Maps between Critical Groups of Group Representations

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May 20, 2017

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Introduction to Chip Firing

\[ [0, 0, 4]^t \quad [1, 1, 2]^t \quad [2, 2, 0]^t \quad [3, 0, 1]^t \]
Definitions for the Graph Case

Let there be $v_i$ chips on node $i$ of our graph $G$. Define a chip configuration, $v = [v_0, v_1, .., v_l]^t \in \mathbb{N}^{l+1}$. 
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- A firing on a graph $G$ is defined by sending a single chip from a node $i$ to all of its neighbours.
- \textit{Stable Configurations} are chip configurations $v < d^C$ which do not permit additional firings.
- \textit{Recurrent Configurations} are stable configurations $v$ such that for all chip-configurations $w$, selectively adding chips to $w$ and stabilizing yields $v$. 
A Laplacian Matrix, \( L(G) = D - A \), where \( D \) is the degree matrix such that \( D_{ij} = \text{deg}(\text{node } i) \) if \( i = j \) and \( D_{ij} = 0 \) if \( i \neq j \). \( A \) is an adjacency matrix such that \( A_{ij} \) is the number of edges from node \( i \) to node \( j \).

Define a dynamical firing on node \( i \) that sends \( v \) to \( v - r_i \), where \( r_i \) corresponds to the \( i \)th row of \( L(G) \), the Laplacian Matrix of \( G \). Let \( d^C \) be the diagonal of \( L(G) \).
Example of Graph Laplacian

\[ L(G) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \]

\[ [0, 0, 4]^t \quad [1, 1, 2]^t \quad [2, 2, 0]^t \quad [3, 0, 1]^t \]
Critical Groups

Definition
Let $G$ be a digraph on $n$ vertices with global sink $s$. The critical group of $G$ is the group quotient:

$$K(G) = \mathbb{Z}^n / \text{im}(L^t(G))$$

Theorem
Let $G$ be a digraph with a global sink. The set of all recurrent chip on $G$ is an abelian group under the operation $(v, w) \rightarrow \text{stab}(v + w)$, and it is isomorphic via the inclusion map to the critical group $L(G)$. 
Revisiting the Graph Example

\[ L(G) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \]

\[ K(G) = \mathbb{Z}/3\mathbb{Z} \]

Recurrent Configurations: \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

Each recurrent must have order 3:
- \( [1, 1]^t \) is the zero recurrent of order 1
- \( [0, 1]^t \) has order 3 since \( \text{stab}([0, 3]^t) = [1, 1] \)
- \( [1, 0]^t \) has order 3 since \( \text{stab}([3, 0]^t) = [1, 1] \)
A group $G$ is a set of elements that are closed under a certain binary operator (the group operation), associativity, identity, and invertibility. For example, the 3rd roots of unity form a group under the operation of normal multiplication.
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A representation of a group $G$ on a vector space $V$, is a homomorphism or map $p : G \rightarrow GL(V)$ such that:

$$p(g_1)p(g_2) = p(g_1g_2)$$

for all $g_1, g_2 \in G$
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An explicit example for the cyclic group $C_3$ with elements $1, g, g^2$ is:

$$1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad g \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad g^2 \rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$
Analogy to the Group Case

For a representation $V$ of the group $G$: Define the McKay-Cartan Matrix to be $\tilde{C} = nl - M$, where $n$ is the dimension of $V$ and:

$$\chi V \chi_i = \sum m_{ij} \chi_j$$

The chip-firing game applies to any $\mathbb{Z}$-matrices and we introduce the McKay-Cartan matrix here instead of the graph Laplacian Matrix. The reduced McKay-Cartan Matrix can be defined as the submatrix by removing the first row and column. In this way, the critical group is defined analogously as:

$$K(V) = \mathbb{Z}^n / \text{im}(C^t(V))$$
The Abelian Group Case

In the case of abelian groups, we have a complete classification of the map $p$ and found an exact correspondence of $p$ to regular covering maps on graphs. From Reiner-Tseng, we also have a combinatorial interpretation of the kernel of our map. In fact, we have discovered the following theorem:

**Definition**

The Cayley Graph of a group $G$ with generating set $S$ has elements of the group as its node and edges between $g$ and $gs$ for elements $s \in S$. The nodes of our Cayley Graph are the irreducible representations of $G$ and the edges correspond to the choice of our faithful representation $V$.

**Theorem**

There is a surjection of critical groups from $K(V)$ to $K(\text{Res}_H^G V)$ corresponding to the map $p$, a graph covering map on the Cayley Graphs of each group.
Let $G = C_6 = \langle g \mid g^6 = e \rangle$ and consider the representation $V = V_{w^2} \oplus V_w$, where $V_{w^k}$ sends $g \rightarrow w^k$ where $w^6 = 1$. Consider the subgroup $H = C_2$ and the regular covering can be depicted by:
General Maps Between Critical Groups

Define the map \( p : \mathbb{Z}^{l+1} \rightarrow \mathbb{Z}^{l'+1} \), with standard matrix \( A \), such that:

\[
\text{Res}V_i = \bigoplus W_j A_{ij}
\]

Theorem

The following diagrams commute:

\[
\begin{array}{ccc}
\mathbb{Z}^{l+1} & \xrightarrow{C^t(V)} & \mathbb{Z}^{l+1} \\
\downarrow p & & \downarrow p \\
\mathbb{Z}^{l'+1} & \xrightarrow{C^t(\text{Res}V)} & \mathbb{Z}^{l'+1}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{Z}^{l'+1} & \xrightarrow{C^t(\text{Res}V)} & \mathbb{Z}^{l'+1} \\
\downarrow p^t & & \downarrow p^t \\
\mathbb{Z}^{l+1} & \xrightarrow{C^t(V)} & \mathbb{Z}^{l+1}
\end{array}
\]

Hence, we have a map, \( \pi : K(V) \rightarrow K(\text{Res}V) \) on cosets:

\[
\pi : u + \text{im}(C^t(V)) \rightarrow p(u) + \text{im}(C^t(\text{Res}V))
\]
Proof Outline

- From characters, $p$ corresponds to restriction of virtual representations, considered in $\mathbb{Z}^{l+1}$
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- From characters, $p$ corresponds to restriction of virtual representations, considered in $\mathbb{Z}^{l+1}$
- $p^t$ corresponds to induction of virtual representations in $\mathbb{Z}^{l'+1}$.
- Check that $\text{Res}V_1 \otimes \text{Res}V \cong \text{Res}(V_1 \otimes V)$ (Commutativity with $M$)
- Check that $\text{Ind}W_1 \otimes V \cong \text{Ind}(W_1 \otimes \text{Res}V)$
It is known that $p$ is surjective as a linear map and also as a map of cosets: $p : K(S_n) \rightarrow K(S_{n-1})$. 
The Symmetric Group

- It is known that \( p \) is surjective as a linear map and also as a map of cosets: \( p : K(S_n) \to K(S_{n-1}) \).
- Whenever \( p \) is surjective as a linear map, \( p^t \) must be injective as a map of cosets: \( p^t : K(S_{n-1}) \to K(S_n) \).
The Symmetric Group

- It is known that $p$ is surjective as a linear map and also as a map of cosets: $p : K(S_n) \rightarrow K(S_{n-1})$.
- Whenever $p$ is surjective as a linear map, $p^t$ must be injective as a map of cosets: $p^t : K(S_{n-1}) \hookrightarrow K(S_n)$.
- Induction and restriction of irreducible representations is well-understood with the concept of Young Diagrams (Binary Matrix for $p$).
Theorem from Gaetz

Let $\gamma$ be the reflection representation of $S_n$ and let $p(j)$ denote the number of partitions of the integer $j$. Then:

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)-p(n-1)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{1 \leq j \leq n, p(j) - p(j-1) \geq i} j$$

Lemma

The kernel of our map, $p$, for $\gamma$ is as follows:

$$\ker(p) = (\mathbb{Z}/n\mathbb{Z})^{p(n)-p(n-1)-1}$$
Future Work

- Describe the kernels of our maps in terms of "voltage critical groups," with some combinatorial structure (group elements in the graph case)
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- Give explicit formulas and/or bounds on critical groups for specific representations of the symmetric group (Using repeated character values and injectivity of $p^t$)

Theorem from Gaetz

If $\chi_\gamma$ is real-valued, as in the symmetric group case, and $\chi_\gamma(c)$ is an integer character value achieved by $m$ different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n - \chi_\gamma(c))\mathbb{Z})^{m-1}$
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- Investigate other potential maps such as dualization (commutes with the same diagram)
- Identify connections to chip firing
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