}\subseteq[N]}\sum_{\{i'_0,\dots,i'_\ell\}\subseteq[N]}\sum_{\text{cyc}}\sum_{\text{cyc}}\frac{(x_{i_0}^m\partial_{i_0}^{m-\ell}S)(x_{i'_0}^m\partial_{i'_0}^{m'-\ell'}S)}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})(x_{i'_0}-x_{i'_1})\cdots(x_{i'_0}-x_{i'_\ell})}$$

and

$$(\mathcal{D}_{N,k})^2S=\sum_{m=1}^{k}\sum_{m'=1}^{k}\left\{ {k \atop m} \right\}\left\{ {k \atop m'} \right\}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{m}\binom{m}{\ell}\ell!\binom{m'}{\ell}\ell'!$$
$$\sum_{\{i_0,\dots,i_\ell\}\subseteq[N]}\sum_{\{i'_0,\dots,i'_\ell\}\subseteq[N]}\sum_{\text{cyc}}\sum_{\text{cyc}}\frac{(x_{i_0}^m\partial_{i_0}^{m-\ell}x_{i'_0}^m\partial_{i'_0}^{m'-\ell'})S}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})(x_{i'_0}-x_{i'_1})\cdots(x_{i'_0}-x_{i'_\ell})}.$$

The only difference in the two is the numerator of the final summand. We have the following claim that is an analogue of Proposition 3.5.

Proposition 3.6. *The leading order terms of*

$$\lim_{x_i\to 1}\sum_{\text{cyc}}\sum_{\text{cyc}}\frac{(x_{i_0}^m\partial_{i_0}^{m-\ell}S)(x_{i'_0}^m\partial_{i'_0}^{m'-\ell'}S)}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})(x_{i'_0}-x_{i'_1})\cdots(x_{i'_0}-x_{i'_\ell})}$$

and

$$\lim_{x_i\to 1}\sum_{\text{cyc}}\sum_{\text{cyc}}\frac{(x_{i_0}^m\partial_{i_0}^{m-\ell}x_{i_0'}^m\partial_{i_0'}^{m'-\ell'})S}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})(x_{i_0'}-x_{i_1'})\cdots(x_{i_0'}-x_{i_\ell'})}$$

are identical.

Proof. As in the proof of Proposition 3.5, set $x_r = 1$ for all $r \notin \{i_0, \dots, i_\ell\} \cup \{i_0', \dots, i_\ell'\}$. Then, using (11) and (12), the leading order term of $(x_{i_0}^m\partial_{i_0}^{m-\ell}S)(x_{i_0'}^m\partial_{i_0'}^{m'-\ell'}S)$ is

$$M^{m-\ell+m'-\ell'}x_{i_0}^m x_{i_0'}^{m'}H'(x_{i_0})^{m-\ell}H'(x_{i_0'})^{m'-\ell'}S^2.$$

Now, the leading order term of $(x_{i_0}^m\partial_{i_0}^{m-\ell}x_{i_0'}^m\partial_{i_0'}^{m'-\ell'})S$ is the same as the leading order term of

$$M^{m'-\ell'}x_{i_0}^m\partial_{i_0}^{m-\ell}(x_{i_0}^m H'(x_{i_0'})S).$$

When applying the product rule, we get a new factor of M when we apply the derivative on the S, and we don't for any of the other terms. Therefore, the leading order of this is

$$M^{m-\ell+m'-\ell'}x_{i_0}^m x_{i_0'}^{m'}H'(x_{i_0})^{m-\ell}H'(x_{i_0'})^{m'-\ell'}S.$$

When taking $x_i \to 1$ and using lemma 3.3, we will get the same thing since $S(1^N)^2 = S(1^N) = 1$. Thus, the leading order terms match as desired. □

This shows (9), and the proof of Theorem 3.1 is complete.

3.3. **Proof of Theorem 3.2.** Repeating the proof of Theorem 2.5, we see that

$$\mathbb{E}\left(\int_{\mathbb{R}} x^k\mu^{(N)}(dx)\right)^m = \frac{1}{M^{mk}}(\mathcal{D}_{N,k}-\mathcal{D}_{N-1,k})^m\mathcal{S}_{\rho^{(N)}}(x_1,\dots,x_N)\bigg|_{x_1=\cdots=x_N=1}. \tag{15}$$

We have from lemma 3.4 that

$$\mathcal{D}_{N,k}-\mathcal{D}_{N-1,k} = \sum_{m=1}^{k}\left\{ {k \atop m} \right\}\sum_{\ell=0}^{m}\binom{m}{\ell}\ell!\sum_{\substack{\{i_0,\dots,i_\ell\}\subseteq[N]\\ N\in\{i_0,\dots,i_\ell\}}}\sum_{\text{cyc}}\frac{x_{i_0}^m\partial_{i_0}^{m-\ell}}{(x_{i_0}-x_{i_1})\cdots(x_{i_0}-x_{i_\ell})}.$$

Thus, the entire proof of Theorem 3.1 carries over, except the step at (14). Now, instead of $\binom{N}{\ell+1}$, we will have $(\ell+1)\binom{N}{\ell}$. The loss of order of M by 1 here is compensated by the same loss in (15). However, the factor of $\frac{\eta^{\ell+1}}{(\ell+1)!}$ in Theorem 3.1 that came from $\binom{N}{\ell+1}$ now becomes $\frac{\eta^\ell}{\ell!}$. Thus, the moments of μ converge to

$$\sum_{\ell=0}^{k}\eta^\ell\frac{1}{\ell!}\binom{k}{\ell}\left(\frac{\partial}{\partial z}\right)^\ell\left(z^kH'(z)^{k-\ell}\right),$$

as desired. This completes the proof of Theorem 3.2.

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