Continuum Modelling of Traffic Systems with Autonomous Vehicles

Kaiying Hou, Brian Rhee
Mentor: Andrew Rzeznik

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Traffic Flow

Figure: Red Blood Cells

Figure: Traffic Flow
Autonomous Vehicle
Continuum Variables

\( \rho(x, t) \): Density defined as the numbers of cars per unit length.

Figure: The density function from a PDE. Here, the \( \rho \) function is the density of cars.
Flux

\[ J(x, t) : \text{Flux defined as the amount of car that pass through } x \text{ per unit time.} \]

\[ J = \frac{\text{cars}}{\text{time}} = \frac{\text{cars}}{\text{length}} \frac{\text{length}}{\text{time}} = \rho v \]
Conservation Equation

\[ \rho_t + J_x = 0 \]

Integral Form:

\[ \frac{d}{dt} \int_{x_0}^{x_0+dx} \rho(x, t) \, dx = J(x_0, t) - J(x_0 + dx, t) \]
Constitutive Laws

Let \( v = v(\rho) \)

- **Greenshield’s Law**
  \[
  v = v_m (1 - \frac{\rho}{\rho_m}) \\
  J = v_m (1 - \frac{\rho}{\rho_m}) \rho
  \]

- **Burger’s Equation**
  \[
  v(\rho) = \frac{1}{2} \rho \\
  J(\rho) = \frac{1}{2} \rho^2
  \]
Solving Conservation Equations

\[ \rho_t + J'(\rho) \rho_x = 0 \]

\[ \frac{dx}{dt} = J'(\rho) \]

\[ x = x_0 + J'(\rho) t \]
Shockwave

\[ s'(t) = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L} \]
Finite Volume Method

Let function \( f(x, t) \) at \( x = x_0 \) and \( t = t_0 \) be written as \( f_{x_0}^{t_0} \)

\[
\frac{d}{dt} \int_{x_0}^{x_0 + dx} \rho(x, t) \, dx = J(x_0, t) - J(x_0 + dx, t)
\]
\[
\Delta x \left( \frac{\rho^{t+1}}{\Delta t} - \frac{\rho^t}{\Delta x} \right) = J_{x-\frac{1}{2}} - J_{x+\frac{1}{2}}
\]
\[
\frac{\rho^{t+1}}{\Delta x} = \frac{\rho^t}{\Delta x} + \frac{\Delta t}{\Delta x} \left( J_{x-\frac{1}{2}} - J_{x+\frac{1}{2}} \right)
\]
Upwinding
Constitutive Laws for Autonomous Vehicles

Linear Piecewise: \( v(\rho) = \begin{cases} 
 v_m & (\rho \leq \rho_c) \\
 -\frac{v_m}{\rho_m-\rho_c} \rho + \frac{v_m\rho_m}{\rho_m-\rho_c} & (\rho_c < \rho \leq \rho_m)
\end{cases} \)

Arctan: \( y = \tan^{-1}(x) \)
\[ \therefore v(\rho) = A\tan^{-1}(C\rho + D)+B \]

ERF: \( \text{erf}(x) = \int_{-\infty}^{x} e^{-t^2} dt \)
\[ \therefore v(\rho) = A\text{erf}(C\rho + D)+B \]

![Graph of velocity vs. density for linear piecewise function](image1)

![Graph of velocity vs. density for arctan function](image2)

![Graph of velocity vs. density for erf function](image3)
System of Conservation Equation

\( \rho(x, t) \): the density of regular cars
\( \sigma(x, t) \): the density of autonomous cars

\[ \rho_t + J_1(\rho, \sigma)_x = 0 \]
\[ \sigma_t + J_2(\rho, \sigma)_x = 0 \]
Lax-Friedrichs Method

\[ J_{x-\frac{1}{2}} = \frac{1}{2} (J_{x-1} + J_x) - \frac{\Delta x}{2\Delta t} (\overline{\rho}_x - \overline{\rho}_{x-1}) \]

\[ \overline{\rho}^{t+1}_x = \overline{\rho}^t_x + \frac{\Delta t}{\Delta x} (J_{x-\frac{1}{2}} - J_{x+\frac{1}{2}}) \]
Future Goals

- Different Methods
- Solving the Coupled Conservation Equations
- Multiple Lanes
- Adding Diffusive Terms
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References